

Deformed dispersion relation constraint with hydrogen atom 1S-2S transition*

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Abstract: We use the latest results of the ultra-high accuracy 1S-2S transition experiments in the hydrogen atom to constrain the forms of the deformed dispersion relation in the non-relativistic limit. For the leading correction of the non-relativistic limit, the experiment sets a limit at an order of magnitude for the desired Planck-scale level, thereby providing another example of the Planck-scale sensitivity in the study of the dispersion relation in controlled laboratory experiments. For the next-to-leading term, the bound is two orders of magnitude away from the Planck scale, however it still amounts to the best limit, in contrast to the previously obtained bound in the non-relativistic limit from the cold-atom-recoil experiments.

Keywords: deformed dispersion relation, hydrogen atom, transition experiment, Planck-scale sensitivity

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1 Introduction

Establishing a complete and self-consistent quantum theory of gravity constitutes one of the main challenges in modern physics. To date, a complete understanding of quantum gravity is lacking, however some phenomenological attempts to explore quantum gravity effects have attracted many researchers' attention [1-56]. Since most quantum gravity effects are expected to occur at the ultra-high Planck energy scale ($E_p = \sqrt{\hbar c^5/G} \cong 1.2 \times 10^{19}$ GeV), there are only slight traces on processes that we can approach experimentally. Hence, it is particularly challenging to gain experimental insight into the quantum gravity scale. However, through tremendous and determined efforts in the past decade, we now have at least a few research lines in the quantum gravity phenomenology, in which we have determined that quantum properties of gravity can be studied with the desired Planck-scale sensitivity. For example, due to the ultra-high levels of accuracy of atom interferometry, the cold-atom-recoil experiments have been used to establish meaningful bounds on parameters characterizing quantum gravity effects.

Moreover, the exceptional sensitivity of the experiments set a limit within a single order of magnitude of the desired Planck-scale level, thereby providing the first example of the Planck-scale sensitivity in the study of the dispersion relation in controlled laboratory experiments [45, 57]. In this study, we attempt to find another example to approach or reach the desired Planck-scale sensitivity using the latest results of the hydrogen atom 1S-2S transition experiments to constrain the forms of the deformed dispersion relation in the non-relativistic limit. In Refs. [58, 59], quantum gravity corrections to Lamb Shift have computed the framework of the generalized uncertainty principle (GUP), where the accuracy of the precision measurement of the Lamb Shift of about 1×10^{-12} leads to the upper bounds on parameters of quantum gravity effects $\beta_0 < 10^{36}$. In Ref. [60], an upper bound ($\beta_0 < 10^{34}$) of quantum gravity effects has been obtained using the high-precision spectrometry of the 1S-2S two photon transition in atomic hydrogen. In contrast, the progress of frequency conversion technology, such as frequency doubling and frequency division in laser research, makes the precision of Lamb Shift experiments in the hydrogen atom and deuterium atom ultra high. In Refs. [61,

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[62], the accuracy of the precision measurement of the hydrogen 1S-2S frequency (Lamb Shift experiments) reaches 10^{-15} . In our case, we use the latest results of the hydrogen atom 1S-2S transition experiments to observe the Planck-scale sensitivity of quantum gravity.

The remainder of this paper is organized as follows. In Sec. 2, we briefly introduce the deformed dispersion relation in the non-relativistic limit. Then, by comparing the results of a detailed calculation of the deformed dispersion relation effects on the 1S-2S transition in the hydrogen atom with its accuracy of precision measurement, upper bounds on the parameters of the deformed dispersion relation are obtained in Sec. 3. Sec. 4 provides the conclusions.

2 Deformed dispersion relation in non-relativistic limit

In 2002, Amelino-Camelia has constructed the famous doubly special relativity (DSR), which has two observer-independent constants, i.e., the speed of light c and Planck length L_p of relativity [1]. In the DSR, the deformed dispersion relation naturally leads to the Planck scale departure from Lorentz symmetry, which is referred to as the Lorentz invariance violation of dispersion relations. Related studies advocated that the general effect of spacetime quantization is the correction of the classical-spacetime dispersion relation between the energy E and momentum p of a microscopic particle with mass m , usually in the form

$$E^2 = p^2 + m^2 + p^2 \left(\xi_n \frac{E}{M_p} \right)^n, \quad (1)$$

where the speed of light c is set to 1. These modifications of the dispersion relation over the past decade have been extensively studied by analyzing observational astrophysics data, which of course involve the ultra-relativistic ($p \gg m$) system of particle kinematics [33, 44, 63-67].

In the non-relativistic limit ($p \ll m$), the deformed dispersion relation (1) should be taken the form [45, 48]

$$E \approx m + \frac{p^2}{2m} + \frac{1}{2M_p} (\xi_1 m p + \xi_2 p^2 + \xi_3 \frac{p^3}{m}). \quad (2)$$

The dispersion relation includes correction terms that are linear in $1/M_p$. To really introduce quantum gravity effects in some neighborhood of the Planck scale, the model-dependent dimensionless parameters ξ_1, ξ_2, ξ_3 should have values approximately of order one. The results from loop quantum gravity [35, 68-69] and non-commutative geometry [70, 71] have likewise shown that at least some of these parameters should be non-zero. In our case, it is reasonable to use the deformed dispersion relation in the non-relativistic limit, because $p \ll m$ (the energy of the electron for $n = 1$ state of hydrogen is about 13.6 eV,

whereas its mass $m \approx 0.5 \times 10^6$ eV).

Unfortunately, just as quantum gravity research usually have challenges, it is also an extremely difficult challenge to translate the theoretically favoured values of these parameters of the deformed dispersion relation into a range of possible magnitudes of the effects. From the deformed dispersion relation (2), we find that if the Planck scale is at the characteristic scale of quantum gravity effects, the values of these parameters (i.e. ξ_1, ξ_2, ξ_3) should indeed be close to 1, such that the effects of the deformed terms characterized quantum gravity effects is extremely small due to the overall factor $1/M_p$. Although some studies have shown that the quantum gravity scale may be slightly smaller than the Planck scale, and that it may even be three orders of magnitude smaller than the Planck scale, which is consistent with the grand unification scale in particle physics [48, 72, 73]. Therefore, these parameters characterizing quantum gravity effects are obtained by three orders of magnitude, however the prospect of detectable quantum gravity effects remains very small.

Recently, The Planck-scale sensitivity in the deformed dispersion relation (2) has been studied using cold atom recoil experiments in Ref. [45], and meaningful bounds on the parameters ξ_1 and ξ_2 have been obtained. Results show that $\xi_1 = -1.8 \pm 2.1$ and $|\xi_2| < 10^9$, using the experimental data of Caesium-atom recoil measurements in Ref. [74] and electron-anomaly measurements in Ref. [75]. As discussed above, the range of values of ξ_1 indicates that the cold-atom recoil experiments can be considered as the first example of controlled laboratory experiments probing the form of the dispersion relation with a sensitivity that is meaningful from a Planck scale perspective. However, the bound on parameter ξ_2 in the dispersion relation remained a few orders of magnitude away from the Planck scale.

Therefore, our main objective in this study is to show that the experiment of the ultra-high accuracy 1S-2S transition in hydrogen atom can be used to establish improved bounds on the parameters ξ_1 and ξ_2 , which characterized the non-relativistic limit of the deformed dispersion relation (2).

3 Bounds on parameters of deformed dispersion relation

The hydrogen atom has played a central role in the development of quantum mechanics. As it is the simplest of atoms, it is used for the development and testing of fundamental theories through ever-refined comparisons between experimental data and theoretical predictions. Hydrogen spectroscopy is closely related to the successive advances in the understanding of the atomic structure.

In recent years, with the advance of experimental technology, the absolute frequency of the 1S-2S transition in atomic hydrogen via two photon spectroscopy has been measured with particularly high precision, such that it can be used to achieve various accurate measurements. For example, the Rydberg constant R_∞ and the proton charge radius have been improved through the advance of measurement precision of the 1S-2S two photo transition [76]. A value of $R_\infty = 10973731.56854(10) \text{ m}^{-1}$ was obtained. 1S-2S hydrogen spectroscopy can also be used to search new limits on the drift of fundamental constants [77, 78]. Another important application of the 1S-2S two photo transition is used to test the electron boost invariance [78]. Inspired by these achievements with the absolute 1S-2S transition frequency in atomic hydrogen, we consider the possibility of studying quantum gravity effects applying hydrogen atomic spectroscopy.

In our case, we ignore the hyperfine structure, such that the hydrogen energy levels are given by [79]

$$E(n, J, L) = E_{DC}(n, J) + E_{RM}(n, J) + E_{LS}(n, J, L), \quad (3)$$

where E_{DC} and E_{RM} represent the Dirac-Coulomb energy and the energy contributed by the leading recoil corrections due to the finite mass of the nucleus, respectively. These two energy contributions play a major role in the hydrogen energy, which are functions of the Rydberg constant R_∞ , the fine structure constant α , and the ratio of the electron and nuclear mass m_e/m_N . The last term E_{LS} represents the energy contributed by the Lamb shift, which contains the QED corrections and corrections for the finite size and polarizability of the nucleus. Comments on the contributions of hydrogen atoms are provided in Refs. [79-81]. In our case, we follow the expression derived by Bethe for the energy level shift. It has been pointed out by Bethe [82] that the displacement of the 2S level of hydrogen observed by Lamb and Retherford [83] can be simply explained as a shift in the energy of the atom arising from its interaction with the radiation field. Subsequently, by calculating the mean square amplitude of oscillation of an electron coupled to the zero-point fluctuations of the electromagnetic field, the shift of nS energy levels has been given by [84]

$$\Delta E_n = \frac{4\alpha^2}{3m^2} \left(\ln \frac{1}{\alpha} \right) |\psi_n(0)|^2 = \frac{8\alpha^3}{3\pi n^3} \left(\ln \frac{1}{\alpha} \right) \left(\frac{1}{2} \alpha^2 m \right) \delta_{l0}. \quad (4)$$

Since the scale of quantum electrodynamic effect is related to the principle quantum number n as $1/n^3$, the 1S Lamb shift is the largest within atomic hydrogen.

Our main objective here is to expose sensitivity to a meaningful range of values of the parameters ξ_1 and ξ_2 , hence we focus on the Planck scale corrections with coefficient ξ_1 and ξ_2 . In the non-relativistic limit $p \ll m$, since the contribution of the relativistic correction terms to the energy in the relativistic Dirac Hamiltonian is far less than that of the non-relativistic Schrodinger Hamiltonian,

we only consider the effect of the Planck scale corrections on the non-relativistic Schrodinger Hamiltonian. Thus, the Planck scale correction terms are regarded as the perturbation terms of the levels energy of hydrogen atom with a well-defined quantum Hamiltonian. In the deformed dispersion relation (2), the leading correction and the next-to-leading correction are respectively denoted by Hamiltonians \hat{H}' and \hat{H}'' , where

$$\hat{H}' = \xi_1 \frac{m}{2M_p} \hat{p}, \quad \hat{H}'' = \xi_2 \frac{\hat{p}^2}{2M_p}. \quad (5)$$

Here, we compute the bounds on parameters ξ_1 and ξ_2 by studying the Planck scale correction of the hydrogen energy levels.

3.1 Bounds on parameter ξ_1

Since the hydrogen atom is spherically symmetric, the Coulomb potential of the hydrogen atom is given by

$$V(r) = -k/r, \quad (6)$$

where $k = e^2/4\pi\epsilon_0 = \alpha\hbar$, e is electronic charge. To the first order, the perturbing Hamiltonian \hat{H}' shift the energy to

$$E_n = E_n^{(0)} + \xi_1 \frac{m}{2M_p} \langle nlm | \hat{p} | nlm \rangle, \quad (7)$$

where $E_n^{(0)} = -k/2an^2$, a is the Bohr radius. As discussed above, the 1S Lamb shift is the largest in atomic hydrogen, hence we are concerned only with the effects of the Planck scale correction on the shift of 1S energy levels. We have $l = m = 0$, and utilize the following to calculate the energy shift: $R_{10}(r) = 2a^{-3/2}e^{-r/a}$, $Y_{00} = 1/\sqrt{4\pi}$. We derive

$$\langle 100 | \hat{p} | 100 \rangle = -i\hbar \langle 100 | \frac{\partial}{\partial r} | 100 \rangle = \frac{i\hbar}{a}. \quad (8)$$

Thus, the shift of energy levels due to the leading correction in the DSR framework is expressed as

$$\Delta E = |\xi_1 \frac{m}{2M_p} \langle 100 | \hat{p} | 100 \rangle| = \xi_1 \frac{m\hbar}{2M_p a}. \quad (9)$$

The additional contribution due to the correction of the parameter ξ_1 term in proportion to the original value 1S Lamb shift is given by

$$\frac{\Delta E}{\Delta E_1} = \xi_1 \frac{3\pi m}{8M_p \alpha^4 \ln \frac{1}{\alpha}} \approx 3.5 \times 10^{-15} \xi_1, \quad (10)$$

where some values in Table 1 have been used. As discussed above, if the Planck scale is the characteristic scale of quantum gravity effects, the parameter ξ_1 should indeed be close to 1, and thus the additional contribution in proportion to the original value (10) is approximately equal to 3.5×10^{-15} . The current accuracy of the precision measurement of the hydrogen 1S-2S transition reaches the 4.5×10^{-15} regime [62]. This interestingly means that the hydrogen 1S-2S transition experiment we considered here can indeed probe the Planck-scale sensitivity on the basis

Table 1. Quantities used in our calculation.

quantity	value	precision	source
α^{-1}	137.035999139(31)	2.3×10^{-10}	[85]
m	$0.5109989461(31)/c^2$ MeV	6.2×10^{-9}	[85]
M_p	$1.220910(29) \times 10^{19}/c^2$ GeV	2.3×10^{-5}	[85]

of the deformed dispersion relation (2). Therefore, we can finally set out to determine the constraint on the parameter ξ_1 by imposing that the corrections are smaller than the experimental error on the value of the hydrogen 1S-2S transition, i.e. $|\xi_1| \leq 1.3$. This estimate is closely related to the degree of coincidence between the physical observation and the theoretical prediction. Since this estimate is determined using the fine-structure constant α as input, the uncertainty of this estimate is orders of magnitude above the experiment of the hydrogen 1S-2S transition, i.e., 1.26×10^{-7} .

3.2 Bounds on parameter ξ_2

Following the same steps that we performed above for the correction term with coefficient ξ_1 , it is easy to verify that the correction term with coefficient ξ_2 would produce the following modification of the hydrogen 1S energy levels

$$\Delta E' = |\langle 100 | \hat{H}'' | 100 \rangle| = \xi_2 \frac{1}{2M_p} |\langle 100 | \hat{p}^2 | 100 \rangle|. \quad (11)$$

Using the expression

$$\hat{p}^2 = 2m \left[\hat{H}_0 + \frac{k}{r} \right], \quad (12)$$

where $\langle 100 | \hat{H}_0 | 100 \rangle = E_1^{(0)}$, we have

$$\langle 100 | \hat{p}^2 | 100 \rangle = \frac{mk}{a} = \frac{m\hbar\alpha}{a}. \quad (13)$$

The shift of energy levels due to the next-to-leading correction in the DSR framework is expressed as

$$\Delta E' = \xi_2 \frac{m\alpha\hbar}{2M_p a}. \quad (14)$$

Thus, the additional contribution due to the correction of the parameter ξ_2 term, in proportion to the original value 1S Lamb shift, is given by

$$\frac{\Delta E'}{\Delta E_1} = \xi_2 \frac{3\pi m}{32M_p \alpha^3 \ln \frac{1}{\alpha}} \approx 2.6 \times 10^{-17} \xi_2. \quad (15)$$

According to the current accuracy of precision measurement of the hydrogen 1S-2S transition, this result allows us to establish that $|\xi_2| < 10^2$, which means that we indeed can probe the spacetime structure down to length scales on the order of $10^{-33}m$ ($\sim \xi_2/M_p$). This bound is the best limit for the scenario of the deformation of Lorentz symmetry in the non-relativistic limit, since previous attempts to constrain the parameter ξ_2 is at level $|\xi_2| < 10^9$

using the cold atom recoil experiments [45]. By comparing Eq. (9) with Eq. (14), the magnitude of the energy shifts of the hydrogen atom caused by the leading correction term and the next-to-leading correction term is found to differ by the fine structure constant α ($\sim 10^2$). However, in the study of constraining bounds on quantum gravity effects in the deformed dispersion relation using the cold atom recoil experiment, the leading correction term, and the next-to-leading correction term cause the energy correction to differ by a factor $m/(h\nu_* + p)$ ($\sim 10^9$)(see details in Ref. [45]).

The correction caused by the quadratic term of momentum (p^2/M_p) expressed by the parameter ξ_2 will indeed become increasingly important at high energy. Therefore, some researchers investigating quantum-gravity have used certain observations in astrophysics to provide Planck-scale sensitivity for some quantum gravity scenarios. These studies have also established meaningful bounds on scenarios with relatively strong ultra-relativistic corrections, such as the proposals of Refs. [86-90], which obtain the bound of the term of order p^2/M_p (≤ 100) through gamma-ray bursts (GRBs) and flaring active galactic nuclei (AGNs). The bounds of the term of order p^2/M_p (≤ 1) can be obtained using neutrino events detected by the IceCube Collaboration in Refs. [90-93]. This means that our bound is two orders of magnitude higher than these meaningful bounds established in astrophysics observations. Thus, the hydrogen 1S-2S transition experiments are considered capable of investigating the desired Planck scale sensitivity.

4 Conclusion

We use the latest results of ultra-high accuracy 1S-2S transition experiments in the hydrogen atom to establish upper bounds on parameters ξ_1 and ξ_2 characterizing the non-relativistic limits of the deformed dispersion relation. The results show that the exceptional sensitivity of the experiments sets a limit on the parameter ξ_1 within a single order of magnitude of the desired Planck-scale level, thereby providing another example of the Planck-scale sensitivity in the study of the dispersion relation in controlled laboratory experiments. At the same time, the bound of parameter ξ_2 is two orders of magnitude away from the Planck scale, however it still amounts to the best limit, in contrast to the previously obtained bounds in the non-relativistic limit from cold-atom-recoil experiments

[45, 57]. We can expect that, as the hydrogen atom 1S-2S transition experiments continue to improve, more strin-

gent bounds on parameters ξ_1 and ξ_2 could be found in the near future.

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