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Tighter monogamy relations of multiqubit entanglement in terms of Rényi-α entanglement

Li-Min Gao[1], **Feng-Li Yan**[1] **and Ting Gao**[2]

[1] College of Physics, Hebei Normal University, Shijiazhuang 050024, China
[2] School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, China

E-mail: flyan@hebtu.edu.cn and gaoting@hebtu.edu.cn

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Abstract

We explore the existence of monogamy relations in terms of Rényi-α entanglement. By using the power of the Rényi-α entanglement, we establish a class of tight monogamy relations of multiqubit entanglement with larger lower bounds than the existing monogamy relations for $\alpha \geqslant 2$, the power $\eta > 1$, and $2 > \alpha \geqslant \frac{\sqrt{7}-1}{2}$, the power $\eta > 2$, respectively.

Keywords: monogamy relation, multiqubit entanglement, Rényi-α entanglement

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement is one of the most important features of quantum mechanics, which distinguishes quantum mechanics from classical theory. A key property of entanglement is known as monogamy relations [1, 2], that is, entanglement cannot be freely shared unconditionally among the multipartite quantum systems. Monogamy relation provides a way to characterize multipartite entanglement sharing and distribution. The first mathematical characterization of monogamy relation was expressed as a form of inequality for three-qubit state in terms of squared concurrence [1]. Furthermore, Osborne and Verstraete generalized this monogamy inequality to arbitrary multiqubit systems [3]. Later, the monogamy inequality was also generalized to other entanglement measures [4–10]. In fact, monogamy of entanglement is fundamentally important in the context of quantum cryptography since it restricts on the amount of information that an eavesdropper could potentially obtain about the secret key extraction. Moreover, monogamy of entanglement also has many important applications in quantum information theory [11], condensed-matter physics [12] and even black-hole physics [13].

As a generalization of entanglement of formation, the Rényi-α entanglement [14] is a well-defined entanglement measure and it has been widely used in the study of quantum information theory [15–21]. Recently it has been shown that if $\alpha \geqslant \frac{\sqrt{7}-1}{2}$, the squared Rényi-$\alpha$ entanglement satisfies the monogamy relation in N-qubit systems [9]. It has also been shown that when $\alpha \geqslant 2$, the Rényi-α entanglement obeys the monogamy relation in multiqubit systems [22]. In general, tightening the monogamy relations can provide a precise characterization of the entanglement in multipartite systems. In particular, the monogamy relations are saturation for W-class states and this implies that this type of multipartite entanglement can be completely characterized [23, 24]. Furthermore, a class of tight monogamy relations was derived by raising the power of the entanglement measures [25–29]. In this paper, we focus on tightening the monogamy relations in terms of Rényi-α entanglement by raising the power of the Rényi-α entanglement for multiqubit systems. It is shown that these new monogamy relations are tighter than the results in [9, 28, 29].

2. The Rényi-α entanglement

The Rényi-α entanglement of a bipartite pure state $|\psi\rangle_{AB}$, is defined as [14]

$$E_\alpha(|\psi\rangle_{AB}) = \frac{1}{1-\alpha}\log_2(\mathrm{tr}\rho_A^\alpha), \tag{1}$$

for any $\alpha > 0$ and $\alpha \neq 1$, $\rho_A = \mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|)$. If α tends to 1, the Rényi-α entanglement converges to the von Neumann entropy. For a bipartite mixed state ρ_{AB}, the Rényi-α entanglement is defined via the convex-roof extension

$$E_\alpha(\rho_{AB}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}), \tag{2}$$

where the minimum is taken over all possible pure-state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$.

Let us recall the definition of concurrence. For a bipartite pure state $|\phi\rangle_{AB}$, the concurrence is [30]

$$C(|\phi\rangle_{AB}) = \sqrt{2(1 - \mathrm{tr}\rho_A^2)}, \tag{3}$$

where $\rho_A = \mathrm{tr}_B(|\phi\rangle_{AB}\langle\phi|)$. For a mixed state ρ_{AB}, the concurrence is defined via the convex-roof extension

$$C(\rho_{AB}) = \min \sum_j p_j C(|\phi_j\rangle_{AB}), \tag{4}$$

where the minimum is taken over all possible pure-state decompositions of $\rho_{AB} = \sum_j p_j |\phi_j\rangle_{AB}\langle\phi_j|$.

For an arbitrary N-qubit state $\rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, $\rho_{A|B_1\cdots B_{N-1}}$ denote the state $\rho_{AB_1\cdots B_{N-1}}$ viewed as a bipartite state with partitions A and $B_1B_2\cdots B_{N-1}$. The concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ satisfies [3]

$$C^2(\rho_{A|B_1\cdots B_{N-1}}) - C^2(\rho_{AB_1}) - \cdots - C^2(\rho_{AB_{N-1}}) \geqslant 0, \tag{5}$$

where $\rho_{AB_i} = \mathrm{tr}_{B_1\cdots B_{i-1}B_{i+1}\cdots B_{N-1}}(\rho_{A|B_1\cdots B_{N-1}})$, $\mathcal{H}_A$, $\mathcal{H}_{B_1}$,$\cdots$,$\mathcal{H}_{B_{N-1}}$ are Hilbert spaces of the systems A, B_1,$\cdots$, B_{N-1}, respectively.

It has been proved that [20, 22], when $\alpha \geqslant \frac{\sqrt{7}-1}{2}$, for a two-qubit state, the Rényi-α entanglement has an analytical formula

$$E_\alpha(\rho_{AB}) = g_\alpha(C(\rho_{AB})), \tag{6}$$

here the function $g_\alpha(x)$ is a monotonically increasing and convex function expressed as

$$g_\alpha(x) = \frac{1}{1-\alpha}\log_2\left[\left(\frac{1-\sqrt{1-x^2}}{2}\right)^\alpha + \left(\frac{1+\sqrt{1-x^2}}{2}\right)^\alpha\right], \tag{7}$$

in $0 \leqslant x \leqslant 1$.

The function $g_\alpha(x)$ in equation (7) for $\alpha \geqslant 2$, has one important property such that [22]

$$g_\alpha\left(\sqrt{x^2+y^2}\right) \geqslant g_\alpha(x) + g_\alpha(y), \tag{8}$$

for $0 \leqslant x, y, x^2 + y^2 \leqslant 1$.

When $\alpha \geqslant \frac{\sqrt{7}-1}{2}$, it is easy to see in [9] that the function $g_\alpha(x)$ satisfies the following inequality

$$\left[g_\alpha\left(\sqrt{x^2+y^2}\right)\right]^2 \geqslant [g_\alpha(x)]^2 + [g_\alpha(y)]^2, \tag{9}$$

for $0 \leqslant x, y, x^2 + y^2 \leqslant 1$.

3. Tighter monogamy relations for Rényi-α entanglement

In the following, we establish a class of tight monogamy relations of Rényi-α entanglement related to the power η. We first provide the following lemma.

Lemma 1. *For $x \in [0, 1]$ and $t \geqslant 1$, then*

$$(1+x)^t \geqslant 1 + \frac{t}{2}x + \left(2^t - \frac{t}{2} - 1\right)x^t \geqslant 1 + (2^t - 1)x^t. \tag{10}$$

Proof. Note that the inequality (10) holds with equality for $x = 0$, we need to prove (10) only for $x \neq 0$. Let us consider the function $f(t, x) = \frac{(1+x)^t - \frac{t}{2}x - 1}{x^t}$. Then, $\frac{\partial f}{\partial x} = \frac{tx^{t-1}\left[1 + \frac{(t-1)}{2}x - (1+x)^{t-1}\right]}{x^{2t}}$. When $t \geqslant 1$ and $0 \leqslant x \leqslant 1$, it is easy to obtain that $1 + \frac{(t-1)}{2}x \leqslant (1+x)^{t-1}$. Thus, $\frac{\partial f}{\partial x} \leqslant 0$, $f(t, x)$ is a decreasing function of x, i.e. $f(t, x) \geqslant f(t, 1) = 2^t - \frac{t}{2} - 1$. It follows that $(1+x)^t \geqslant 1 + \frac{t}{2}x + \left(2^t - \frac{t}{2} - 1\right)x^t$.

Since $x \geqslant x^t$, for $x \in [0, 1]$ and $t \geqslant 1$, one gets $1 + \frac{t}{2}x + \left(2^t - \frac{t}{2} - 1\right)x^t = 1 + \frac{t}{2}(x - x^t) + (2^t - 1)x^t \geqslant 1 + (2^t - 1)x^t$. Altogether, we can get $(1+x)^t \geqslant 1 + \frac{t}{2}x + \left(2^t - \frac{t}{2} - 1\right)x^t \geqslant 1 + (2^t - 1)x^t$.

Now we provide our main results of this paper.

Lemma 2. *For an N-qubit state $\rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $C(\rho_{AB_i}) \geqslant C(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \cdots, N-2$, $N \geqslant 3$, then*

$$\begin{aligned} E_\alpha^\eta(\rho_{A|B_1\cdots B_{N-1}}) &\geqslant E_\alpha^\eta(\rho_{AB_1}) + (2^\eta - 1)E_\alpha^\eta(\rho_{AB_2}) \\ &+ \cdots + (2^\eta - 1)^{N-4}E_\alpha^\eta(\rho_{AB_{N-3}}) \\ &+ (2^\eta - 1)^{N-3}\Big\{E_\alpha^\eta(\rho_{AB_{N-2}}) \\ &+ \frac{\eta}{2}E_\alpha^{\eta-1}(\rho_{AB_{N-2}})E_\alpha(\rho_{AB_{N-1}}) \\ &+ \left(2^\eta - \frac{\eta}{2} - 1\right)E_\alpha^\eta(\rho_{AB_{N-1}})\Big\}, \end{aligned} \tag{11}$$

for $\alpha \geqslant 2$ and the power $\eta \geqslant 1$; and

$$\begin{aligned} &E_\alpha^\gamma(\rho_{A|B_1\cdots B_{N-1}}) \geqslant E_\alpha^\gamma(\rho_{AB_1}) + (2^t - 1)E_\alpha^\gamma(\rho_{AB_2}) + \cdots \\ &+ (2^t - 1)^{N-4}E_\alpha^\gamma(\rho_{AB_{N-3}}) + (2^t - 1)^{N-3}\Big\{E_\alpha^\gamma(\rho_{AB_{N-2}}) \\ &+ \frac{t}{2}E_\alpha^{\gamma-2}(\rho_{AB_{N-2}})E_\alpha^2(\rho_{AB_{N-1}}) \\ &+ \left(2^t - \frac{t}{2} - 1\right)E_\alpha^\gamma(\rho_{AB_{N-1}})\Big\}, \end{aligned} \tag{12}$$

for $2 > \alpha \geqslant \frac{\sqrt{7}-1}{2}$ and the power $\gamma \geqslant 2$, where $t = \frac{\gamma}{2}$.

Proof. For $\alpha \geqslant 2$, by the inequality (8), for $\eta \geqslant 1$, we have

$$\left[g_\alpha\left(\sqrt{x^2+y^2}\right)\right]^\eta \geqslant [g_\alpha(x)+g_\alpha(y)]^\eta. \tag{13}$$

Without loss of generality, we assume $x \geqslant y$, the inequality (10) of lemma 1 ensures

$$\begin{aligned}\left[g_\alpha\left(\sqrt{x^2+y^2}\right)\right]^\eta &\geqslant [g_\alpha(x)]^\eta+\frac{\eta}{2}[g_\alpha(x)]^{\eta-1}g_\alpha(y)\\ &+\left(2^\eta-\frac{\eta}{2}-1\right)[g_\alpha(y)]^\eta.\end{aligned} \tag{14}$$

Let us first consider an N-qubit pure state $|\Psi\rangle_{A|B_1\cdots B_{N-1}}$. The entanglement $E_\alpha(|\Psi\rangle_{A|B_1\cdots B_{N-1}})$ and $C(|\Psi\rangle_{A|B_1\cdots B_{N-1}})$ are related by the function $g_\alpha(x)$ in equation (7) since the subsystem $B_1\cdots B_{N-1}$ can be regarded as a logic qubit. Thus, we can obtain

$$\begin{aligned}
&E_\alpha^\eta(|\Psi\rangle_{A|B_1\cdots B_{N-1}})\\
&\quad=[g_\alpha(C(|\Psi\rangle_{A|B_1\cdots B_{N-1}}))]^\eta\\
&\quad\geqslant\left[g_\alpha\left(\sqrt{C^2(\rho_{AB_1})+\cdots+C^2(\rho_{AB_{N-1}})}\right)\right]^\eta\\
&\quad\geqslant[g_\alpha(C(\rho_{AB_1}))]^\eta\\
&\qquad+\frac{\eta}{2}[g_\alpha(C(\rho_{AB_1}))]^{\eta-1}\\
&\qquad\times g_\alpha\left(\sqrt{C^2(\rho_{AB_2})+\cdots+C^2(\rho_{AB_{N-1}})}\right)\\
&\qquad+\left(2^\eta-\frac{\eta}{2}-1\right)\left[g_\alpha\left(\sqrt{C^2(\rho_{AB_2})+\cdots+C^2(\rho_{AB_{N-1}})}\right)\right]^\eta\\
&\quad\geqslant[g_\alpha(C(\rho_{AB_1}))]^\eta+(2^\eta-1)[g_\alpha(C(\rho_{AB_2}))]^\eta+\cdots\\
&\qquad+(2^\eta-1)^{N-4}[g_\alpha(C(\rho_{AB_{N-3}}))]^\eta\\
&\qquad+(2^\eta-1)^{N-3}\{[g_\alpha(C(\rho_{AB_{N-2}}))]^\eta\\
&\qquad+\frac{\eta}{2}[g_\alpha(C(\rho_{AB_{N-2}}))]^{\eta-1}g_\alpha(C(\rho_{AB_{N-1}}))\\
&\qquad+\left(2^\eta-\frac{\eta}{2}-1\right)[g_\alpha(C(\rho_{AB_{N-1}}))]^\eta\}\\
&\quad=E_\alpha^\eta(\rho_{AB_1})+(2^\eta-1)E_\alpha^\eta(\rho_{AB_2})\\
&\qquad+\cdots+(2^\eta-1)^{N-4}E_\alpha^\eta(\rho_{AB_{N-3}})\\
&\qquad+(2^\eta-1)^{N-3}\{E_\alpha^\eta(\rho_{AB_{N-2}})\\
&\qquad+\frac{\eta}{2}E_\alpha^{\eta-1}(\rho_{AB_{N-2}})E_\alpha(\rho_{AB_{N-1}})\\
&\qquad+\left(2^\eta-\frac{\eta}{2}-1\right)E_\alpha^\eta(\rho_{AB_{N-1}})\Big\},
\end{aligned} \tag{15}$$

where we have utilized the monogamy inequality (5) and the monotonically increasing property of the function $g_\alpha(x)$ to obtain the first inequality, the second inequality is due to inequality (14) by letting $x=C(\rho_{AB_1})$ and $y=\sqrt{C^2(\rho_{AB_2})+\cdots+C^2(\rho_{AB_{N-1}})}$. The third inequality is obtained from the iterative use of inequality (14). Here we are using the fact that $C(\rho_{AB_i})\geqslant C(\rho_{A|B_{i+1}\cdots B_{N-1}})\geqslant\sqrt{C^2(\rho_{AB_{i+1}})+\cdots+C^2(\rho_{AB_{N-1}})}$, $i=1,2,\cdots,N-2$ and $1+\frac{\eta}{2}x+\left(2^\eta-\frac{\eta}{2}-1\right)x^\eta\geqslant1+(2^\eta-1)x^\eta$ for $\eta\geqslant1$. Since for any two-qubit state ρ_{AB}, when $\alpha\geqslant\frac{\sqrt{7}-1}{2}$, $E_\alpha(\rho_{AB})=g_\alpha(C(\rho_{AB}))$, we obtain the last equality.

Let us now consider an N-qubit mixed state $\rho_{A|B_1\cdots B_{N-1}}$. Assume that $\rho_{A|B_1\cdots B_{N-1}}=\sum_k p_k|\varphi_k\rangle_{A|B_1\cdots B_{N-1}}\langle\varphi_k|\in\mathcal{H}_A\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_{N-1}}$ is the optimal pure-state decomposition for $E_\alpha(\rho_{A|B_1\cdots B_{N-1}})$. Thus, we can deduce

$$\begin{aligned}
E_\alpha(\rho_{A|B_1\cdots B_{N-1}})&=\textstyle\sum_k p_kE_\alpha(|\varphi_k\rangle_{A|B_1\cdots B_{N-1}})\\
&=\textstyle\sum_k p_kg_\alpha(C(|\varphi_k\rangle_{A|B_1\cdots B_{N-1}}))\\
&\geqslant g_\alpha\left(\textstyle\sum_k p_kC(|\varphi_k\rangle_{A|B_1\cdots B_{N-1}}\right)\\
&\geqslant g_\alpha\left(\textstyle\sum_l p_lC(|\chi_l\rangle_{A|B_1\cdots B_{N-1}}\right)\\
&=g_\alpha(C(\rho_{A|B_1\cdots B_{N-1}})),
\end{aligned} \tag{16}$$

where the first inequality follows from the convex property of the function $g_\alpha(x)$, the second equality is satisfied because $\{p_l,|\chi_l\rangle_{A|B_1\cdots B_{N-1}}\}$ is the optimal pure-state decomposition for $C(\rho_{A|B_1\cdots B_{N-1}})$.

Consequently we can write

$$\begin{aligned}
&E_\alpha^\eta(\rho_{A|B_1\cdots B_{N-1}})\\
&\quad\geqslant[g_\alpha(C(\rho_{A|B_1\cdots B_{N-1}}))]^\eta\\
&\quad\geqslant\left[g_\alpha(\sqrt{C^2(\rho_{AB_1})+\cdots+C^2(\rho_{AB_{N-1}})})\right]^\eta\\
&\quad\geqslant[g_\alpha(C(\rho_{AB_1}))]^\eta+(2^\eta-1)[g_\alpha(C(\rho_{AB_2}))]^\eta+\cdots\\
&\qquad+(2^\eta-1)^{N-4}[g_\alpha(C(\rho_{AB_{N-3}}))]^\eta\\
&\qquad+(2^\eta-1)^{N-3}\{[g_\alpha(C(\rho_{AB_{N-2}}))]^\eta\\
&\qquad+\frac{\eta}{2}[g_\alpha(C(\rho_{AB_{N-2}}))]^{\eta-1}g_\alpha(C(\rho_{AB_{N-1}}))\\
&\qquad+\left(2^\eta-\frac{\eta}{2}-1\right)[g_\alpha(C(\rho_{AB_{N-1}}))]^\eta\Big\}\\
&\quad=E_\alpha^\eta(\rho_{AB_1})+(2^\eta-1)E_\alpha^\eta(\rho_{AB_2})\\
&\qquad+\cdots+(2^\eta-1)^{N-4}E_\alpha^\eta(\rho_{AB_{N-3}})\\
&\qquad+(2^\eta-1)^{N-3}\Big\{E_\alpha^\eta(\rho_{AB_{N-2}})\\
&\qquad+\frac{\eta}{2}E_\alpha^{\eta-1}(\rho_{AB_{N-2}})E_\alpha(\rho_{AB_{N-1}})\\
&\qquad+\left(2^\eta-\frac{\eta}{2}-1\right)E_\alpha^\eta(\rho_{AB_{N-1}})\Big\},
\end{aligned} \tag{17}$$

here in the second inequality we have used the monogamy inequality (5) and the monotonically increasing property of the function $g_\alpha(x)$. Iterative use of inequality (14), we have the third inequality. We also use the fact that $C(\rho_{AB_i})\geqslant C(\rho_{A|B_{i+1}\cdots B_{N-1}})\geqslant\sqrt{C^2(\rho_{AB_{i+1}})+\cdots+C^2(\rho_{AB_{N-1}})}$, $i=1,2,\ldots,N-2$ and $1+\frac{\eta}{2}x+\left(2^\eta-\frac{\eta}{2}-1\right)x^\eta\geqslant1+(2^\eta-1)x^\eta$ for $\eta\geqslant1$. Because when $\alpha\geqslant\frac{\sqrt{7}-1}{2}$, $E_\alpha(\rho_{AB})=g_\alpha(C(\rho_{AB}))$ for any two-qubit state ρ_{AB}, one gets the last equality. Combining (16) and (17) completes the proof of inequality (11).

The proof of inequality (12) is very similar to that of the inequality (11). By the inequality (9), for $2>\alpha\geqslant\frac{\sqrt{7}-1}{2}$,

$\gamma \geqslant 2$, $t = \frac{\gamma}{2} \geqslant 1$, we can obtain

$$\left[g_\alpha\left(\sqrt{x^2+y^2}\right)\right]^\gamma \geqslant \{[g_\alpha(x)]^2 + [g_\alpha(y)]^2\}^t. \tag{18}$$

Again, without loss of generality one may assume $x \geqslant y$, by the inequality (10) of lemma 1, one finds

$$\left[g_\alpha\left(\sqrt{x^2+y^2}\right)\right]^\gamma \geqslant [g_\alpha(x)]^\gamma + \frac{t}{2}[g_\alpha(x)]^{\gamma-2}[g_\alpha(y)]^2 + \left(2^t - \frac{t}{2} - 1\right)[g_\alpha(y)]^\gamma. \tag{19}$$

Now, using the inequality (19), following a similar procedure as above, we can obtain the inequality (12). The proof of lemma 2 is completed.

In particular, we consider the case $N = 3$. Note that when $\alpha \geqslant 2$ and the power $\eta \geqslant 1$, if $E_\alpha(\rho_{AB_1}) \geqslant E_\alpha(\rho_{AB_2})$, then we arrive at

$$E_\alpha^\eta(\rho_{A|B_1B_2}) \geqslant E_\alpha^\eta(\rho_{AB_1}) + \frac{\eta}{2}E_\alpha^{\eta-1}(\rho_{AB_1})E_\alpha(\rho_{AB_2}) + \left(2^\eta - \frac{\eta}{2} - 1\right)E_\alpha^\eta(\rho_{AB_2}). \tag{20}$$

If $E_\alpha(\rho_{AB_1}) \leqslant E_\alpha(\rho_{AB_2})$, then

$$E_\alpha^\eta(\rho_{A|B_1B_2}) \geqslant E_\alpha^\eta(\rho_{AB_2}) + \frac{\eta}{2}E_\alpha^{\eta-1}(\rho_{AB_2})E_\alpha(\rho_{AB_1}) + \left(2^\eta - \frac{\eta}{2} - 1\right)E_\alpha^\eta(\rho_{AB_1}). \tag{21}$$

Also, note that when $2 > \alpha \geqslant \frac{\sqrt{7}-1}{2}$ and the power $\gamma \geqslant 2$, if $E_\alpha(\rho_{AB_1}) \geqslant E_\alpha(\rho_{AB_2})$, we can write

$$E_\alpha^\gamma(\rho_{A|B_1B_2}) \geqslant E_\alpha^\gamma(\rho_{AB_1}) + \frac{\gamma}{4}E_\alpha^{\gamma-2}(\rho_{AB_1})E_\alpha^2(\rho_{AB_2}) + \left(2^{\frac{\gamma}{2}} - \frac{\gamma}{4} - 1\right)E_\alpha^\gamma(\rho_{AB_2}). \tag{22}$$

If $E_\alpha(\rho_{AB_1}) \leqslant E_\alpha(\rho_{AB_2})$, then

$$E_\alpha^\gamma(\rho_{A|B_1B_2}) \geqslant E_\alpha^\gamma(\rho_{AB_2}) + \frac{\gamma}{4}E_\alpha^{\gamma-2}(\rho_{AB_2})E_\alpha^2(\rho_{AB_1}) + \left(2^{\frac{\gamma}{2}} - \frac{\gamma}{4} - 1\right)E_\alpha^\gamma(\rho_{AB_1}). \tag{23}$$

Moreover, based on lemma 2, if $C(\rho_{AB_i}) \geqslant C(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \cdots, m$ and $C(\rho_{AB_j}) \leqslant C(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1,\cdots,N-2$, $\forall\, 1 \leqslant m \leqslant N-3$, $N \geqslant 4$, we have the following lemma.

Lemma 3. *For an* N*-qubit state* $\rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, *if* $C(\rho_{AB_i}) \geqslant C(\rho_{A|B_{i+1}\cdots B_{N-1}})$ *for* $i = 1, 2,\cdots,m$, *and* $C(\rho_{AB_j}) \leqslant C(\rho_{A|B_{j+1}\cdots B_{N-1}})$ *for* $j = m+1,\cdots,N-2$, $\forall\, 1 \leqslant m \leqslant N-3$, $N \geqslant 4$, *then*

$$\begin{aligned} &E_\alpha^\eta(\rho_{A|B_1\cdots B_{N-1}}) \\ &\quad\geqslant E_\alpha^\eta(\rho_{AB_1}) + (2^\eta - 1)E_\alpha^\eta(\rho_{AB_2}) \\ &\qquad + \cdots + (2^\eta - 1)^{m-1}E_\alpha^\eta(\rho_{AB_m}) \\ &\qquad + (2^\eta - 1)^{m+1}[E_\alpha^\eta(\rho_{AB_{m+1}}) + \cdots + E_\alpha^\eta(\rho_{AB_{N-3}})] \\ &\qquad + (2^\eta - 1)^m\left\{\left(2^\eta - \frac{\eta}{2} - 1\right)E_\alpha^\eta(\rho_{AB_{N-2}})\right. \\ &\qquad \left. + \frac{\eta}{2}E_\alpha(\rho_{AB_{N-2}})E_\alpha^{\eta-1}(\rho_{AB_{N-1}}) + E_\alpha^\eta(\rho_{AB_{N-1}})\right\}, \end{aligned} \tag{24}$$

for $\alpha \geqslant 2$ *and the power* $\eta \geqslant 1$; *and*

$$\begin{aligned} &E_\alpha^\gamma(\rho_{A|B_1\cdots B_{N-1}}) \\ &\quad\geqslant E_\alpha^\gamma(\rho_{AB_1}) + (2^t - 1)E_\alpha^\gamma(\rho_{AB_2}) \\ &\qquad + \cdots + (2^t - 1)^{m-1}E_\alpha^\gamma(\rho_{AB_m}) \\ &\qquad + (2^t - 1)^{m+1}[E_\alpha^\gamma(\rho_{AB_{m+1}}) + \cdots + E_\alpha^\gamma(\rho_{AB_{N-3}})] \\ &\qquad + (2^t - 1)^m\left\{\left(2^t - \frac{t}{2} - 1\right)E_\alpha^\gamma(\rho_{AB_{N-2}})\right. \\ &\qquad + \frac{t}{2}E_\alpha^2(\rho_{AB_{N-2}})E_\alpha^{\gamma-2}(\rho_{AB_{N-1}}) \\ &\qquad \left. + E_\alpha^\gamma(\rho_{AB_{N-1}})\right\}, \end{aligned} \tag{25}$$

for $2 > \alpha \geqslant \frac{\sqrt{7}-1}{2}$ *and the power* $\gamma \geqslant 2$, *where* $t = \frac{\gamma}{2}$.

Proof. For $\alpha \geqslant 2$, from lemma 2, we have

$$\begin{aligned} &E_\alpha^\eta(\rho_{A|B_1\cdots B_{N-1}}) \\ &\quad\geqslant [g_\alpha(C(\rho_{AB_1}))]^\eta + (2^\eta - 1)[g_\alpha(C(\rho_{AB_2}))]^\eta \\ &\qquad + \cdots + (2^\eta - 1)^{m-2}[g_\alpha(C(\rho_{AB_{m-1}}))]^\eta \\ &\qquad + (2^\eta - 1)^{m-1}\{[g_\alpha(C(\rho_{AB_m}))]^\eta \\ &\qquad + \frac{\eta}{2}[g_\alpha(C(\rho_{AB_m}))]^{\eta-1}g_\alpha(C(\rho_{A|B_{m+1}\cdots B_{N-1}})) \\ &\qquad + \left(2^\eta - \frac{\eta}{2} - 1\right)[g_\alpha(C(\rho_{A|B_{m+1}\cdots B_{N-1}}))]^\eta\} \\ &\quad\geqslant [g_\alpha(C(\rho_{AB_1}))]^\eta + (2^\eta - 1)[g_\alpha(C(\rho_{AB_2}))]^\eta \\ &\qquad + \cdots + (2^\eta - 1)^{m-1}[g_\alpha(C(\rho_{AB_m}))]^\eta \\ &\qquad + (2^\eta - 1)^m[g_\alpha(C(\rho_{A|B_{m+1}\cdots B_{N-1}}))]^\eta. \end{aligned} \tag{26}$$

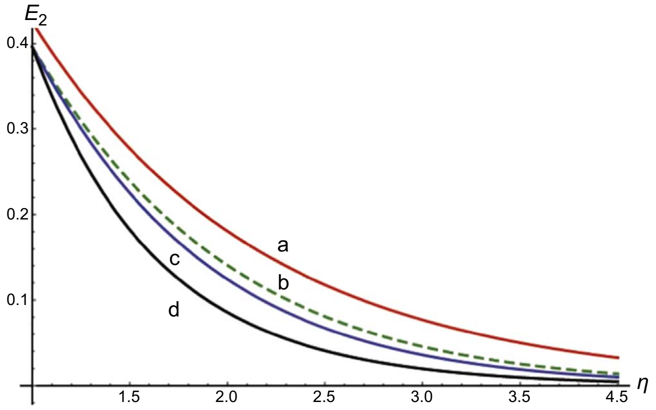

Figure 1. The y axis is the Rényi-α entanglement of $|\varphi\rangle$ with $\alpha = 2$ and its lower bound. The (red solid) line a represents the Rényi-2 entanglement of $|\varphi\rangle_{A|BC}$ in Example. The (green dashed) line b denotes the lower bound given by inequality (21). The (blue) line c expresses the lower bound from the result in [28, 29] when $\alpha = 2$. The (black) line d is the lower bound from the result in [9].

When $C(\rho_{AB_j}) \leqslant C(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1,\cdots,N-2$, applying the preceding procedure, one finds

$$\begin{aligned}
&[g_\alpha(C(\rho_{A|B_{m+1}\cdots B_{N-1}}))]^\eta \\
&\quad\geqslant \left(2^\eta - \frac{\eta}{2} - 1\right)[g_\alpha(C(\rho_{AB_{m+1}}))]^\eta \\
&\qquad + \frac{\eta}{2} g_\alpha(C(\rho_{AB_{m+1}}))[g_\alpha(C(\rho_{A|B_{m+2}\cdots B_{N-1}}))]^{\eta-1} \\
&\qquad + [g_\alpha(C(\rho_{A|B_{m+2}\cdots B_{N-1}}))]^\eta \\
&\quad\geqslant (2^\eta - 1)\{[g_\alpha(C(\rho_{AB_{m+1}}))]^\eta + \cdots + [g_\alpha(C(\rho_{AB_{N-3}}))]^\eta\} \\
&\qquad + \left(2^\eta - \frac{\eta}{2} - 1\right)[g_\alpha(C(\rho_{AB_{N-2}}))]^\eta \\
&\qquad + \frac{\eta}{2} g_\alpha(C(\rho_{AB_{N-2}}))[g_\alpha(C(\rho_{AB_{N-1}}))]^{\eta-1} \\
&\qquad + [g_\alpha(C(\rho_{AB_{N-1}}))]^\eta.
\end{aligned} \tag{27}$$

Combining inequalities (26) and (27), we obtain the inequality (24). The inequality (25) can be proved in a similar way.

To see the tightness of our monogamy relations of multiqubit entanglement, we give an example below.

Example. Under local unitary operations, the three-qubit pure state can be written as [31]

$$\begin{aligned}|\varphi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle \\ + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,\end{aligned} \tag{28}$$

where $0 \leqslant \phi \leqslant \pi$, $\lambda_s \geqslant 0$, $s = 0, 1, 2, 3, 4$, and $\sum_{s=0}^{4}\lambda_s^2 = 1$. Suppose that $\lambda_0 = \frac{1}{2}$, $\lambda_1 = \frac{\sqrt{47}}{14}$, $\lambda_2 = \frac{4}{7}$, $\lambda_3 = \frac{3}{7}$, $\lambda_4 = 0$.

Straightforward calculation of the Rényi-α entanglement shows that $E_2(|\varphi\rangle_{A|BC}) = 0.424\,89$, $E_2(\rho_{A|B}) = 0.138\,98$, $E_2(\rho_{A|C}) = 0.257\,16$, for $\alpha = 2$. One can explicitly see that our lower bound is larger than the results in [9, 28, 29], as shown in figure 1.

One can obtain that $E_{\frac{3}{2}}(|\varphi\rangle_{A|BC}) = 0.497\,25$, $E_{\frac{3}{2}}(\rho_{A|B}) = 0.181\,27$, $E_{\frac{3}{2}}(\rho_{A|C}) = 0.318\,78$, for $\alpha = \frac{3}{2}$. It is clear from figure 2 that our lower bound is larger than the results in [9].

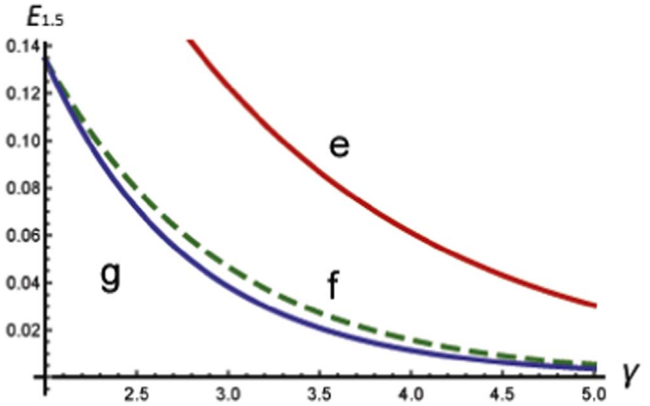

Figure 2. The y axis is the Rényi-α entanglement of $|\varphi\rangle$ with $\alpha = 1.5$ and its lower bound. The (red solid) line e represents the Rényi-1.5 entanglement of $|\varphi\rangle_{A|BC}$ in Example. The (green dashed) line f denotes the lower bound given by inequality (23). The (blue) line g is the lower bound from the result in [9].

4. Conclusion

In this paper we have investigated the tight monogamy relations in terms of Rényi-α entanglement. By using the power of the Rényi-α entanglement, we have provided a class of tight monogamy relations for $\alpha \geqslant 2$, the power $\eta > 1$ and $2 > \alpha \geqslant \frac{\sqrt{7}-1}{2}$, the power $\eta > 2$, respectively. We have also shown that these new monogamy relations of multiparty entanglement with larger lower bounds than the former results [9, 28, 29].

Multipartite entanglement can be regarded as a fundamental problem in the theory of quantum entanglement. It has attracted much attention over the past two decades. Our results provide a finer characterization of multiqubit entanglement sharing and distribution based on the Rényi-α entanglement. The framework can also be applied to other entanglement measures [4–10]. Our results cannot only provide a useful methodology to study further the monogamous property of multipartite quantum entanglement, but also may contribute to a fully understanding of the multipartite quantum entanglement.

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