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Solitons and soliton molecules in two nonlocal Alice–Bob Sawada–Kotera systems

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Abstract

Two nonlocal Alice–Bob Sawada–Kotera (ABSK) systems, accompanied by the parity and time reversal invariance are studied. The Lax pairs of two systems are uniformly written out in matrix form. The periodic waves, multiple solitons, and soliton molecules of the ABSK systems are obtained via the bilinear method and the velocity resonant mechanism. Though the interactions among solitons are elastic, the interactions between soliton and soliton molecules are not elastic. In particular, the shapes of the soliton molecules are changed explicitly after interactions.

Keywords: soliton molecules, nonlocal Sawada–Kotera equations, nonelastic interactions, periodic and solitary waves

(Some figures may appear in colour only in the online journal)

1. Introduction

In 2013, the nonlinear Schrödinger (NLS) equation was extended to a nonlocal form by Ablowitz and Musslimani [1], which can be written as

$$\mathrm{i}A_t + A_{xx} \pm A^2B = 0, \quad B = \hat{P}\hat{C}A = A^*(-x, t),$$

where $\hat{P}$ is the parity operator and $\hat{C}$ is the charge conjugation. By using the discrete symmetry group generated by $\{\hat{P}, \hat{C}, \hat{T}\}$ (*PTC* symmetry group), where $\hat{T}$ is the time reversal operator, many nonlinear systems can be extended to nonlocal forms (or so-called Alice–Bob (AB) systems), including nonlocal Korteweg–de Vries (KdV) systems [2–5], nonlocal modified KdV (MKdV) systems [2, 3, 6, 7], discrete and continuous nonlocal NLS systems [8, 9], etc. The *PTC* symmetry group is very important in many physical fields such as quantum chromodynamics [10], electric circuits [11], optics [12, 13], Bose–Einstein condensates [14], and atmospheric and oceanic dynamics [4].

The Sawada–Kotera (SK) equation

$$u_t + 5u^2u_x + 5uu_{3x} + 5u_xu_{2x} + u_{5x} = 0, \tag{1}$$

where $u_t \equiv \frac{\partial}{\partial t}u$, $u_{ix} \equiv \frac{\partial^i u}{\partial x^i}$, has been found to be important in some physical fields and in mathematics [15].

The SK equation (1) is integrable because of the existence of the Lax pair

$$\begin{aligned}&\psi_{3x} + u\psi_x = \lambda\psi,\\ &\psi_t = 9\psi_{5x} + 15(u\psi_{2x})_x + 5(u^2 + 2u_{2x})\psi_x.\end{aligned} \tag{2}$$

By using the *PTC* symmetry group, the SK system can be extended to a nonlocal system, written as

$$\begin{aligned}A_t = &-A_{5x} + \frac{5}{2}\{[(\sigma-1)B - (\sigma+3)A]A_{2x}\\ &+ (\sigma-1)(A-B)B_{2x}\}_x + 10[(\sigma-1)B\\ &-(\sigma+1)A]AA_x + 5(\sigma-1)(A^2-B^2)B_x,\\ B = &\hat{P}\hat{T}A = A(-x, -t),\end{aligned} \tag{3}$$

with the corresponding Lax pair

$$\Psi_{3x} + U\Psi_x = \Lambda\Psi, \tag{4}$$

$$\Psi_t = 9\Psi_{5x} + 15[U\Psi_{2x}]_x + 5[U^2 + 2U_{2x}]\Psi_x, \tag{5}$$

where

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad U = \begin{pmatrix}u & \sigma v\\ v & u\end{pmatrix},$$
$$\Lambda = \begin{pmatrix}\lambda_1 & \sigma\lambda_2\\ \lambda_2 & \lambda_1\end{pmatrix}, \tag{6}$$

and $u = A + B$, $v = A - B$. λ_1, λ_2, σ are arbitrary constants.

The nonlocal SK system (3) possesses different properties with different values of σ. Specifically, this nonlocal system will degenerate to a local system when $\sigma = 1$,

$$A_t = -A_{5x} - 10(AA_{2x})_x - 20A^2A_x. \tag{7}$$

Without loss of generality, we take $\sigma = 0$ and $\sigma = -1$ to find their periodic waves, multiple soliton solutions and soliton molecules in sections 2 and 3. Section 4 includes a short summary and some discussions.

2. Exact solutions of (3) with $\sigma = 0$

By taking $\sigma = 0$, the model equation (3) becomes

$$\begin{aligned} A_t &= -A_{5x} - \frac{5}{2}[(B + 3A)A_{2x} + (A - B)B_{2x}]_x \\ &\quad - 10(B + A)AA_x - 5(A^2 - B^2)B_x, \\ B &= \hat{P}\hat{T}A = A(-x, -t). \end{aligned} \tag{8}$$

To solve this equation, a symmetric-antisymmetric separation approach [2, 3] is the most useful and simplest method. We separate A into a symmetric part u and an anti-symmetric part v with respect to the operator '$\hat{P}\hat{T}$' as

$$\begin{aligned} A &= \frac{1}{2}(A + B) + \frac{1}{2}(A - B) = \frac{1}{2}(u + v), \\ B &= \hat{P}\hat{T}A = \frac{1}{2}(u - v),\ \hat{P}\hat{T}u = u,\ \hat{P}\hat{T}v = -v. \end{aligned} \tag{9}$$

Substituting (9) into (8) and separating the resulting equation into symmetric and antisymmetric parts, we obtain

$$u_t = -5u^2u_x - 5u_xu_{2x} - 5uu_{3x} - u_{5x},\ \hat{P}\hat{T}u = u, \tag{10}$$

$$v_t = -5(u^2v - uv_{2x} - u_{2x}v)_x - v_{5x},\ \hat{P}\hat{T}v = -v. \tag{11}$$

It is clear that (10) and (11) are simply the SK equation and its symmetry equation. In other words, the approach used in (10) and (11) is simply a special integrable SK coupling, or a special dark SK equation [16]. One special trivial symmetry v can be taken as $v = cu_x$ which is related to the space translation invariance.

2.1. Periodic waves of (8)

The periodic waves of (8) can be obtained by solving the traveling wave solutions of equations (10) and (11), and using the symmetric condition $\hat{P}\hat{T}u = u$ and the antisymmetric condition $\hat{P}\hat{T}v = -v$. By using the Jacobi elliptic function expansion method [17, 18], two specific examples take the following forms:

Case 1.

$$\begin{aligned} A &= s_0k^2 - 3k^2m^2\mathrm{sn}^2(\xi, m) \\ &\quad - c\,\mathrm{sn}(\xi, m)\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m),\ \xi = kx + \omega t, \end{aligned} \tag{12}$$

where, $\mathrm{sn}(x, m)$, $\mathrm{cn}(x, m)$ and $\mathrm{dn}(x, m)$ are the Jacobi elliptic functions of x with module m,

$$\omega = (-16m^4 + 20m^2s_0 - 44m^2 - 5s_0^2 + 20s_0 - 16)k^5, \tag{13}$$

and s_0, k, m and c are arbitrary constants.

Case 2.

$$\begin{aligned} A &= 4(m^2 + 1)k^2 - 6k^2m^2\mathrm{sn}^2(\xi, m) \\ &\quad - c\ \mathrm{sn}(\xi, m)\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m),\ \xi = kx + \omega t, \end{aligned} \tag{14}$$

where

$$\omega = -16(m^4 - m^2 + 1)k^5, \tag{15}$$

and k, m and c are arbitrary constants.

Two types of single soliton solutions of the ABSK equation (3) with $\sigma = 0$ can be obtained from the periodic waves by fixing $m = 1$.

2.2. Multiple soliton solutions and the soliton molecule of (8)

To obtain multiple soliton solutions, we assume the solution of (8) has the form

$$\begin{aligned} A &= 3[\ln(f) + c(\ln(f))_x]_{xx} + \frac{u_0}{2}, \\ B &= 3[\ln(f) - c(\ln(f))_x]_{xx} + \frac{u_0}{2},\ \hat{P}\hat{T}f = f. \end{aligned} \tag{16}$$

Substituting (16) into (8), we obtain

$$\begin{aligned} &[c(f^2\partial_x^2 - 4ff_x\partial_x - 2ff_{xx} + 6f_x^2) + 3f^2\partial_x - 6ff_x] \\ &\times(D_x^6 + D_xD_t + 5u_0^2D_x^2 + 5u_0D_x^4)f\cdot f = 0, \end{aligned} \tag{17}$$

where Hirota's bilinear operators D_x and D_t are defined as

$$\begin{aligned} D_x^nD_t^mf\cdot g &= (\partial_x - \partial_{x'})^n \\ &\quad\times(\partial_t - \partial_{t'})^nf(x, t)\cdot g(x', t')|_{x'=x,t'=t}. \end{aligned} \tag{18}$$

From equation (17) we know that (16) solves (8), with f being given by the bilinear equation

$$(D_x^6 + D_xD_t + 5u_0^2D_x^2 + 5u_0D_x^4)f\cdot f = 0. \tag{19}$$

The n-soliton solution $f = f_n$, can be written as [2, 3]

$$f_n = \sum_{\{\nu\}}K_{\{\nu\}}\cosh\left(\sum_{i=1}^{n}\nu_i\xi_i\right), \tag{20}$$

$$\xi_i = \frac{k_ix + \omega_it}{2},\ \omega_i = -k_i(k_i^4 + 5u_0(k_i^2 + u_0)), \tag{21}$$

where

$$\begin{aligned} &K_{\{\nu\}} = \textstyle\prod_{i<j}a_{ij}, \\ &a_{ij}^2 = (k_i^2 - \nu_i\nu_jk_ik_j + k_j^2 + 3u_0)(k_i - \nu_i\nu_jk_j)^2, \end{aligned}$$

and the summation of $\{\nu\} = \{\nu_1, \nu_2,\ldots,\nu_n\}$ should be completed for all non-dual permutations of $\nu_i = 1, -1, i = 1, 2,\ldots,n$. In addition, $\{\nu\}$ and $-\{\nu\}$ are defined as dual because the cosh function is an even function.

To be specific, for $n = 1$, we have

$$f_1 = \cosh(\xi),\quad \xi = \frac{1}{2}(kx + \omega t), \tag{22}$$

where $\omega = -k[k^4 + 5u_0(k^2 + u_0)]$. Substituting (22) into (16) yields

$$A = \frac{u_0}{2} + \frac{3k^2}{4}\mathrm{sech}^2(\xi) - \frac{ck^3}{4}\tanh(\xi)\mathrm{sech}^2(\xi). \tag{23}$$

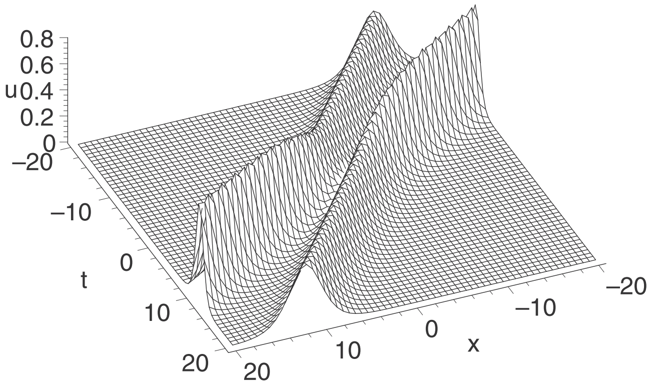

Figure 1. Two-soliton solution of (8).

For $n=2$, we have

$$f_2 = a_{12}^{+}\cosh(\xi_1-\xi_2) + a_{12}^{-}\cosh(\xi_1+\xi_2), \tag{24}$$

where

$$(a_{12}^{\pm})^2 = (k_1^2 \pm k_1k_2 + k_2^2 + 3u_0)(k_1 \pm k_2)^2, \tag{25}$$

and

$$\xi_i = \frac{1}{2}(k_ix+\omega_it),$$
$$\omega_i = -k_i[k_i^4 + 5u_0(k_i^2+u_0)],\ i=1,2. \tag{26}$$

Substituting (26), (25) and (24) into (16), we obtain a two-soliton solution of (8)

$$A = u_2 + c(u_2)_x,\ u_2 = 3\times\frac{a_{12}^{-}a_{12}^{+}[k_2^2\cosh(2\xi_1) + k_1^2\cosh(2\xi_2)] + (k_1^2-k_2^2)^2(k_1^2+k_2^2+3u_0)}{2[a_{12}^{+}\cosh(\xi_1+\xi_2) + a_{12}^{-}\cosh(\xi_1-\xi_2)]^2}. \tag{27}$$

Figure 1 displays the two-soliton solution expressed by (27) for the ABSK equation (8) with the parameter selections $k_1=1$, $k_2=0.8$, $u_0=0.1$ and $c=1$.

To find soliton molecules for the nonlocal system (8), we can apply a special type of velocity resonant mechanism, as proposed in [19–22]:

$$1 = \frac{k_i^4 + 5u_0(k_i^2+u_0)}{k_j^4 + 5u_0(k_j^2+u_0)},\ k_i \neq \pm k_j, \tag{28}$$

i.e.

$$k_i^2 + k_j^2 + 5u_0 = 0,\ k_i \neq \pm k_j. \tag{29}$$

It is clear from the above that two-soliton molecules are permitted; however, n-soliton molecules for $n \geqslant 3$ do not exist. In other words, there is no solution for the three-soliton molecule condition

$$k_i^2 + k_j^2 + 5u_0 = k_i^2 + k_l^2 + 5u_0$$
$$= k_j^2 + k_l^2 + 5u_0 = 0,\ k_i \neq \pm k_j,$$
$$k_i \neq \pm k_l,\ k_j \neq \pm k_l.$$

Figure 2 displays a single two-soliton molecule with the parameter selections

$$c = k_1 = 1,\ k_2 = 0.48,\ u_0 = -0.24608.$$

For $n=3$, we have

$$f_3 = K_0\cosh(\xi_1+\xi_2+\xi_3) + K_1\cosh(-\xi_1+\xi_2+\xi_3) + K_2\cosh(\xi_1-\xi_2+\xi_3) + K_3\cosh(\xi_1+\xi_2-\xi_3), \tag{30}$$

where

$$K_0 = a_{12}^{-}a_{13}^{-}a_{23}^{-},\quad K_1 = a_{12}^{+}a_{13}^{+}a_{23}^{-},$$
$$K_2 = a_{12}^{+}a_{13}^{-}a_{23}^{+},\quad K_3 = a_{12}^{-}a_{13}^{+}a_{23}^{+},$$
$$(a_{ij}^{\pm})^2 = (k_i^2 \pm k_ik_j + k_j^2 + 3u_0)(k_i \pm k_j)^2,$$
$$\xi_i = \frac{k_ix - k_i[k_i^4 + 5u_0(k_i^2+u_0)]t}{2}. \tag{31}$$

Figure 3 displays a three-soliton solution (16) with $f=f_3$ (30) and parameter selections

$$k_1 = 0.7,\ k_2 = 0.6,\ k_3 = 0.5,\ u_0 = 0.05,\ c = 1.$$

From figure 3, we note that the interaction between three solitons is naturally elastic, and does not change their shape and velocities apart from phase shifts.

Figure 4 displays the interaction between a soliton molecule and a typical soliton as described by (16) with $f=f_3$ (30) and parameter selections

$$k_1 = 1,\ k_2 = 0.45,\ k_3 = 0.3,\ u_0 = -0.2405,\ c = 1.$$

Figure 4 shows that the interaction between a soliton and a soliton molecule is nonelastic, meaning that the shape of the molecule is changed though the shape of the soliton is not changed.

For $n=4$, we write the soliton solution for the function $f=f_4$ in the equivalent Hirota's form:

$$\begin{aligned} f_4 = {} & 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_4} \\ & + A_{12}e^{\xi_1+\xi_2} + A_{13}e^{\xi_1+\xi_3} \\ & + A_{14}e^{\xi_1+\xi_4} + A_{23}e^{\xi_2+\xi_3} \\ & + A_{24}e^{\xi_2+\xi_4} + A_{34}e^{\xi_3+\xi_4} \\ & + A_{123}e^{\xi_1+\xi_2+\xi_3} + A_{124}e^{\xi_1+\xi_2+\xi_4} \\ & + A_{134}e^{\xi_1+\xi_3+\xi_4} + A_{234}e^{\xi_2+\xi_3+\xi_4} \\ & + A_{1234}e^{\xi_1+\xi_2+\xi_3+\xi_4}, \end{aligned} \tag{32}$$

where

$$\xi_i = \frac{k_ix - k_i[k_i^4 + 5u_0(k_i^2+u_0)]t}{2},$$
$$(a_{ij}^{\pm})^2 = (k_i^2 \pm k_ik_j + k_j^2 + 3u_0)(k_i \pm k_j)^2$$
$$A_{ij} = \frac{(a_{ij}^{-})^2}{(a_{ij}^{+})^2},$$
$$A_{ijm} = A_{ij}A_{im}A_{jm},\quad A_{ijmn} = A_{ij}A_{im}A_{in}A_{jm}A_{jn}A_{mn}.$$

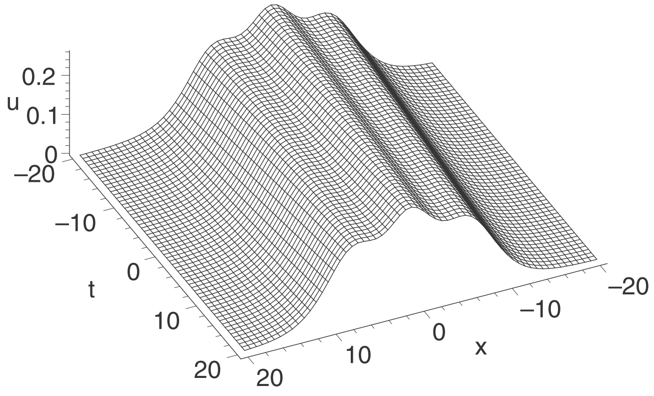

Figure 2. One-soliton molecule of (8).

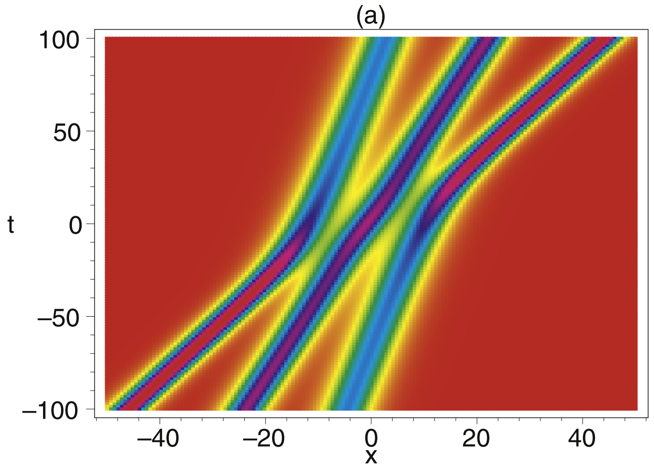

Figure 3. Three-soliton solution of (8).

Figure 5 shows a density plot of the interaction between two soliton molecules, as described by (16) with $f = f_4$ (32) and parameter selections

$$k_1 = 1,\ k_2 = 0.45,\ k_3 = 0.4,$$
$$k_4 = \frac{\sqrt{417}}{20},\ u_0 = -0.2405,\ c = 1.$$

Figure 5 shows that the interaction between two soliton molecules is also nonelastic, meaning that the shapes are changed for both molecules.

3. Exact solutions of (3) with $\sigma = -1$

In this section, we focus on the exact solutions of (3) with $\sigma = -1$,

$$\begin{aligned} A_t &= -A_{5x} - 5[2(B + A)A_{2x} + (A - B)B_{2x}]_x \\ &\quad - 20ABA_x - 10(A^2 - B^2)B_x, \\ B &= \hat{P}\hat{T}A = A(-x, -t). \end{aligned} \tag{33}$$

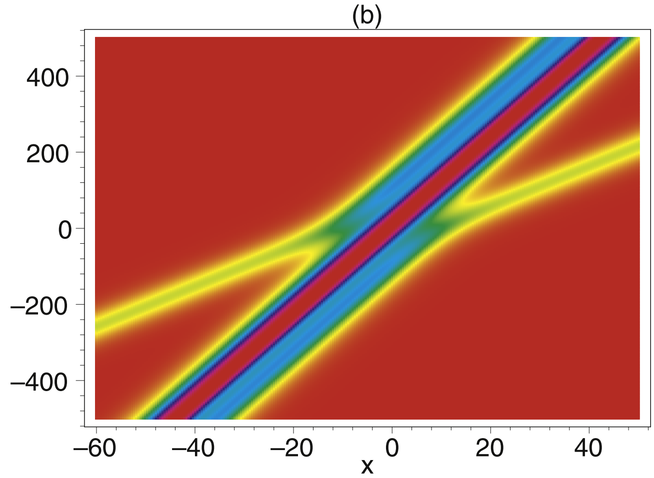

Figure 4. Interaction between one soliton and one soliton molecule within the ABSK system (8).

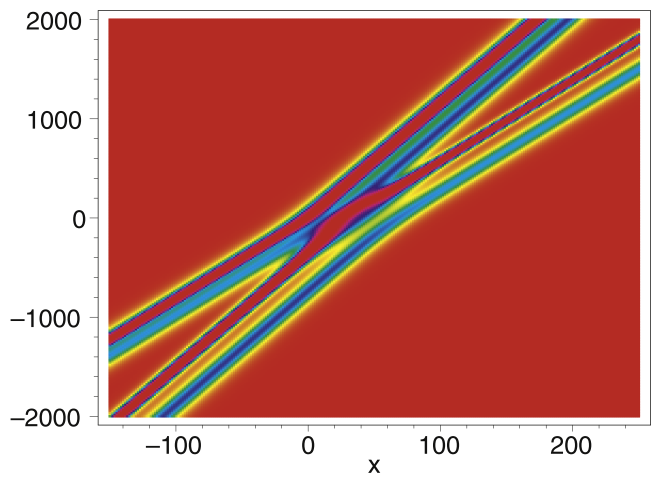

Figure 5. Interaction between two soliton molecules within the ABSK system (8).

Applying the same transformation (9) and the symmetric-antisymmetric separation approach to solve (33) obtains

$$u_t + u_{5x} + 5(uu_{2x} - vv_{2x} - v^2u)_x + 5u^2u_x = 0, \tag{34}$$
$$v_t + v_{5x} + 5(vu_{2x} + uv_{2x} + vu^2)_x - 5v^2v_x = 0, \tag{35}$$

with the symmetric and antisymmetric conditions

$$\hat{P}\hat{T}u = u, \qquad \hat{P}\hat{T}v = -v.$$

Thus, we can prove that the equation system (34)–(35) can be solved by the complex SK equation

$$U_t + (U_{4x} + 5UU_{2x})_x + 5U^2U_x = 0,\ U = u + \mathrm{i}v,\ \mathrm{i} = \sqrt{-1}. \tag{36}$$

3.1. Periodic wave of (33)

Because of the properties of (36), using the Jacobi elliptic function expansion method [17, 18] for (36) allows one of the special periodic waves of (33) to be written as ($\xi = kx + \omega t$)

so that

$$A = a_0 + \frac{3k^2m^2}{[m^2\mathrm{sn}^2(\xi, m)S_0^2 + C_0^2]^2} \times [C_0^2\mathrm{cn}^2(\xi, m)\mathrm{dn}^2(\xi, m)S_0^2 + 2C_0D_0\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m)\mathrm{sn}(\xi, m)S_0 - D_0^2\mathrm{sn}^2(\xi, m)], \tag{37}$$

where the constants C_0, D_0 and S_0 are defined by

$$C_0 = \mathrm{cn}(c_0, \sqrt{1-m^2}), \quad D_0 = \mathrm{dn}(c_0, \sqrt{1-m^2}), \quad S_0 = \mathrm{sn}(c_0, \sqrt{1-m^2}),$$

and the other constants a_0 k, m, c_0 and ω are related by

$$\omega + 20k[a_0 - k^2(m^2+1)]^2 - 4k^5(m^4 + m^2 + 1) = 0.$$

3.2. Multiple solitons and soliton molecules of (33)

Because the symmetric and antisymmetric separation equations (34) and (35) can be solved by the complex SK equation (36), the complex SK equation can be bilinearized to

$$(D_x^6 + D_xD_t + 5u_0^2D_x^2 + 5u_0D_x^4)F\cdot F = 0 \tag{38}$$

by using the transformation

$$U = u_0 + 6[\ln(F)]_{xx}. \tag{39}$$

If we allow $F = f + \mathrm{i}g$ and $U = u + \mathrm{i}v$, we obtain a soliton solution of (33), with the form

$$A = 3\left[\frac{1}{2}\ln(f^2 + g^2) + \arctan\left(\frac{g}{f}\right)\right]_{xx} + \frac{u_0}{2}, \quad B = \hat{P}\hat{T}A,\ \hat{P}\hat{T}f = f,\ \hat{P}\hat{T}g = -g, \tag{40}$$

where functions f and g are determined by the bilinear equations

$$(D_x^6 + D_xD_t + 5u_0^2D_x^2 + 5u_0D_x^4)(f\cdot f - g\cdot g) = 0, \tag{41}$$

$$(D_x^6 + D_xD_t + 5u_0^2D_x^2 + 5u_0D_x^4)f\cdot g = 0 \tag{42}$$

with the explicit special solutions

$$f = f_n = \cos(c)\textstyle\sum_{\{\nu\}}K_{\{\nu\}}\cosh\left(\sum_{i=1}^n\nu_i\xi_i\right), \quad g = g_n = \sin(c)\sum_{\{\nu\}}K_{\{\nu\}}\sinh\left(\sum_{i=1}^n\nu_i\xi_i\right), \tag{43}$$

where $K_{\{\nu\}}$, ξ_i are the same as (21) and (2.2), $\omega_i = -k_i[k_i^4 + 5u_0(k_i^2 + u_0)]$ and c is an arbitrary constant.

For $n = 1$, we take

$$f = \cosh(\xi)\cos(c), \quad g = \sinh(\xi)\sin(c). \tag{44}$$

Substituting (44) and (21) into (40), we obtain a one-soliton solution as follows:

$$A = \frac{u_0}{2} + \frac{3k_1^2[1 + \cos(2c)\cosh(2\xi_1) - \sin(2c)\sinh(2\xi_1)]}{[\cos(2c) + \cosh(2\xi_1)]^2}. \tag{45}$$

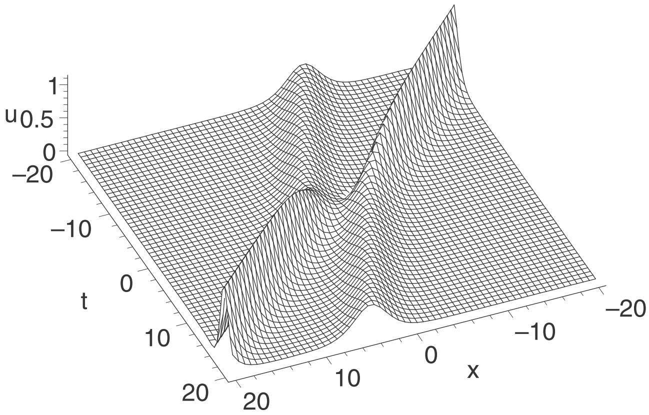

Figure 6. Two-soliton solution expressed by (40) with (46) with the parameter selections $k_1 = 1$, $k_2 = 0.8$, $u_0 = 0$ and $c = \pi/6$ within the ABSK system (33).

When $c = 0$, the solution (45) is a $\hat{P}\hat{T}$ invariant soliton, whereas (45) is also a $\hat{P}\hat{T}$ symmetry breaking soliton for non-zero c.

For $n = 2$, we have the forms of $f = f_2$ and $g = g_2$

$$f_2 = \cos(c)(a_{12}^-\cosh(\xi_1 + \xi_2) + a_{12}^+\cosh(\xi_1 - \xi_2)), \quad g_2 = \sin(c)(a_{12}^-\sinh(\xi_1 + \xi_2) + a_{12}^+\sinh(\xi_1 - \xi_2)), \tag{46}$$

where

$$a_{12}^{\pm} = \sqrt{k_1^2 \pm k_1k_2 + k_2^2 + 3u_0}\,(k_1 \pm k_2), \tag{47}$$

and

$$\xi_i = \frac{k_ix - k_i[k_i^4 + 5u_0(k_i^2 + u_0)]t}{2},\ i = 1, 2. \tag{48}$$

Substituting (46), (47) and (48) into (40), we obtain a two-soliton solution of (33).

Figure 6 shows the interaction of two solitons for the $\hat{P}\hat{T}$ symmetry breaking case $c = \pi/6$, where the other parameters are fixed as $k_1 = 1$, $k_2 = 0.8$ and $u_0 = 0$.

If the velocity resonance condition $k_1^2 + k_2^2 + 5u_0 = 0$ is satisfied, the two soliton solution (40), together with (46), becomes a two-soliton molecule. Figure 7 displays the structure of the two-soliton molecule under the parameter selections

$$u_0 = -\frac{k_1^2}{5} - \frac{k_2^2}{5},\ k_1 = 1,\ k_2 = 0.4,\ c = \frac{\pi}{3}.$$

For $n = 3$, we have

$$\begin{aligned} f_3 &= \cos(c)(a_{12}^-a_{13}^-a_{23}^-\cosh(\xi_1 + \xi_2 + \xi_3) + a_{12}^+a_{13}^+a_{23}^-\cosh(\xi_1 - \xi_2 - \xi_3) + a_{12}^+a_{13}^-a_{23}^+\cosh(\xi_1 - \xi_2 + \xi_3) + a_{23}^+a_{13}^+a_{12}^-\cosh(\xi_1 + \xi_2 - \xi_3)),\\ g_3 &= \sin(c)(a_{12}^-a_{13}^-a_{23}^-\sinh(\xi_1 + \xi_2 + \xi_3) + a_{12}^+a_{13}^+a_{23}^-\sinh(\xi_1 - \xi_2 - \xi_3) + a_{12}^+a_{13}^-a_{23}^+\sinh(\xi_1 - \xi_2 + \xi_3) + a_{23}^+a_{13}^+a_{12}^-\sinh(\xi_1 + \xi_2 - \xi_3)), \end{aligned} \tag{49}$$

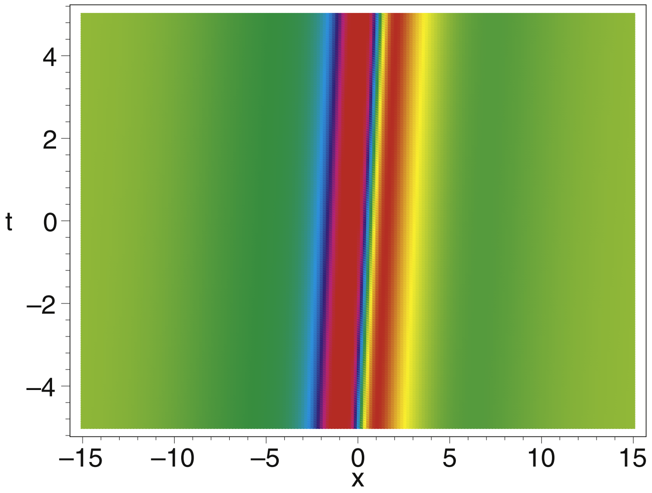

Figure 7. Two-soliton molecule expressed by (40) with (46) and the parameter selections $k_1 = 1$, $k_2 = 0.4$, $u_0 = -0.232$ and $c = \pi/3$ within the ABSK system (33).

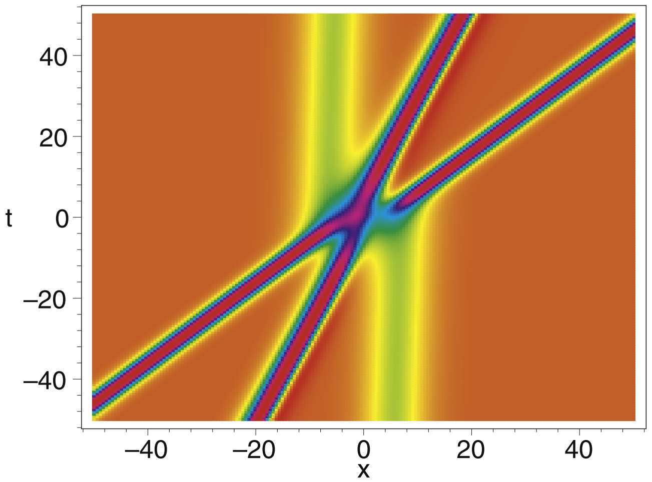

Figure 8. Interaction among three solitons expressed by (40) with (49) and the parameter selections $k_1 = 1$, $k_2 = 0.8$, $k_3 = 0.4$, $u_0 = 0$ and $c = \pi/6$ within the ABSK system (33).

where

$$a_{ij}^{\pm} = \sqrt{k_i^2 \pm k_i k_j + k_j^2 + 3u_0}\,(k_i \pm k_j), \quad 1 \leqslant i < j \leqslant 3, \tag{50}$$

and

$$\xi_i = \frac{k_i x - k_i(k_i^4 + 5u_0(k_i^2 + u_0))t}{2}, \, i = 1, 2, 3. \tag{51}$$

Substituting (49), (50) and (51) into (40), we can obtain a three soliton solution of (33).

Figure 8 displays the elastic interaction property between three solitons expressed by (40) with (49), and parameter selections

$$k_1 = 1, \, k_2 = 0.8, \, k_3 = 0.4, \, u_0 = 0, \, c = \frac{\pi}{6}.$$

Figure 9 displays the interaction between one soliton and one two-soliton molecule expressed by (40) with (49), and the velocity resonant condition $5u_0 + k_1^2 + k_2^2 = 0$, where the parameters are fixed as

$$k_1 = 1, \, k_2 = 0.48, \, k_3 = 0.4, \, u_0 = -0.24608, \, c = \frac{\pi}{3}.$$

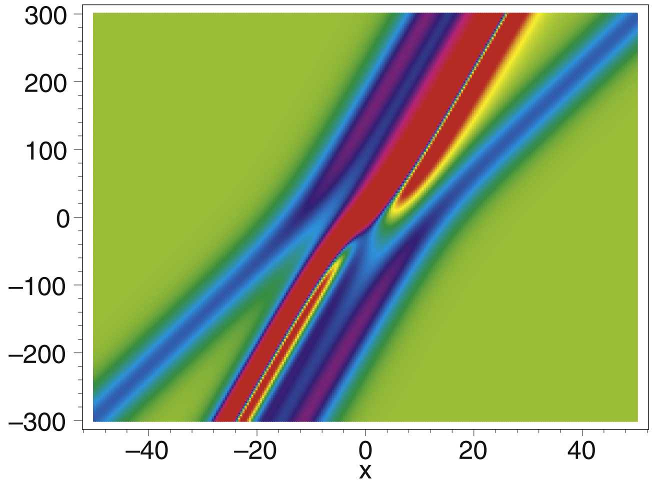

Figure 9. Interaction between one soliton and one two-soliton molecule expressed by (40) with (49) and the parameter selections $k_1 = 1$, $k_2 = 0.48$, $k_3 = 0.4$, $u_0 = -0.24608$ and $c = \pi/3$ within the ABSK system (33).

From figure 9, it is clear that the interaction between one soliton and one soliton molecule is nonelastic because the shape of the molecule has been altered by the interaction.

4. Summary and discussions

In this paper, we focus on the nonlocal Alice–Bob Sawada–Kotera systems, and select two typical models in order to discuss their integrability and exact solutions. Other models with different values of σ could use the same approach to obtain solutions. Moreover, other fifth-order integrable nonlocal systems such as the nonlocal Kaup-Kupershmidt system and the nonlocal fifth-order KDV system [23] could also integrate into a whole fifth-order nonlocal system with a nonlocal SK system by introducing further parameters.

Using the Jacobi elliptic function expansion method and bilinear approach, we obtain periodic waves and multiple soliton solutions for two typical nonlocal Sawada–Kotera models. It is interesting to note that by using the so-called velocity resonance mechanism introduced in [19, 21], we find that soliton molecules can also be found in nonlocal ABSK systems. Based on the velocity resonance mechanism, other researchers have successfully obtained soliton molecules in many local systems [20, 21, 24, 25]. Soliton molecules have been experimentally observed by many scientists [26, 27].

It is well known that with the exception of soliton fission and soliton fusion for some special models [28], interactions between solitons are usually elastic. In fact, soliton interactions based on local and nonlocal SK equations are particularly elastic. However, in this paper we find that the interactions between solitons and soliton molecules may be nonelastic because the shape of the soliton molecules is explicitly changed due to interaction.

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