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Interaction phenomena between lump and solitary wave of a generalized (3 + 1)-dimensional variable-coefficient nonlinear-wave equation in liquid with gas bubbles*

Jian-Guo Liu[1,4], Wen-Hui Zhu[2,4], Yan He[1,4] and Ya-Kui Wu[3,4]

[1] College of Computer, Jiangxi University of Traditional Chinese Medicine, Jiangxi 330004, China
[2] Institute of artificial intelligence, Nanchang Institute of Science and Technology, Jiangxi 330108, China
[3] School of science, Jiujiang University, Jiangxi 332005, China

E-mail: 20101059@jxutcm.edu.cn, 415422402@qq.com, 274667818@qq.com and 2180501008@cnu.edu.cn

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Abstract

In this paper, a generalized (3 + 1)-dimensional variable-coefficient nonlinear-wave equation is studied in liquid with gas bubbles. Based on the Hirota's bilinear form and symbolic computation, lump and interaction solutions between lump and solitary wave are obtained, which include a periodic-shape lump solution, a parabolic-shape lump solution, a cubic-shape lump solution, interaction solutions between lump and one solitary wave, and between lump and two solitary waves. The spatial structures called the bright lump wave and the bright-dark lump wave are discussed. Interaction behaviors of two bright-dark lump waves and a periodic-shape bright lump wave are also presented. Their interactions are shown in some 3D plots.

Keywords: solitary wave, lump wave, variable-coefficient nonlinear-wave equation, interaction behaviors

(Some figures may appear in colour only in the online journal)

1. Introduction

In some branches of science and engineering such as fluid mechanics, quantum mechanics, particle physics, mass transfer, plasma physics, nano liquids and biological mathematics [1–5], nonlinear partial differential equations (NPDES) are used to describe many nonlinear phenomena and wave propagation characteristics. As the lump solutions of the NPDES are the special, powerful destructive ocean wave in the real world, it is important to search for the lump solutions of the NPDES, especially the constant-coefficient NPDES that have attracted the attention of many scholars [6–10].

Recently, a generalized (3 + 1)-dimensional nonlinear-wave equation has been presented as [11]

$$[4u_t + 4uu_x + u_{xxx} - 4u_x]_x + 3(u_{yy} + u_{zz}) = 0, \qquad (1)$$

which describes a liquid with gas bubbles in the three-dimensional case.

However, the variable-coefficient NPDES provide us with more real phenomena in the inhomogeneities of media and non-uniformities of boundaries than corresponding constant-coefficient counterparts in some physical cases [12–15]. In this paper, a generalized (3 + 1)-dimensional variable-coefficient nonlinear-wave equation is investigated [16, 17]

$$\alpha(t)u_x^2 + \alpha(t)uu_{xx} + \beta(t)u_{xxxx} + \gamma(t)u_{xx} + \delta(t)u_{yy} + \varrho(t)u_{zz} + u_{xt} = 0, \qquad (2)$$

where $u = u(x, y, z, t)$ is the wave-amplitude function. The bilinear form, Bäcklund transformation, Lax pair, infinitely-many

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[4] Authors to whom any correspondence should be addressed.

conservation laws, multi-soliton solutions, traveling-wave solutions and one-periodic wave solutions are presented by virtue of the binary Bell polynomials, the Hirota method, the polynomial expansion method and the Hirota-Riemann method [18]. However, lump and interaction solutions between the lump and solitary wave of equation (2) have not been obtained yet, which will make the main work of our paper.

This paper will be organized as follows: section 2 obtains the lump solutions of equation (2) with the aid of the Hirota's bilinear form [19–24] and demonstrates their physical structures by some 3D plots; section 3 presents the interaction solutions between lump and one solitary wave; section 4 derives the interaction solutions between lump and two solitary waves; and section 5 gives the conclusion.

2. Lump solutions of equation (2)

Setting $u = 12\,[\ln \xi(x, y, z, t)]_{xx}$ and $\alpha(t) = \beta(t)$, and using the multi-dimensional Bell polynomials, the bilinear form of equation (2) can be introduced as (see [18])

$$[\mathrm{D}_x\mathrm{D}_t + \beta(t)\mathrm{D}_x^4 + \gamma(t)\mathrm{D}_x^2 + \delta(t)\mathrm{D}_y^2 + \varrho(t)\mathrm{D}_z^2]\xi\cdot\xi = 0. \quad (3)$$

This is equivalent to

$$\begin{aligned}&\xi[\beta(t)\xi_{xxxx} + \gamma(t)\xi_{xx} + \delta(t)\xi_{yy} + \varrho(t)\xi_{zz} + \xi_{xt}]\\&\quad + 3\beta(t)\xi_{xx}^2 - 4\beta(t)\xi_x\xi_{xxx} - \gamma(t)\xi_x^2\\&\quad - \delta(t)\xi_y^2 - \varrho(t)\xi_z^2 - \xi_t\xi_x = 0.\end{aligned} \quad (4)$$

In order to seek the lump solutions of equation (2), we suppose

$$\begin{aligned}&\zeta = x\alpha_1 + y\alpha_2 + z\alpha_3 + \alpha_4(t),\\&\varsigma = x\alpha_5 + y\alpha_6 + z\alpha_7 + \alpha_8(t),\\&\xi = \zeta^2 + \varsigma^2 + \alpha_9(t),\end{aligned} \quad (5)$$

where α_1, α_2, α_3, α_5, α_6 and α_7 are unknown constants. $\alpha_4(t)$, $\alpha_8(t)$ and $\alpha_9(t)$ are undefined real functions. Substituting equation (5) into equation (4) through Mathematica software, we get

$$\begin{aligned}&(I)\colon \alpha_8(t) = \eta_1 - \int\frac{(\alpha_1^2+\alpha_5^2)[\alpha_1^2\gamma(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)] + \alpha_1^3\alpha_4'(t)}{\alpha_1^2\alpha_5}\,\mathrm{d}t,\\&\alpha_9(t) = \eta_2 + \int\Bigg[2[\alpha_1(\alpha_1\gamma(t) + \alpha_4'(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)][\alpha_1[\eta_3\\&\quad - \int\frac{(\alpha_1^2+\alpha_5^2)[\alpha_1^2\gamma(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)] + \alpha_1^3\alpha_4'(t)}{\alpha_1^2\alpha_5}\,\mathrm{d}t\Bigg] - \alpha_5 * \alpha_4(t)]]/(\alpha_1\alpha_5)\,\mathrm{d}t,\ \alpha(t) = \beta(t) = 0,\\&\alpha_6 = \frac{\alpha_2\alpha_5}{\alpha_1},\ \alpha_7 = \frac{\alpha_3\alpha_5}{\alpha_1},\end{aligned} \quad (6)$$

with $\alpha_1 \neq 0$, $\alpha_5 \neq 0$. Substituting equations (5) and (6) into the transformation $u = 12\,[\ln \xi(x, y, z, t)]_{xx}$, we have the following lump solution of equation (2)

$$\begin{aligned}u^{(I)} = &\Bigg[12\Bigg[2(\alpha_1^2+\alpha_5^2)\Bigg[\eta_2 + \int[2[\alpha_1[\alpha_1\gamma(t) + \alpha_4'(t)] + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)][\alpha_1[\eta_1\\&- \int\frac{(\alpha_1^2+\alpha_5^2)[\alpha_1^2\gamma(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)] + \alpha_1^3\alpha_4'(t)}{\alpha_1^2\alpha_5}\,\mathrm{d}t\Bigg] - \alpha_5\alpha_4(t)]]/(\alpha_1\alpha_5)\,\mathrm{d}t\\&+ \Bigg[\eta_1 - \int\frac{(\alpha_1^2+\alpha_5^2)[\alpha_1^2\gamma(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)] + \alpha_1^3\alpha_4'(t)}{\alpha_1^2\alpha_5}\,\mathrm{d}t\\&+ \frac{\alpha_5(\alpha_1x + \alpha_2y + \alpha_3z)}{\alpha_1}\Bigg]^2 + [\alpha_4(t) + \alpha_1x + \alpha_2y + \alpha_3z]^2\Bigg] - 4[\alpha_5[\eta_1\\&- \int\frac{(\alpha_1^2+\alpha_5^2)[\alpha_1^2\gamma(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)] + \alpha_1^3\alpha_4'(t)}{\alpha_1^2\alpha_5}\,\mathrm{d}t + \alpha_5x\Bigg] + \alpha_1(\alpha_4(t) + \alpha_2y + \alpha_3z) + \alpha_1^2x + \frac{\alpha_5^2(\alpha_2y + \alpha_3z)}{\alpha_1}\Bigg]^2\Bigg]\Bigg]/[[\eta_2\\&+ \int[2[\alpha_1(\alpha_1\gamma(t) + \alpha_4'(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)][\alpha_1[\eta_1 - \int\frac{(\alpha_1^2+\alpha_5^2)[\alpha_1^2\gamma(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)] + \alpha_1^3\alpha_4'(t)}{\alpha_1^2\alpha_5}\,\mathrm{d}t\Bigg]\\&- \alpha_5\alpha_4(t)]]/(\alpha_1\alpha_5)\,\mathrm{d}t + \Bigg[\eta_1 - \int\frac{(\alpha_1^2+\alpha_5^2)[\alpha_1^2\gamma(t) + \alpha_2^2\delta(t) + \alpha_3^2\varrho(t)] + \alpha_1^3\alpha_4'(t)}{\alpha_1^2\alpha_5}\,\mathrm{d}t\\&+ \frac{\alpha_5(\alpha_1x + \alpha_2y + \alpha_3z)}{\alpha_1}\Bigg]^2 + (\alpha_4(t) + \alpha_1x + \alpha_2y + \alpha_3z)^2\Bigg]^2\Bigg],\end{aligned} \quad (7)$$

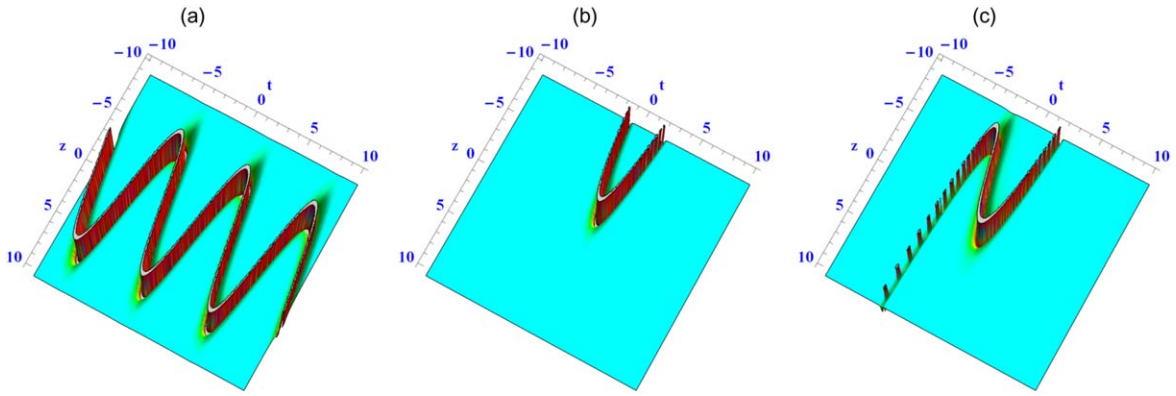

Figure 1. Lump solution $u^{(I)}$ with $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_5 = -3$, $x = -5$, $\eta_1 = \eta_2 = y = 0$, when $\gamma(t) = \delta(t) = \varrho(t) = \cos t$, $\alpha_4(t) = \sin t$ in (a), $\gamma(t) = \delta(t) = \varrho(t) = t$, $\alpha_4(t) = \sin t$ in (b) and $\gamma(t) = \cosh t$, $\varrho(t) = \exp t$ and $\alpha_4(t) = \delta(t) = t$ in (c).

where $\alpha_4(t)$ is arbitrary function, η_1 and η_2 are integral constants.

The physical structures for $(u^{(I)})$ are described in figure 1 by the 3D plots. Figure 1 shows the propagation of solution $(u^{(I)})$ when $\gamma(t)$, $\delta(t)$, $\varrho(t)$ and $\alpha_4(t)$ select different functions. When $\gamma(t) = \delta(t) = \varrho(t) = \cos t$ and $\alpha_4(t) = \sin t$, a periodic-shape rational solution is listed in figure 1(a). When $\gamma(t) = \delta(t) = \varrho(t) = t$ and $\alpha_4(t) = \sin t$, a parabolic-shape rational solution is presented in figure 1(b). When $\gamma(t) = \cosh t$, $\varrho(t) = \exp t$ and $\alpha_4(t) = \delta(t) = t$, a cubic-shape rational solution is shown in figure 1(c).

$$
\begin{aligned}
(II)\colon\ \alpha_8(t) &= \eta_3 + \int \alpha_5 \left[-\frac{3(\alpha_1^2 + \alpha_5^2)\beta(t)}{\alpha_9} - \frac{\alpha_3^2 \varrho(t)}{\alpha_1^2} - \gamma(t) \right] \mathrm{d}t, \\
\alpha_9(t) &= \alpha_9,\ \alpha_6 = -\frac{\alpha_1\alpha_2}{\alpha_5},\ \alpha_7 = \frac{\alpha_3\alpha_5}{\alpha_1}, \\
\delta(t) &= -\frac{3\alpha_5^2(\alpha_1^2 + \alpha_5^2)\beta(t)}{\alpha_2^2\alpha_9}, \\
\alpha_4(t) &= \eta_4 - \int \left[\frac{\alpha_1[3(\alpha_1^2 + \alpha_5^2)\beta(t) + \alpha_9\gamma(t)]}{\alpha_9} + \frac{\alpha_3^2\varrho(t)}{\alpha_1} \right] \mathrm{d}t,
\end{aligned} \tag{8}
$$

with $\alpha_1 \neq 0$, $\alpha_5 \neq 0$, $\alpha_9 \neq 0$ and $\alpha_2\alpha_9 \neq 0$. Substituting equations (5) and (8) into the transformation $u = 12\,[\ln \xi(x, y, z, t)]_{xx}$, we have the following lump solution of equation (2)

$$
\begin{aligned}
u^{(II)} = {} & \Bigg[12 \Bigg[2(\alpha_1^2 + \alpha_5^2) \Bigg[\alpha_9 + \Bigg[\int \alpha_5 \Bigg[-\frac{3(\alpha_1^2 + \alpha_5^2)\beta(t)}{\alpha_9} \\
& - \frac{\alpha_3^2\varrho(t)}{\alpha_1^2} - \gamma(t) \Bigg] \mathrm{d}t + \eta_3 + \alpha_5 x - \frac{\alpha_1\alpha_2 y}{\alpha_5} + \frac{\alpha_3\alpha_5 z}{\alpha_1} \Bigg]^2 \\
& + \Bigg[\int \Bigg[-\frac{\alpha_1[3(\alpha_1^2 + \alpha_5^2)\beta(t) + \alpha_9\gamma(t)]}{\alpha_9} \\
& - \frac{\alpha_3^2\varrho(t)}{\alpha_1} \Bigg] \mathrm{d}t + \eta_4 + \alpha_1 x + \alpha_2 y + \alpha_3 z \Bigg]^2 \Bigg] \\
& - \Bigg[4 \Bigg[\alpha_1^2 \Bigg[\int \Bigg[-\frac{\alpha_1[3(\alpha_1^2 + \alpha_5^2)\beta(t) + \alpha_9\gamma(t)]}{\alpha_9} \\
& - \frac{\alpha_3^2\varrho(t)}{\alpha_1} \Bigg] \mathrm{d}t \Bigg] + \alpha_5\alpha_1[\eta_3 \\
& + \int \alpha_5 \Bigg[-\frac{3(\alpha_1^2 + \alpha_5^2)\beta(t)}{\alpha_9} - \frac{\alpha_3^2\varrho(t)}{\alpha_1^2} - \gamma(t) \Bigg] \mathrm{d}t \\
& + \alpha_5 x] + \alpha_1^3 x \\
& + \alpha_1^2(\eta_4 + \alpha_3 z) + \alpha_3\alpha_5^2 z]^2 \big/ (\alpha_1^2)]] \\
& \Bigg/ \Bigg[\Bigg[\alpha_9 + \Bigg[\eta_3 + \alpha_5 x - \frac{\alpha_1\alpha_2 y}{\alpha_5} + \frac{\alpha_3\alpha_5 z}{\alpha_1} \\
& + \int \alpha_5 \Bigg[-\frac{3(\alpha_1^2 + \alpha_5^2)\beta(t)}{\alpha_9} - \frac{\alpha_3^2\varrho(t)}{\alpha_1^2} - \gamma(t) \Bigg] \mathrm{d}t \Bigg]^2 \\
& + [\eta_4 + \alpha_1 x + \alpha_2 y \\
& - \int \Bigg[\frac{\alpha_1[3(\alpha_1^2 + \alpha_5^2)\beta(t) + \alpha_9\gamma(t)]}{\alpha_9} + \frac{\alpha_3^2\varrho(t)}{\alpha_1} \Bigg] \mathrm{d}t \\
& + \alpha_3 z]^2]^2],
\end{aligned} \tag{9}
$$

where η_3 and η_4 are integral constants.

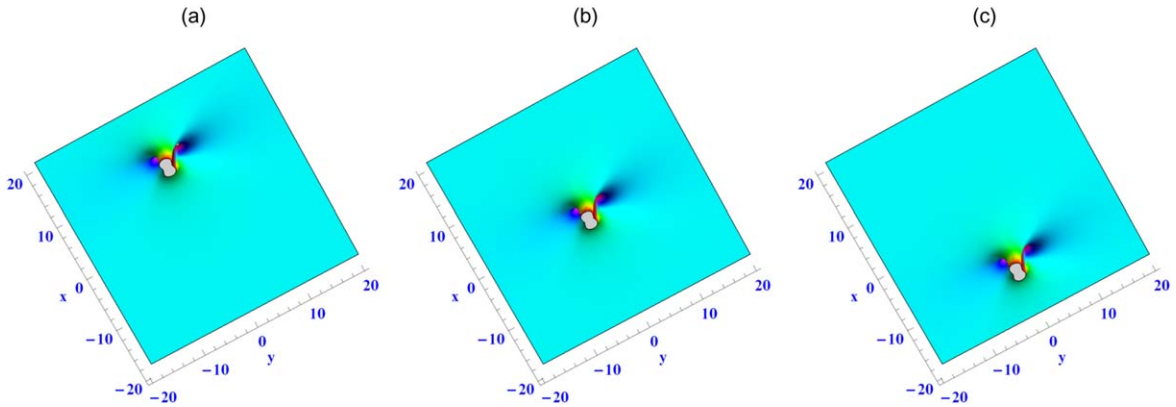

Figure 2. Lump solution (9) with $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_5 = \alpha_9 = -3$, $\eta_3 = \eta_4 = 0$, $z = -10$, when $t = -1$ in (a), $t = 0$ in (b) and $t = 1$ in (c).

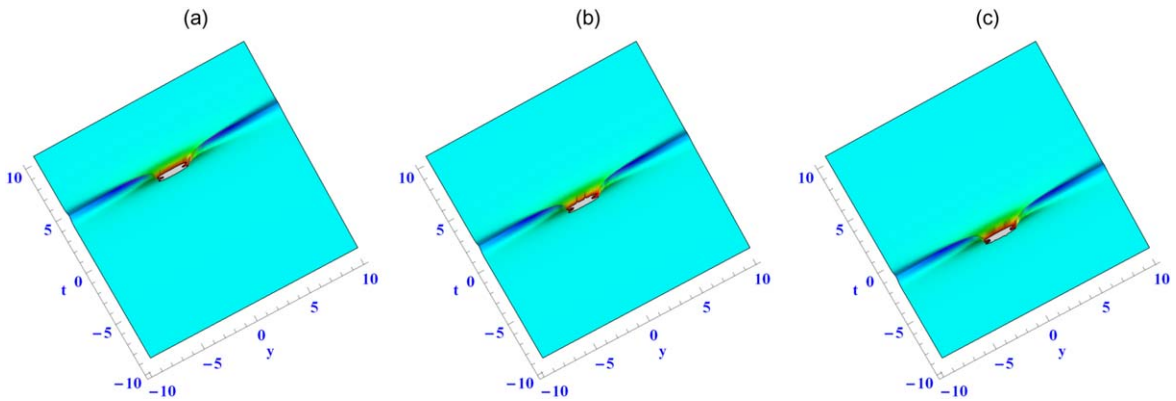

Figure 3. Lump solution (9) with $\alpha_1 = -1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_5 = 3$, $\alpha_9 = -3$, $\eta_3 = \eta_4 = z = 0$, when $x = -30$ in (a), $x = 0$ in (b) and $x = 30$ in (c).

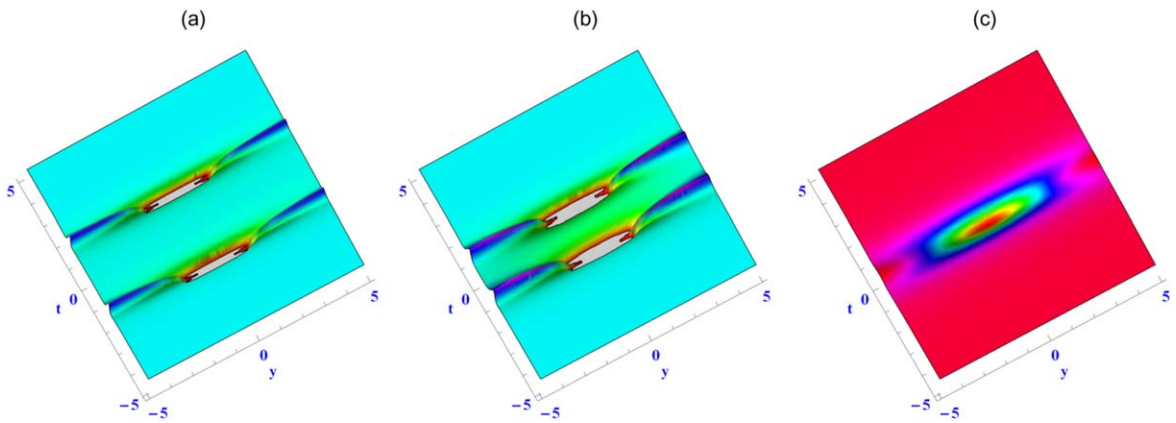

Figure 4. Lump solution (9) with $\alpha_1 = -1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_5 = 3$, $\alpha_9 = -3$, $\eta_3 = \eta_4 = z = 0$, when $x = -8$ in (a), $x = 0$ in (b) and $x = 8$ in (c).

Then, the physical structures for $(u^{(II)})$ are shown in figures 2–5 with some 3D plots. When $\gamma(t) = -1$, $\varrho(t) = \beta(t) = 1$, the spatial structure called the bright lump wave is seen in figure 2 at $t = -1$; 0; 1, the spatial structure called the bright-dark lump wave is shown in figure 3 at $x = -30$; 0; 30. When $\gamma(t) = -t$, $\varrho(t) = \beta(t) = t$, interaction behaviors of two bright-dark lump waves are presented in figure 4 at $x = -8$; 0; 8. As the value of x

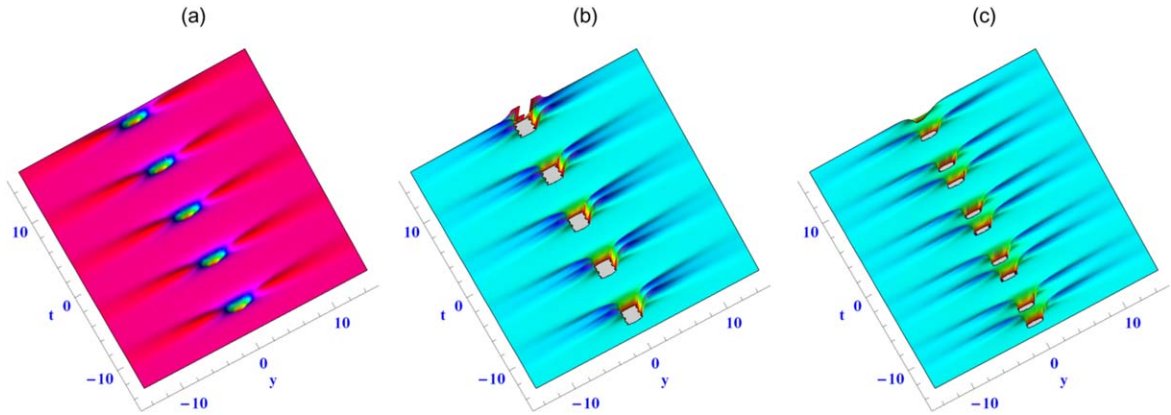

Figure 5. Lump solution (9) with $\alpha_1 = -1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_5 = 3$, $\alpha_9 = -3$, $\eta_3 = \eta_4 = z = 0$, when $x = -3$ in (a), $x = 0$ in (b) and $x = 3$ in (c).

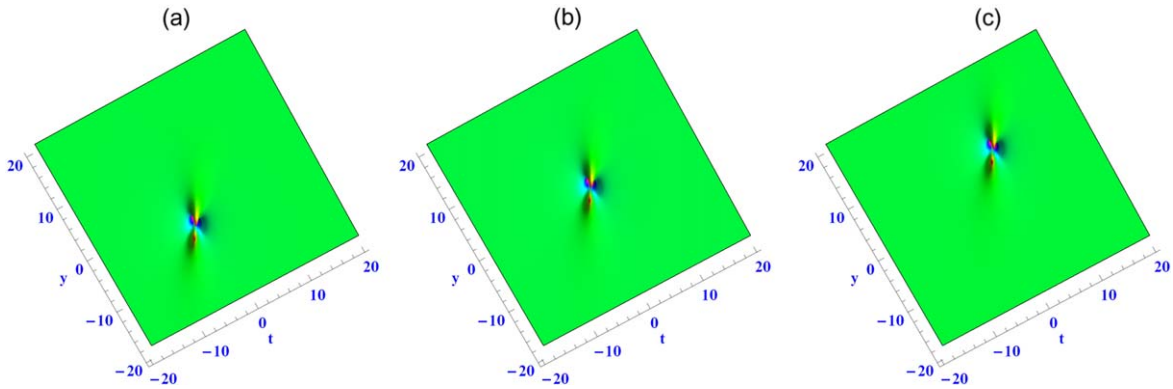

Figure 6. Lump solution (11) with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = -1$, $\alpha_6 = \alpha_7 = \eta_5 = 3$, $\eta_6 = -2$, $\alpha_9 = 2$, $z = 0$, when $x = -10$ in (a), $x = 0$ in (b) and $x = 10$ in (c).

changes, the two bright-dark lump waves move towards each other, and finally merge together. When $\gamma(t) = \varrho(t) = \beta(t) = \cos t$, a periodic-shape bright lump wave is found in figure 5 at $x = -3$; 0; 3.

with $(\alpha_3\alpha_5 - \alpha_1\alpha_7)^2\alpha_9 \neq 0$, $\alpha_1^2 + \alpha_5^2 \neq 0$, η_5 and η_6 are integral constants. Substituting equations (5) and (10) into the transformation $u = 12\,[\ln \xi(x, y, z, t)]_{xx}$, we derive another

$$
\begin{aligned}
(III)\colon\ \varrho(t) &= -\frac{3(\alpha_1^2 + \alpha_5^2)^3\beta(t) + (\alpha_2\alpha_5 - \alpha_1\alpha_6)^2\alpha_9\delta(t)}{(\alpha_3\alpha_5 - \alpha_1\alpha_7)^2\alpha_9},\ \alpha_9(t) = \alpha_9,\\
\alpha_4(t) &= \eta_5 - \int[\alpha_1[\alpha_5^2\gamma(t) + (\alpha_2^2 - \alpha_6^2)\delta(t) + (\alpha_3^2 - \alpha_7^2)\varrho(t)]\\
&\quad + \alpha_1^3\gamma(t) + 2\alpha_5[\alpha_2\alpha_6\delta(t) + \alpha_3\alpha_7\varrho(t)]]/(\alpha_1^2 + \alpha_5^2)\,\mathrm{d}t,\\
\alpha_8(t) &= \eta_6 - \int[\alpha_5[\alpha_5^2\gamma(t) + (\alpha_6^2 - \alpha_2^2)\delta(t) + (\alpha_7^2 - \alpha_3^2)\varrho(t)]\\
&\quad + \alpha_5\alpha_1^2\gamma(t) + 2\alpha_1[\alpha_2\alpha_6\delta(t) + \alpha_3\alpha_7\varrho(t)]]/(\alpha_1^2 + \alpha_5^2)\,\mathrm{d}t
\end{aligned}
\tag{10}
$$

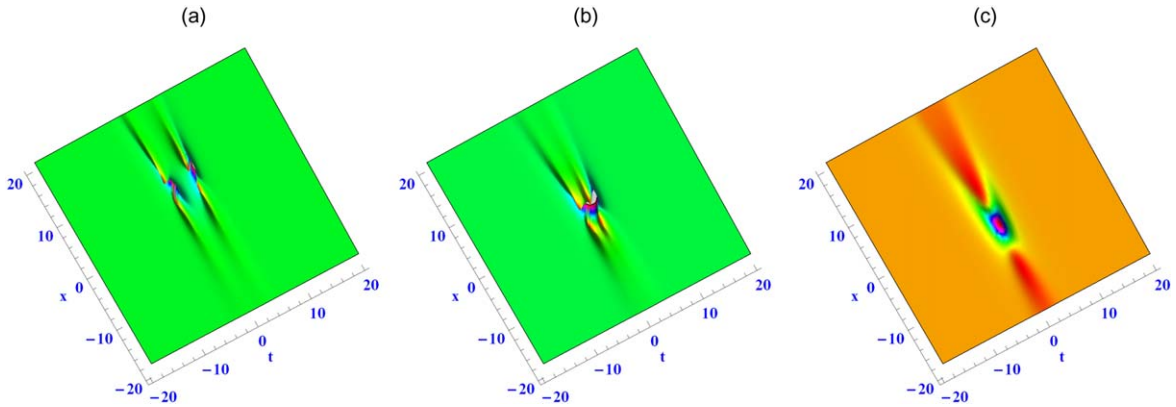

Figure 7. Lump solution (11) with $\alpha_1 = \alpha_3 = \alpha_5 = -1$, $\alpha_2 = 1$, $\alpha_6 = -3$, $\alpha_7 = 3$, $\eta_6 = \eta_5 = z = 0$, $\alpha_9 = 2$, $z = 0$, when $y = -3$ in (a), $y = 0$ in (b) and $y = 3$ in (c).

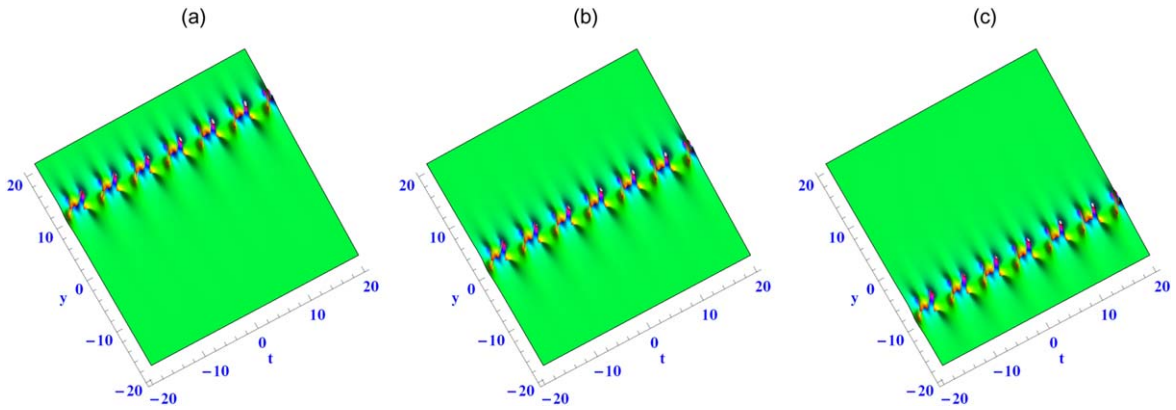

Figure 8. Lump solution (11) with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = -1$, $\alpha_6 = \alpha_7 = \eta_5 = 3$, $\eta_6 = -2$, $\alpha_9 = 2$, $x = 0$, when $z = -10$ in (a), $z = 0$ in (b) and $z = 10$ in (c).

lump solution of equation (2)

$$\begin{aligned} u^{(III)} = {} & [12[2(\alpha_1^2 + \alpha_5^2)[\alpha_9 + [\alpha_4(t) + \alpha_1 x + \alpha_2 y + \alpha_3 z]^2 \\ & + [\alpha_8(t) + \alpha_5 x + \alpha_6 y + \alpha_7 z]^2] \\ & - [2\alpha_1[\alpha_4(t) + \alpha_1 x + \alpha_2 y + \alpha_3 z] \\ & + 2\alpha_5[\alpha_8(t) + \alpha_5 x \\ & + \alpha_6 y + \alpha_7 z]]^2]]/[[\alpha_9 + [\alpha_4(t) \\ & + \alpha_1 x + \alpha_2 y + \alpha_3 z]^2 \\ & + [\alpha_8(t) + \alpha_5 x + \alpha_6 y + \alpha_7 z]^2]^2], \end{aligned} \tag{11}$$

where $\alpha_4(t)$, $\alpha_4(t)$ and $\varrho(t)$ satisfy constraint (10).

The physical structures for $(u^{(III)})$ are shown in figures 6–8 with some 3d plots. When $\gamma(t) = -1$, $\delta(t) = \beta(t) = 1$, the spatial structure called the bright lump wave is seen in figure 6 at $x = -10$; 0; 10. When $\gamma(t) = -t$, $\delta(t) = \beta(t) = t$, interaction behaviors of two bright lump waves are presented in figure 7 at $y = -3$; 0; 3. As the value of y changes, the two bright lump waves move towards each other, and finally merge together. When $\gamma(t) = \delta(t) = \beta(t) = \cos t$, a periodic-shape bright lump wave is found in figure 8 at $z = -10$; 0; 10 and $x = 0$.

3. Interaction solutions between lump and one solitary wave

In order to find the interaction solutions between lump and one solitary wave, we add an exponential function in equation (5) as follows

$$\begin{aligned} \zeta &= x\alpha_1 + y\alpha_2 + z\alpha_3 + \alpha_4(t), \\ \varsigma &= x\alpha_5 + y\alpha_6 + z\alpha_7 + \alpha_8(t), \\ \xi &= \zeta^2 + \varsigma^2 + \alpha_9(t) + \alpha_{14}(t)\exp[\alpha_{13}(t) \\ &\quad + \alpha_{10}x + \alpha_{11}y + \alpha_{12}z], \end{aligned} \tag{12}$$

where α_{10}, α_{11} and α_{12} are unknown constants. $\alpha_{13}(t)$ and $\alpha_{14}(t)$ are unknown real functions. Substituting equation (12)

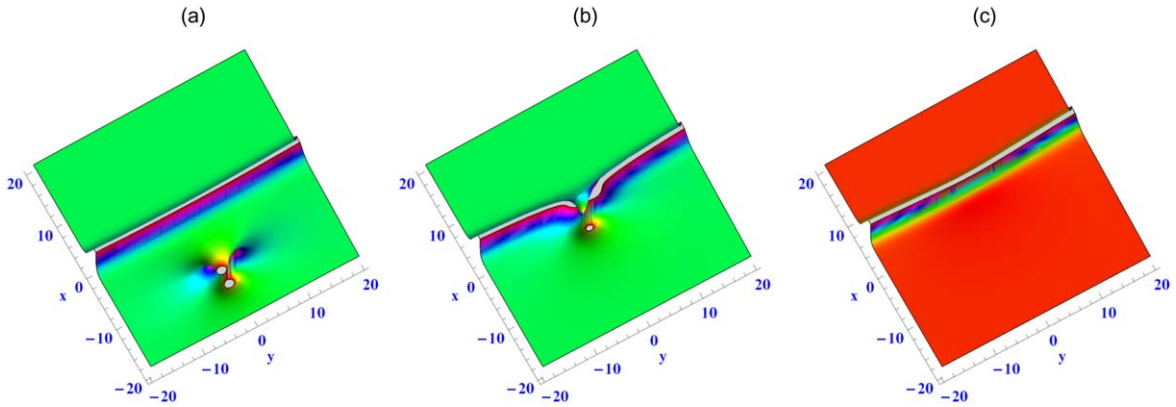

Figure 9. Interaction solution (14) with $\alpha_1 = \eta_8 = \eta_9 = \eta_{10} = \beta(t) = 1$, $\alpha_3 = \gamma(t) = -1$, $\alpha_2 = \alpha_{10} = 2$, $\alpha_5 = -3$, $z = 0$, when $t = -1$ in (a), $t = 0$ in (b) and $t = 1$ in (c).

into equation (4) through Mathematica software, we have

$$\begin{aligned}
&\alpha_6 = -\frac{\alpha_1\alpha_2}{\alpha_5},\ \alpha_7 = -\frac{\alpha_1\alpha_3}{\alpha_5},\\
&\alpha_{11} = \alpha_{12} = 0,\ \alpha_9(t) = \frac{\alpha_1^2 + \alpha_5^2}{\alpha_{10}^2},\\
&\delta(t) = -\frac{3\alpha_5^2\alpha_{10}^2\beta(t) + \alpha_3^2\varrho(t)}{\alpha_2^2},\\
&\alpha_8(t) = \eta_8 - \alpha_5\int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t,\\
&\alpha_4(t) = \eta_9 - \alpha_1\int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t,\\
&\alpha_{13}(t) = \eta_{10} - \alpha_{10}\int[\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t - \ln\alpha_{14}(t),
\end{aligned} \tag{13}$$

with $\alpha_2 \neq 0$, $\alpha_5 \neq 0$ and $\alpha_{10} \neq 0$. Substituting equations (5) and (13) into the transformation $u = 12\,[\ln\xi(x, y, z, t)]_{xx}$, we get

where η_8, η_9 and η_{10} are integral constants. Interaction phenomena between lump and one solitary wave in equation (14) is shown in figures 9 and 10. Obviously, we can see a solitary wave and a lump wave in figure 9(a). In figure 9(b), the solitary and lump wave are slowly approaching at $t = 0$. In figure 9(c), the solitary and lump waves merge together to propagate forward at $t = 1$. Figure 10 displays the effect of variable coefficient $\gamma(t)$ on the interaction phenomena between lump and one solitary wave.

4. Interaction solutions between lump and two solitary waves

In order to derive the interaction solutions between lump and two solitary waves, we add two exponential functions in

$$\begin{aligned}
u^{(IV)} = &\Bigg[12\Bigg[\Bigg[2(\alpha_1^2 + \alpha_5^2) + \alpha_{10}^2\exp\Big[\eta_{10} + \alpha_{10}\Big[x - \int[\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big]\Big]\Bigg]\\
&*\Bigg[\frac{\alpha_1^2 + \alpha_5^2}{\alpha_{10}^2} + \exp\Big[\eta_{10} + \alpha_{10}\Big[x - \int[\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big]\Big]\\
&+ \frac{\Big[\alpha_1(\alpha_2 y + \alpha_3 z) - \alpha_5\Big[\eta_8 + \alpha_5\Big[x - \int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big]\Big]\Big]^2}{\alpha_5^2}\\
&+ \Big[\eta_9 + \alpha_1\Big[x - \int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big] + \alpha_2 y + \alpha_3 z\Big]^2\Bigg]\\
&- \Bigg[2\alpha_1\eta_9 + \alpha_{10}\exp\Big[\eta_{10} + \alpha_{10}\Big[x - \int[\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big]\Big]\\
&+ 2\alpha_5\Big[\eta_8 + \alpha_5\Big[x - \int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big]\Big]\\
&+ 2\alpha_1^2\Big[x - \int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big]\Big]^2\Bigg]\Bigg]\Bigg/\Bigg[\Bigg[\frac{\alpha_1^2 + \alpha_5^2}{\alpha_{10}^2}\\
&+ \exp\Big[\eta_{10} + \alpha_{10}\Big[x - \int[\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big]\Big] + [[\alpha_1(\alpha_2 y + \alpha_3 z)\\
&- \alpha_5\Big[\eta_8 + \alpha_5\Big[x - \int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big]\Big]\Big]^2\Bigg]\Big/\alpha_5^2\\
&+ \Big[\eta_9 + \alpha_1\Big[x - \int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\Big] + \alpha_2 y + \alpha_3 z\Big]^2\Bigg]^2\Bigg],
\end{aligned} \tag{14}$$

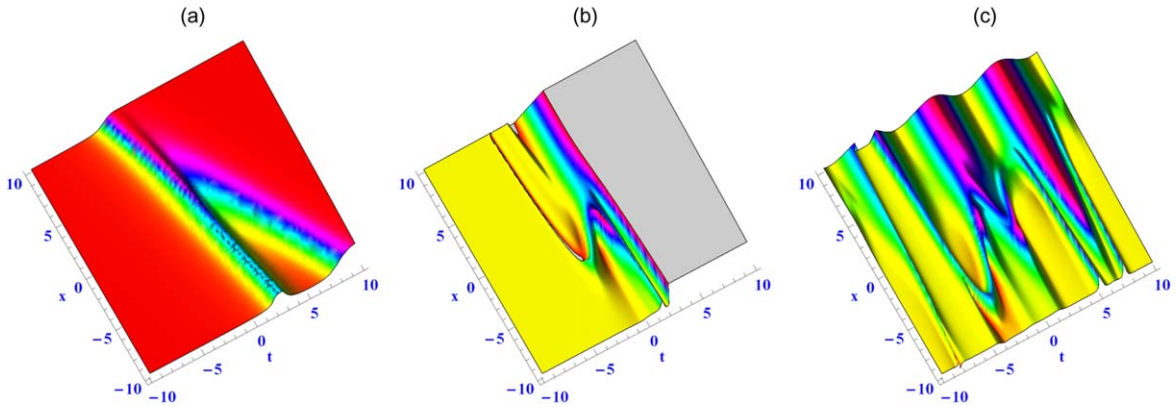

Figure 10. Interaction solution (14) with $\alpha_1 = \eta_8 = \eta_9 = \eta_{10} = \alpha_{10} = \beta(t) = 1$, $\alpha_3 = \alpha_5 = -1$, $\alpha_2 = 2$, $y = z = 0$, when $\gamma(t) = 1$ in (a), $\gamma(t) = t$ in (b) and $\gamma(t) = \cos t$ in (c).

equation (5) as follows

$$\begin{aligned}
&\zeta = x\alpha_1 + y\alpha_2 + z\alpha_3 + \alpha_4(t),\\
&\varsigma = x\alpha_5 + y\alpha_6 + z\alpha_7 + \alpha_8(t),\\
&\xi = \zeta^2 + \varsigma^2 + \alpha_9(t) + \alpha_{14}(t)\exp[\alpha_{13}(t)\\
&\quad + \alpha_{10}x + \alpha_{11}y + \alpha_{12}z]\\
&\quad + \alpha_{15}(t)\exp[-\alpha_{13}(t) - \alpha_{10}x - \alpha_{11}y - \alpha_{12}z], \qquad (15)
\end{aligned}$$

where $\alpha_{15}(t)$ are unknown real functions. Substituting equation (15) into equation (4) through Mathematica software, we obtain

$$\begin{aligned}
&\alpha_6 = -\frac{\alpha_1\alpha_2}{\alpha_5},\ \alpha_7 = -\frac{\alpha_1\alpha_3}{\alpha_5},\\
&\alpha_9(t) = \frac{\alpha_{10}^4\eta_{12} + \alpha_1^4 + 2\alpha_5^2\alpha_1^2 + \alpha_5^4}{(\alpha_1^2 + \alpha_5^2)\alpha_{10}^2},\\
&\varrho(t) = \frac{-\alpha_2^2\delta(t) - 3\alpha_5^2\alpha_{10}^2\beta(t)}{\alpha_3^2},\\
&\alpha_8(t) = \eta_{13} - \alpha_5\int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t,\\
&\alpha_4(t) = \eta_{14} - \alpha_1\int[3\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t,\ \alpha_{11} = \alpha_{12} = 0,\\
&\alpha_{13}(t) = \eta_{15} - \alpha_{10}\int[\alpha_{10}^2\beta(t) + \gamma(t)]\,\mathrm{d}t\\
&\qquad - \ln\alpha_{14}(t),\ \alpha_{15}(t) = \frac{\eta_{12}}{\alpha_{14}(t)},
\end{aligned} \qquad (16)$$

with $\alpha_3 \neq 0$, $\alpha_5 \neq 0$, $\alpha_{14}(t) \neq 0$, $\alpha_1^2 + \alpha_5^2 \neq 0$ and $\alpha_{10} \neq 0$. Substituting equations (5) and (16) into the transformation $u = 12\,[\ln\xi(x, y, z, t)]_{xx}$, we get

$$\begin{aligned}
u^{(V)} = {}& 12[[2\alpha_1^2 + 2\alpha_5^2 + \alpha_{10}^2\alpha_{14}(t)\exp[\alpha_{13}(t) + \alpha_{10}x]\\
&+ \alpha_{10}^2\alpha_{15}(t)\exp[-\alpha_{10}x\\
&- \alpha_{13}(t)]]/[\alpha_9(t) + \alpha_{14}(t)\exp[\alpha_{13}(t) + \alpha_{10}x]\\
&+ \alpha_{15}(t)\exp[-\alpha_{10}x\\
&- \alpha_{13}(t)] + [\alpha_4(t) + \alpha_1x + \alpha_2y + \alpha_3z]^2\\
&+ (\alpha_8(t) + \alpha_5x + \alpha_6y + \alpha_7z)^2]\\
&- [[\alpha_{10}\alpha_{14}(t)\exp[\alpha_{13}(t) + \alpha_{10}x]\\
&- \alpha_{10}\alpha_{15}(t)\exp[-\alpha_{10}x - \alpha_{13}(t)] + 2\alpha_1\\
&* [\alpha_4(t) + \alpha_1x + \alpha_2y + \alpha_3z]\\
&+ 2\alpha_5(\alpha_8(t) + \alpha_5x + \alpha_6y + \alpha_7z)]^2]/[[\alpha_9(t)\\
&+ \alpha_{14}(t)\exp[\alpha_{13}(t) + \alpha_{10}x]\\
&+ \alpha_{15}(t)\exp[-\alpha_{10}x - \alpha_{13}(t)] + [\alpha_4(t)\\
&+ \alpha_1x + \alpha_2y + \alpha_3z]^2\\
&+ [\alpha_8(t) + \alpha_5x + \alpha_6y + \alpha_7z]^2]^2]], \qquad (17)
\end{aligned}$$

where η_{12}, η_{13}, η_{14} and η_{15} are integral constants. Interaction phenomena between lump and two solitary waves in equation (16) is shown in figure 11. Two solitary waves can be found in figure 11(a). A lump wave appears in one of two solitary waves in figure 11(b). In figures 11(c) and 11(d), the lump wave slowly shifts to another solitary wave, until it vanishes in figure 11(e).

5. Conclusion

In this paper, based on the Hirota's bilinear form and Mathematica software [25–35], the lump and interaction solutions between lump and solitary wave of a generalized (3 + 1)-dimensional variable-coefficient nonlinear-wave equation in liquid with gas bubbles are studied. Their physical structures are described in some 3D plots. A periodic-shape rational solution is listed in figure 1(a). A parabolic-shape rational solution is presented in figure 1(b). A cubic-shape rational solution is shown in figure 1(c). In lump solutions ($u^{(II)}$), the spatial structure called the bright lump wave is seen in figure 2; the spatial structure called the bright-dark lump wave is shown in figure 3. Interaction behaviors of two bright-dark lump waves are presented in figure 4. A periodic-shape bright lump wave is found in figure 5. In lump solutions ($u^{(III)}$), the spatial structure called the bright lump wave is seen in figure 6. Interaction behaviors of two bright lump waves are presented in figure 7. A periodic-shape bright lump wave is found in figure 8. Figures 9 and 10 display the

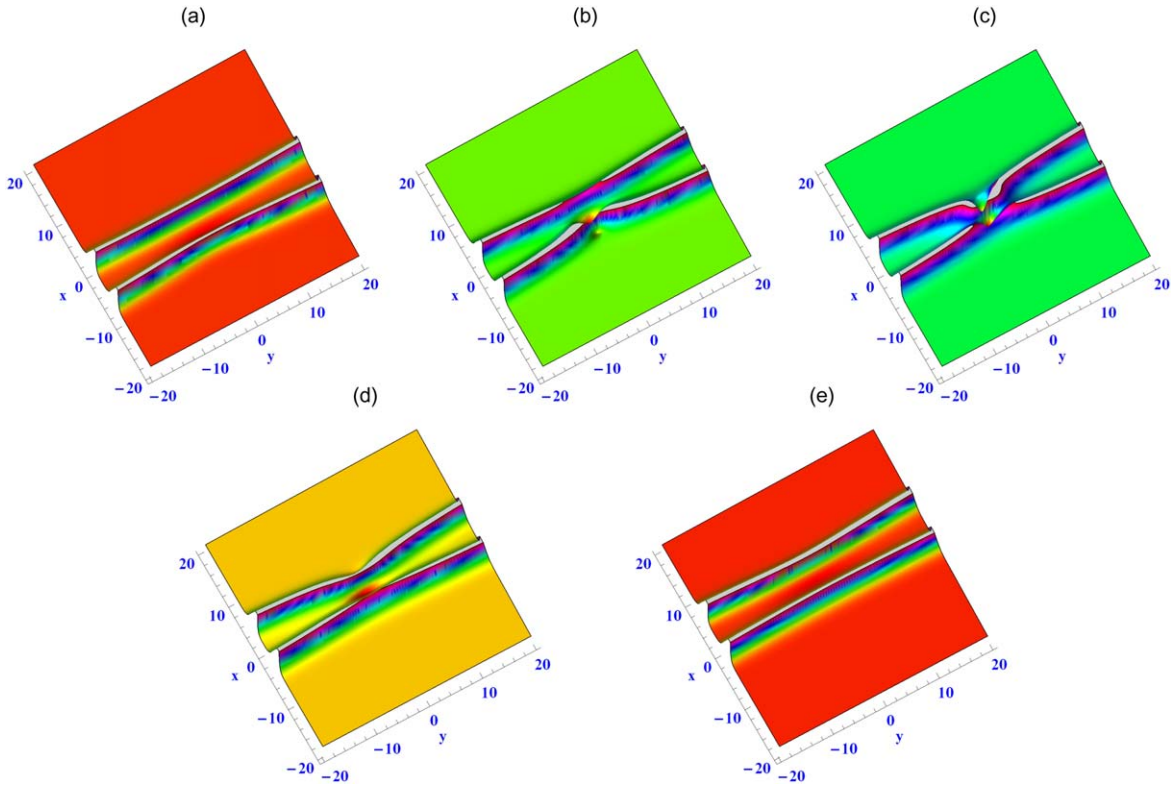

Figure 11. Interaction solution (16) with $\alpha_3 = \gamma(t) = -1$, $\alpha_2 = \alpha_{10} = 2$, $\alpha_5 = -3$, $\alpha_1 = \eta_{12} = \eta_{13} = \eta_{14} = \eta_{15} = \beta(t) = 1$, $z = 0$, when $t = -1$ in (a), $t = -0.3$ in (b), t = 0 in (c), $t = 0.3$ in (d), $t = 1$ in (e).

interaction phenomena between lump and one solitary wave. Figure 11 shows the interaction phenomena between lump and two solitary waves.

References

[1] Deng G F and Gao Y T 2017 *Eur. Phys. J. Plus.* **132** 255
[2] Osman M S 2017 *Nonlinear Dyn.* **87** 1209
[3] He J S, Wang L H, Li L J, Porsezian K and Erdlyi R 2014 *Phys. Rev.* E **89** 062917
[4] Tan W and Dai Z D 2017 *Nonlinear Dyn.* **89** 2723
[5] Osman M S, Machado J A T and Baleanu D 2018 *Opt. Quant. Electron.* **50** 1
[6] Yang J Y, Ma W X and Qin Z 2017 *Anal. Math. Phys.* **8** 427
[7] Li Y Z and Liu J G 2018 *Pramana* **90** 71
[8] Alquran M, Jaradat H M and Syam M I 2017 *Nonlinear Dyn.* **90** 2525
[9] Hu C C, Tian B, Wu X Y, Yuan Y Q and Du Z 2018 *Eur. Phys. J. Plus.* **133** 40
[10] Sun H Q and Chen A H 2016 *Appl. Math. Lett.* **68** 55
[11] Kudryashov N A and Sinelshchikov D I 2012 *Phys. Scr.* **85** 025402
[12] Lan Z Z, Hu W Q and Guo B L 2019 *Appl. Math. Model.* **73** 695
[13] He J S and Li Y S 2011 *Stud. Appl. Lett.* **126** 1–15
[14] Lan Z Z 2019 *Appl. Math. Lett.* **94** 126
[15] He J S, Charalampidis E G, Kevrekidis P G and Frantzeskakis D J 2014 *Phys. Lett.* A **378** 577
[16] Tu J M, Tian S F, Xu M J, Song X Q and Zhang T T 2016 *Nonlinear Dyn.* **83** 1199
[17] Huang W H 2006 *Chaos Soliton. Fract.* **29** 365
[18] Rao J G, Mihalache D, Cheng Y and He J S 2019 *Phys. Lett.* A **383** 1138
[19] Zhou Q, Zhu Q, Liu Y, Yao P, Bhrawy A H, Moraru L and Biswas A 2014 *Optoelectron Adv. Mat.* **8** 837
[20] Ma W X and Zhou Y 2018 *J. Diff Equ.* **264** 2633
[21] Gao X Y 2017 *Appl. Math. Lett.* **73** 143
[22] Gao X Y 2019 *Appl. Math. Lett.* **91** 165
[23] Liu W H, Zhang Y F and Shi D D 2019 *Phys. Lett.* A **383** 97
[24] Qawasmeh A and Alquran M 2014 *Appl. Math. Sci.* **8** 5985
[25] Biswas A, Sonmezoglu A, Ekici M, Mirzazadeh M, Zhou Q, Alshomrani A S, Moshokoa S P and Belic M 2018 *Optik* **161** 301
[26] Huang L L and Chen Y 2017 *Commun. Theor. Phys.* **67** 473
[27] Xie X Y and Meng G Q 2018 *Nonlinear Dyn.* **93** 779
[28] Lan Z Z and Su J J 2019 *Nonlinear Dyn.* **96** 2535
[29] Chen R P and Dai C Q 2017 *Nonlinear Dyn.* **90** 1563
[30] Chen M D, Li X, Wang Y and Li B 2017 *Commun. Theor. Phys.* **67** 595
[31] Huang Q M and Gao Y T 2017 *Nonlinear Dyn.* **89** 2855
[32] Ma W X 2018 *Comput. Appl. Math.* **37** 6359
[33] Ma W X 2018 *J. Geom. Phys.* **133** 10
[34] Li Y Z and Liu J G 2018 *Nonlinear Dyn.* **91** 497
[35] Lan Z Z 2019 *Appl. Math. Lett.* **98** 128