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Analysis of overload-based cascading failure in multilayer spatial networks*

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Many complex networks in real life are embedded in space and most infrastructure networks are interdependent, such as the power system and the transport network. In this paper, we construct two cascading failure models on the multilayer spatial network. In our research, the distance l between nodes within the layer obeys the exponential distribution $P(l) \sim \exp(-l/\zeta)$, and the length r of dependency link between layers is defined according to node position. An entropy approach is applied to analyze the spatial network structure and reflect the difference degree between nodes. Two metrics, namely dynamic network size and dynamic network entropy, are proposed to evaluate the spatial network robustness and stability. During the cascading failure process, the spatial network evolution is analyzed, and the numbers of failure nodes caused by different reasons are also counted, respectively. Besides, we discuss the factors affecting network robustness. Simulations demonstrate that the larger the values of average degree $\langle k \rangle$, the stronger the network robustness. As the length r decreases, the network performs better. When the probability p is small, as ζ decreases, the network robustness becomes more reliable. When p is large, the network robustness manifests better performance as ζ increases. These results provide insight into enhancing the robustness, maintaining the stability, and adjusting the difference degree between nodes of the embedded spatiality systems.

Keywords: cascading failure, multilayer network, load distribution, spatial network, entropy

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1. Introduction

Complex network as a useful tool has been widely used to predict future trends and analyze the properties of various systems, including power systems, transportation networks, communication networks, and supply chain networks.[1–5] More and more scholars from different fields have been devoting to the study of complex networks, involving robustness, vulnerability, controllability, cascading failure, and the evaluation of node importance.[6–10]

Cascading failure refers to the phenomenon that some invalid nodes will cause other nodes to fail, and as a result, the network collapses. Cascading failures widely exist in real life, novel pneumonia (COVID-19), for instance. If one person is infected, he will spread the disease to others and then more and more people get sick. Watts[11] analyzed a binary-decision model and found that cascade propagation is influenced by network density. When the network is dense, the effect of the stability of the individual node on the cascade propagation is the largest; otherwise, global connectivity is the key factor. Buldyrev[12] built a framework for investigating the multilayer network robustness against cascading failure. They showed that the broader the degree distribution, the more vulnerable of the multilayer network to random failure. Lee[13] constructed a cascading failure model on the multilayer network. They obtained that the cascade can be facilitated or inhibited depending on how nodes respond to their connected nodes of other layers. Gao[14] investigated the robustness of a network that consists of n interdependent networks. They showed that the phase transition of the cascading model of the multilayer network is different from the single-layer network. Zhou[15] considered the possibility of restoring a network in which cascading failure would happen. Critical slowing down indicators were used to predict the network collapse and five node addition rules were applied to prevent network paralysis. Jin[16] analyzed the cascading failure in multilayer networks with dependence groups. In a further step, the author obtained the theoretical values of the network and discussed the factors affecting network robustness. More extensive works on the cascading failures of multiplex networks, we refer the readers to Refs. [17,18]. Overload is one of the causes regarding the cascading failure in the network, which means the load of an ineffective node will be distributed to its neighbors, leading the neighbors to be invalid under their limited load capacities. Overload can be used to describe many phenomena, like the orders of products in the supply chain systems. If a com-

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pany closes down, its suppliers with a lot of materials may also close. Zhou[19] proposed a new model to analyze mixed cascading failure in complex systems, taking into account the impacts of network load and dependency. Tang[20] constructed a cascading failure model that focused on load propagation and analyzed the robustness of the supply chain network. Wang[21] studied the cascading failure process under servers overload, which happened in a modern cloud datacenter, and put forward a cascading failure resilience system (CFRS) to handle this problem.

The spatial network is composed of nodes with the positional relationship, such as houses in the community network. At present, the spatial network exists in the research fields of information communication, ecology, sociology, and plays a substantial role in engineering applications.[22–24] Li[25] investigated the cascading failures in a system that was composed of two interdependent square lattice networks. Shekhtman[26] considered the robustness of embedded spatiality networks which could explain the sudden failure phenomena. Danziger[27] constructed a new model and modeled the influence of spatiality on the robustness. With the same strength embedment, the behaviors of single-layer network and multiplex network were different. Chen[28] extended the model of Ref. [27] to a general model in which whether the nodes exist or not depends on the probability λ. The robustness of the extension model was discussed and an algorithm was put forward to change the link distance. From the perspective of network robustness, Shekhtman[29] provided an overview of spatial networks.

In this paper, we will construct two models on multilayer spatial networks against cascading failure, considering the combined influences of overload, percolation, and connection between layers. Different from previous studies, the distance between nodes within the layer obeys the exponential distribution, and the connection between layers generalizes the one-to-one correspondence based on node position. A new entropy method is proposed to investigate the spatial network structure and reflect the difference degree between nodes. Two indicators, namely dynamic network size and dynamic network entropy, are developed to evaluate the network robustness and stability. Moreover, the evolution of spatial networks during the cascading failure process is analyzed and the number of failure nodes caused by different effects is also counted respectively. Furthermore, we analyze and compare some factors affecting the robustness of multilayer spatial networks. The practical significance of the conclusions is also discussed.

This paper is organized as follows. In Section 2, we develop two cascading failure models in the multilayer spatial network. In Section 3, the spatial network is analyzed by entropy. In Section 4, utilizing dynamic network size and dynamic network entropy, the spatial network evolution is analyzed during the cascading failure process, and the affecting factors of the network robustness are discussed and compared as well. In Section 5, we give the conclusion of this paper on the whole.

2. Spatial networks cascading model

In this section, we will construct two cascading failure models on multilayer spatial networks, taking into consideration the combined effects of overload, the connection between layers, and percolation. These models can be applied to describe many systems embedded in space, the server system in the engine room, for example. If a server breaks down, its workload will transfer to other servers that may become ineffective due to overload, and in the end, the system collapses.

2.1. Multilayer spatial network

The spatial network will be firstly introduced. In a single-layer network, the number of nodes is N and the average degree is $\langle k\rangle$. Therefore, the number of links is $m=\langle k\rangle N/2$. The connection between nodes within the layer discovered in the previous papers is shown in Fig. 1(a). Obviously, the spatial effect is the largest when nodes tend to connect to their nearby neighbors. However, there is not always such a tendency, with the motion trajectories of atoms in crystalline structure as an exception. Danziger[27] assumed the distance l between nodes obeys the exponential distribution $P(l)\sim\exp(-l/\zeta)$, where ζ is the spatial parameter adjusting the spatial effect. The exponential distribution $P(l)$ for different values of ζ is shown in Fig. 1(b). When $\zeta\to 0$, nodes will prefer to connect to their nearby neighbors and the spatial effect will also become larger. When $\zeta\to+\infty$, $P(l_1)\approx P(l_2)$ for anytime when $l_1\neq l_2$. That means the distance l between nodes is arbitrary and the spatial effect is not dominant. Figures 1(c) and 1(d) display the spatial networks when $\zeta=0.1$ and $\zeta=10$ respectively.

Secondly, the generation process of the spatial network is also studied. In a 2D lattice, there are $N=L^2$ nodes with integer coordinate $(x,y)\in[0,L)\times[0,L)$. The links of the network are constructed as below. Randomly choose a node i and distance l with probability $P(l)$, and then connect to node j satisfying the formula of

$$\left|\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-l\right|=\min\left\{\left|\sqrt{(x_i-x_z)^2+(y_i-y_z)^2}-l\right|,z=1,2,\ldots,N\right\},\tag{1}$$

where $1\leq j\leq N$, $(x_z,y_z)\in[0,L)\times[0,L)$ is the coordinate of node z, $z=1,2,\ldots,N$. The reason why the minimum is used in Eq. (1) is that the distance between nodes is not exactly equal to l in the lattice. Also, there may be more than one node satisfying Eq. (1). Here we only randomly select one of them. The link construction process will continue until the number of links is m.

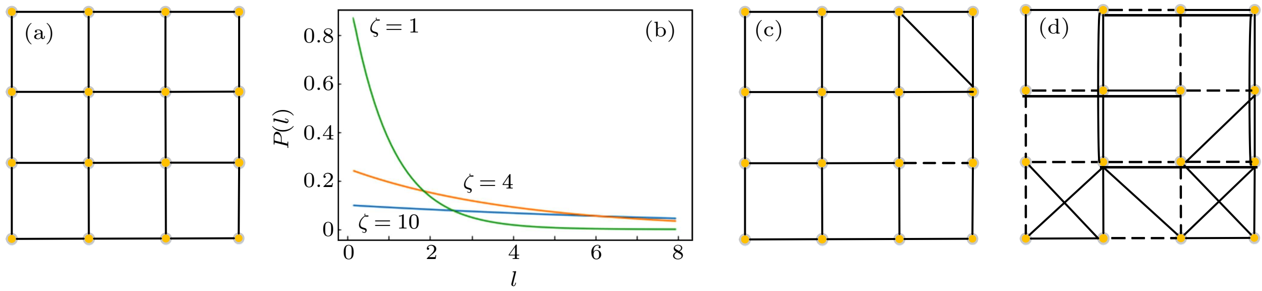

Fig. 1. The connection of nodes within the layer in small networks. The solid line represents the actual connection while the dotted line denotes the lattice line. The parameters are $\langle k\rangle=3$, $N=16$. (a) The connection between nodes defined in the previous papers. (b) The exponential distribution of l: $P(l)\sim\exp(-l/\zeta)$, $\zeta=1, 4, 10$. (c) The distance l between nodes obeys the exponential distribution: $P(l)\sim\exp(-l/0.1)$. (d) The distance l between nodes obeys the exponential distribution: $P(l)\sim\exp(-l/10)$.

Thirdly, we define the connection between layers. The number of layers is M in the network. Take the case of $M=2$, the first layer is denoted as A and the second layer is marked as B. If node A_i of layer A connects to node B_j of layer B, their coordinates are expected to meet the following condition

$$|x_{A_i}-x_{B_j}|\leq r,\ |y_{A_i}-y_{B_j}|\leq r, \tag{2}$$

where $r\geq 0$, $1\leq i\leq N_A$, $1\leq j\leq N_B$, $(x_{A_i},\ y_{A_i})$ and $(x_{B_j},\ y_{B_j})$ are the coordinates of nodes A_i and B_j individually, N_A and N_B are the number of nodes of layer A and layer B respectively. Figure 2(a) shows the connection between layers for different values of r. It is worth noting that a node in one layer may connect to more than one node in other layers when $r>0$, as shown in Fig. 2(b).

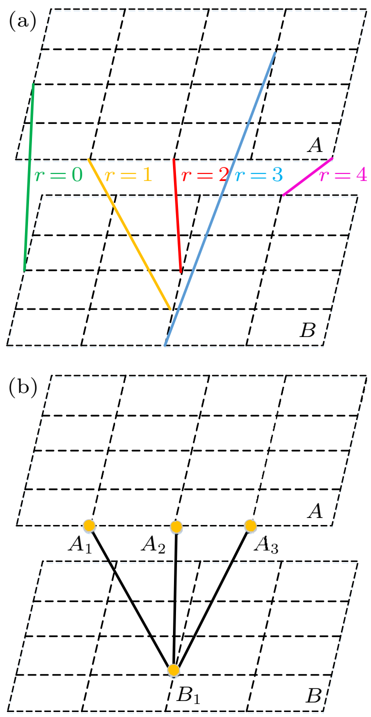

Fig. 2. (a) The connections between layers in a two-layer network when $r=0$, 1, 2, 3, 4 separately. (b) A node in one layer connects to more than one node in other layers. When $r=1$, node B_1 is connected to the nodes of A_1, A_2, A_3.

Remark 1 If $r=0$, $M=2$, and the number of nodes and the average degree in each layer are the same, our multilayer spatial network can degenerate to the network of Ref. [27].

Consequently, it can be summarized from our multilayer spatial network that the distance between nodes within the layer obeys the exponential distribution and the connection between layers satisfies Eq. (2).

2.2. Load distribution process

We apply the degree centrality to represent the initial load of each node, namely,

$$L_i(0)=\frac{k_i}{N-1},\quad i=1,2,\ldots,N, \tag{3}$$

where k_i is the degree of node i. The largest load that can be handled on node i is defined as load capacity C_i, which is related to the initial load $L_i(0)$. In a linear system, C_i (Ref. [30]) is proportional to $L_i(0)$, *i.e.*,

$$C_i=(1+a)L_i(0),\quad i=1,2,\ldots,N, \tag{4}$$

where $a\geq 0$ is the tolerance parameter. However, the real situation in our life is always described as a nonlinear system. Therefore, we adopt a new load capacity[31] defined as

$$C_i=L_i(0)+aL_i^b(0),\quad i=1,2,\ldots,N, \tag{5}$$

where $a,b\geq 0$ are the tolerance parameters that can adjust the load capacity. Besides, equation (5) will transform to Eq. (4) when $b=1$.

It is defined that node i will become invalid when i overloads, i.e., $L_i(t)\geq c\times C_i$, where $c\geq 0$. If a node becomes ineffective, the load of it will be distributed to its valid neighbors. The load distribution strategy is formulated as

$$\Delta L_{i\to j}=\frac{k_j}{\sum_{j\in\Gamma(i)}k_j}\times L_i(t-1),$$

where $\Gamma(i)$ is the valid neighbors of node i. Hence, the load of node j becomes

$$L_j(t)=L_j(t-1)+\Delta L_{i\to j}=L_j(t-1)+\frac{k_jL_i(t-1)}{\sum_{j\in\Gamma(i)}k_j}.$$

Figure 3 gives a typical example of the load distribution process.

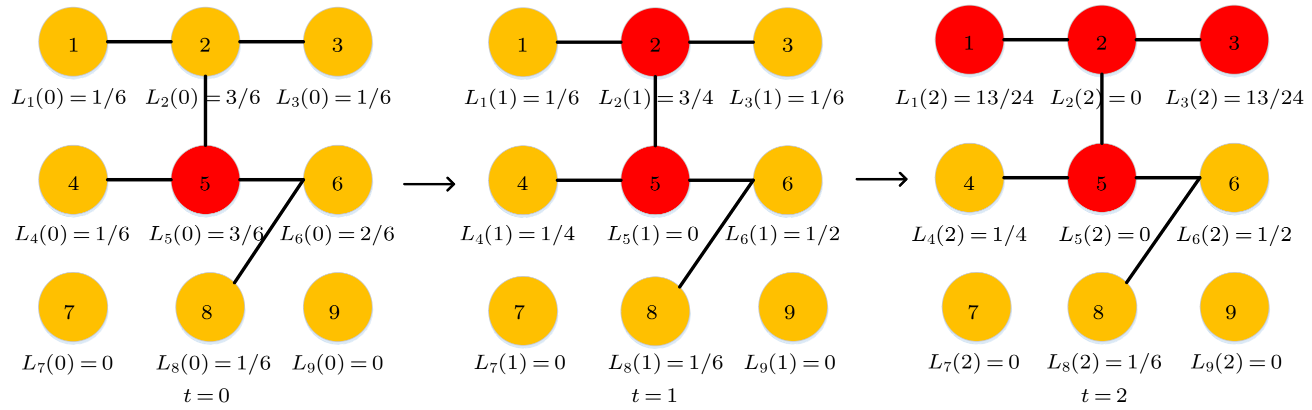

Fig. 3. Load distribution process. The parameters are $a=1/3$, $b=1/2$, $c=1$, $N=9$. The valid nodes are marked in yellow and the disabled ones are depicted in red. When $t=0$, the initial loads of nodes are $L_1(0)=1/6$, $L_2(0)=3/6$, $L_3(0)=1/6$, $L_4(0)=1/6$, $L_5(0)=3/6$, $L_6(0)=2/6$, $L_7(0)=0$, $L_8(0)=1/6$, $L_9(0)=0$. Since node 5 is invalid, its load will distribute to nodes 2, 4, 6 according to the load distribution strategy. When $t=1$, the loads of nodes 2, 4, 6 become $L_2(1)=3/4$, $L_4(1)=1/4$, $L_6(1)=1/2$. Owing to $L_2(1)>c\times C_2$, node 2 becomes ineffective and will distribute its load to nodes 1, 3. Other nodes will not change their state due to $L_i(1)<c\times C_i$, $i=1,3,4,6,7,8,9$. When $t=2$, nodes 1, 3 become invalid because their loads exceed their capacities ($L_1(2)>c\times C_1$, $L_3(2)>c\times C_3$).

2.3. Cascading failure model

In this subsection, we will construct two cascading failure models on spatial networks separately. The cascading failure process of each model is divided into five steps. Two examples are illustrated in Fig. 4.

To begin with, we introduce the first cascading failure model.

Step 1 The random failure process. In the initial state (*i.e.*, $t=0$), the load of each node is defined by Eq. (3). By randomly selecting and initially removing some nodes with probability p, there are $N_x\times p$ nodes will become invalid in layer x, $x=1,2,\ldots,M$. N_x is the number of nodes of layer x. Consequently, the total number of invalid nodes is $\sum_{x=1}^{M}N_x\times p$.

Step 2 The effect of overload within the layer. Based on the load distribution strategy, the loads of the invalid nodes will be distributed to their valid neighbors. Therefore, more nodes will receive extra loads and become invalid due to overload. This process continues until there is no any further node overload.

Step 3 The effect of the connection between layers. After the above steps, the state of each node is either valid or invalid. If a node in one layer is valid, the invalid nodes connected to it in other layers will get recovered. For example, if node A_1 is invalid and node B_1 is valid, as shown in Fig. 2(b), node A_1 will become valid. We name this effect of the first model as effect 1.

Step 4 The percolation process within the layer. With the steps mentioned above, we can obtain the largest connected component of each layer. If a node is not in the largest connected component, it will become invalid.

Step 5 Turn to step 2 until the state of all nodes is stable.

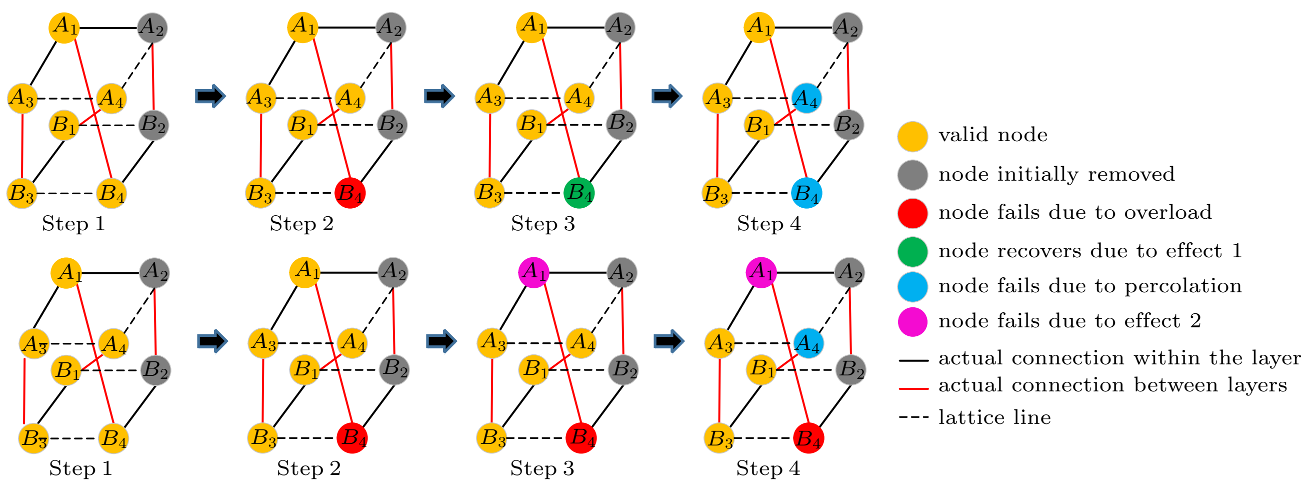

Fig. 4. Cascading failure processes of two models in double-layer spatial networks in the first iteration. The parameters are $a=0.25$, $b=2$, $c=1.5$, $r=1$, $\zeta=0.2$, $p=0.25$, $N_1=N_2=4$, $\langle k\rangle_1=\langle k\rangle_2=1$. N_x and $\langle k\rangle_x$ are the number of nodes and average degree of layer x individually, $x=1,2$. In layer x, the number of nodes initially removed is $N_x\times p=1$. (a) The cascading failure process of the first model. In Step 1, nodes A_2 and B_2 are initially removed. In Step 2, node B_4 fails due to overload ($13/24=c\times C_{B_4}<L_{B_4}(1)=2/3$). In Step 3 (effect 1), node B_4 recovers because A_1 is valid. In Step 4, A_4 and B_4 become invalid owing to the percolation process. (b) The cascading failure process of the second model. The behaviors of nodes are consistent with (a) in Step 1 and Step 2. In Step 3 (effect 2), node A_1 becomes invalid because B_4 is invalid. In Step 4, node A_4 fails owing to the percolation process.

Moreover, we present the second cascading failure model. The difference between these two models is the Step 3. In the second cascading failure model, we define the Step 3 as follows. If a node in one layer is invalid, the valid nodes connected to it in other layers will become invalid. For example, if the state of nodes A_1 and B_1 is invalid and valid separately, as shown in Fig. 2(b), node B_1 will fail. We name this effect of the second model as effect 2. The other steps of the second model are consistent with the first cascading model.

3. Analysis of the spatial network

In this portion, entropy is utilized to investigate the spatial network structure and reflect the difference degree between nodes. Simulations found that the spatial network entropy is influenced by the spatial argument ζ, node number N as well as average degree $\langle k\rangle$. As ζ increases, the spatial network entropy will become smaller. With the increments of N and $\langle k\rangle$, the spatial network entropy will become larger.

In physics, entropy is used to describe the disorder phenomenon of the system. The more chaotic the system, the larger the entropy. Recently, entropy has demonstrated significant applications in cybernetics, number theory, probability theory and has become a substantial tool to solve numerous problems. In complex networks, the definition of the network entropy[32] is

$$E = -\sum_{i=1}^{N} I_i \ln I_i, \tag{6}$$

where $I_i = k_i / \sum_{i=1}^{N} k_i$ is the importance degree of node i, k_i is the degree of node i, $i = 1, 2, \ldots, N$. It is worth noting that $E_{\max} = \ln N$ when $I_i = 1/N$, *i.e.*, the regular network entropy is the largest. $E_{\min} = \ln 4(N-1)/2$ when $k_1 = N-1$, $k_2 = k_3 = \cdots = k_N = 1$, *i.e.*, the star network entropy is the smallest.

The changes of network entropy can be applied to reflect the network structure stability. The smaller the changes of network entropy, the more stable the network structure. In addition, network entropy is a measurement of network disorder that can indicate the difference between nodes. The greater the network entropy, the more disordered the network and the smaller the difference between nodes. For instance, in the regular network,[33] the nodes have the same degrees. Consequently, the network is disordered and there is no difference between nodes. In the ER network,[34,35] the connections between nodes are random. Hence, the nodes of the ER network have a slight difference from each other. In the BA network,[36] the node degree obeys the power-law distribution. That means some nodes have larger degrees and other nodes have smaller ones. Therefore, the network is ordered and there is a great difference between nodes. The elementary statistics of characteristics of the regular network, ER network, and BA network are displayed in Table 1 in which the values of entropy are accurate to four decimal places.

The spatial network entropy is influenced by the spatial argument ζ. As shown in Fig. 5(a), with the increase of ζ, the spatial network entropy will decrease gradually. That means the spatial network will be more ordered and the difference between nodes will become larger and larger. The reason is that when $\zeta \to +\infty$, the distance l between nodes will be arbitrary which is analogue to the ER network. When $\zeta \to 0$, nodes prefer to connect to their nearby neighbors. As shown in Fig. 1(c), the node degree is almost equal in the spatial network which is similar to the regular network. From Fig. 5(a) and Table 1, the difference degree between nodes is described as below:

$$\begin{cases} D_{\mathrm{ER}} < D_{\mathrm{SN}} < D_{\mathrm{RN}}, & 0 < \zeta < 2, \\ D_{\mathrm{BA}} < D_{\mathrm{SN}} < D_{\mathrm{ER}}, & 2 < \zeta, \end{cases} \tag{7}$$

where D_{SN}, D_{RN}, D_{ER}, D_{BA} are the difference degrees between nodes in the spatial network, regular network, ER network, and BA network, correspondingly. Remarkably, as ζ increases, the spatial network entropy will converge to a constant ζ_c. That indicates the network structure is unstable when $\zeta < \zeta_c$ and the network structure is scarcely influenced by ζ when $\zeta \geq \zeta_c$.

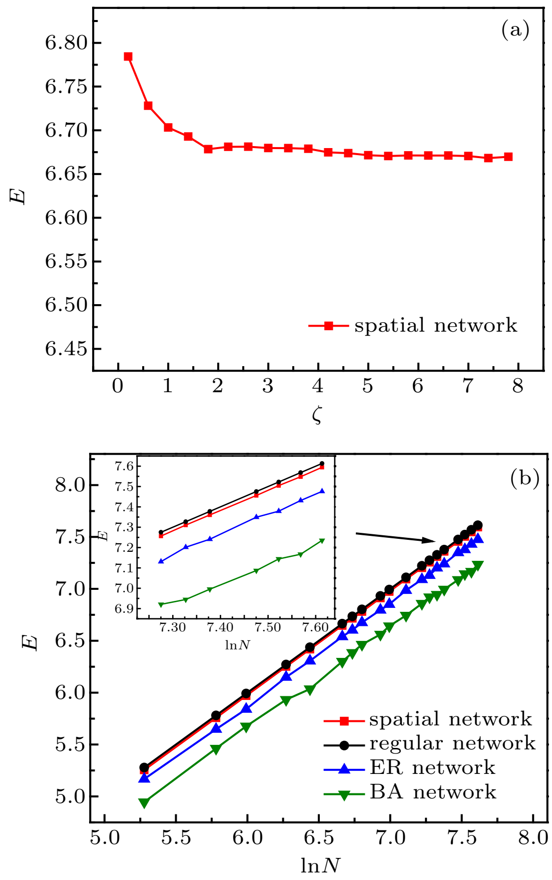

Fig. 5. (a) Entropy E as a function of ζ on the spatial network. The parameters are $N = 900$, $\langle k\rangle = 4$. (b) Entropy E as a function of $\ln N$ on the spatial network, regular network, ER network, and BA network separately. The parameters are $\zeta = 0.2$, $\langle k\rangle = 4$. Each point on the above figures is the average value of 20 experiments.

What is more, the spatial network entropy is affected by the number of nodes, namely N. As shown in Fig. 5(b), with the increase of N, the spatial network entropy will increase. That means the network will be more disordered and the difference degree between nodes will become smaller bit by bit. The reason lies that the importance degree I_i of node i is reducing as N increases. For example, in the regular network, the importance degree of node i is $I_i = 1/N$. According to Eq. (6), with the increase of N, I_i will decrease and E will grow.

Furthermore, the spatial network entropy is subject to the average degree $\langle k\rangle$. As shown in Table 2, these quantities are accurate to four decimal places and each point is the average value of 20 experiments. The higher the average degree $\langle k\rangle$, the larger the spatial network entropy. That means as $\langle k\rangle$ increases, the spatial network will be more disordered and the difference degree between nodes is expected to decrease. It is caused by that the numerator and denominator of $I_i = k_i/\sum_{i=1}^{N} k_i = k_i/\langle k\rangle N$ will surely be larger as $\langle k\rangle$ increases. Since the increment of molecular is far less than the change of denominator, I_i will become smaller and E will become larger. Besides, the regular network entropy is hardly altered by the increase of $\langle k\rangle$. The reason is that $\langle k\rangle$ has nothing to do with the importance degree $I_i = 1/N$ of node i.

Table 1. The elementary statistics of characteristics of the three networks.

	N	m	$\langle k\rangle$	$k_{\max}$	$k_{\min}$	E
Regular network	900	1800	4	4	4	6.8024
ER network	900	1806	4	11	0	6.6767
BA network	900	1796	4	87	2	6.4589

Table 2. The entropy of different networks with the average degree $\langle k\rangle = 2, 4, 6, 8$ respectively.

	$\langle k\rangle = 2$	$\langle k\rangle = 4$	$\langle k\rangle = 6$	$\langle k\rangle = 8$
Spatial network	6.6510	6.7817	6.7823	6.7870
Regular network	6.8023	6.8023	6.8023	6.8023
ER network	6.6658	6.6745	6.7097	6.7353
BA network	6.4177	6.4539	6.4780	6.5110

4. The simulation analysis and discussion

4.1. Analysis of the spatial network evolution

In this subsection, we will analyze the spatial network evolution during the cascading failure process. Two metrics, namely dynamic network size $S(t)$ and dynamic network entropy $E(t)$, are proposed to describe the network stability. The number of invalid nodes triggered by different reasons is counted and analyzed respectively. From the perspective of the description of network evolution, it can be seen from simulations that the changes of $S(t)$ and $E(t)$ are consistent under effect 1 or effect 2 while $S(t)$ performs better than $E(t)$.

Firstly, we present the definitions of dynamic network size $S(t)$ and dynamic network entropy $E(t)$. In terms of network connectivity, the dynamic network size is defined by

$$S(t) = \frac{S_1(t) + S_2(t) + \cdots + S_M(t)}{N_1 + N_2 + \cdots + N_M}, \tag{8}$$

where $S_x(t)$ is the size of the largest connected component of layer x at t moment, N_x is the number of nodes of layer x, $x = 1, 2, \ldots, M$. To verify the effectiveness of the proposed method (8), we define the dynamic network entropy by

$$E(t) = \frac{E_1(t) + E_2(t) + \cdots + E_M(t)}{M}, \tag{9}$$

where at t moment, $E_x(t) = -\sum_{j=1}^{N_x(t)} I_j(t) \ln I_j(t)$ is the dynamic entropy of layer x, $N_x(t)$ is number of valid nodes of layer x, $x = 1, 2, \ldots, M$, $I_j(t) = k_j(t)/\sum_{j=1}^{N_x(t)} k_j(t)$ is the importance degree of node j, $k_j(t)$ is the degree of node j.

Secondly, we analyze the spatial network evolution during the cascading failure process when $p = 0.2$ and $p = 0.7$ separately. As shown in Figs. 6(a) and 6(b), when $p = 0.2$, the network becomes stable after the first iteration (*i.e.*, $t = 1$) and most nodes are active under effect 1. The reason is that there are few nodes initially removed, which will cause fewer nodes invalid owing to overload. Therefore, the probability is large that a failure node connects to effective nodes of other layers. Hence, the probability of node recovery increases. As shown in Fig. 6(c), as a result of effect 1, the number of invalid nodes caused by initial deletion of layer x is not equal to $N_x p$ when the network is stable. Different from the effect 1, figures 6(d) and 6(e) present that the number of iterations increases under effect 2. Figure 6(f) indicates that most nodes are invalid and the effect 2 and percolation process are the primary reasons causing nodes to be invalid.

When $p = 0.7$, as shown in Figs. 7(a) and 7(b), the network becomes stable after the first iteration under effect 1. The number of invalid nodes resulted from the percolation process increases and most nodes shown in Fig. 7(c) are invalid. The reason is that most nodes are initially removed which will cause links ineffective. Therefore, more nodes will not connect to the largest component and will become invalid. Under effect 2, the number of iterations increases shown in Figs. 7(d) and 7(e), and most nodes become invalid shown in Fig. 7(f). The reason behind it is the number of initially removed nodes is large. Therefore, the probability that an ineffective node connects to an invalid one increases within the layer or between layers. Hence, the number of disabled nodes due to overload or effect 2 is relatively few. Since most nodes are invalid after the processes of load distribution within the layer and connection between layers, there are few remaining valid nodes. Consequently, fewer nodes will become invalid due to the percolation process.

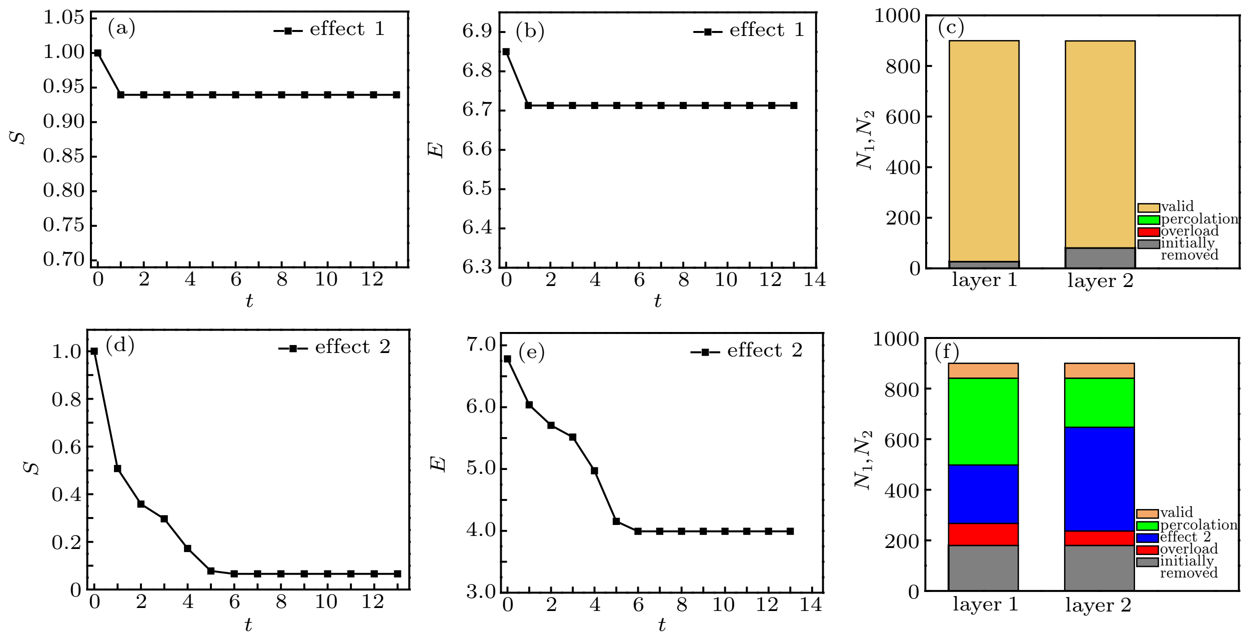

Fig. 6. The evolution processes of two double-layer spatial networks under effect 1 and effect 2 when $p = 0.2$. The parameters are $a = 5$, $b = 0.8$, $c = 0.5$, $r = 1$, $\zeta = 0.2$, $N_1 = N_2 = 900$, $\langle k\rangle_1 = \langle k\rangle_2 = 4$. N_x and $\langle k\rangle_x$ are the number of nodes and average degree of layer x individually, $x = 1$, 2. Each point on the figures is the average value of 20 experiments. Panels (a) and (b) are the changes of dynamic network size $S(t)$ and dynamic network entropy $E(t)$ under effect 1. Panel (c) reveals the nodes state when the network is stable under effect 1. Panels (d)–(f) are similar to panels (a)–(c) but under effect 2.

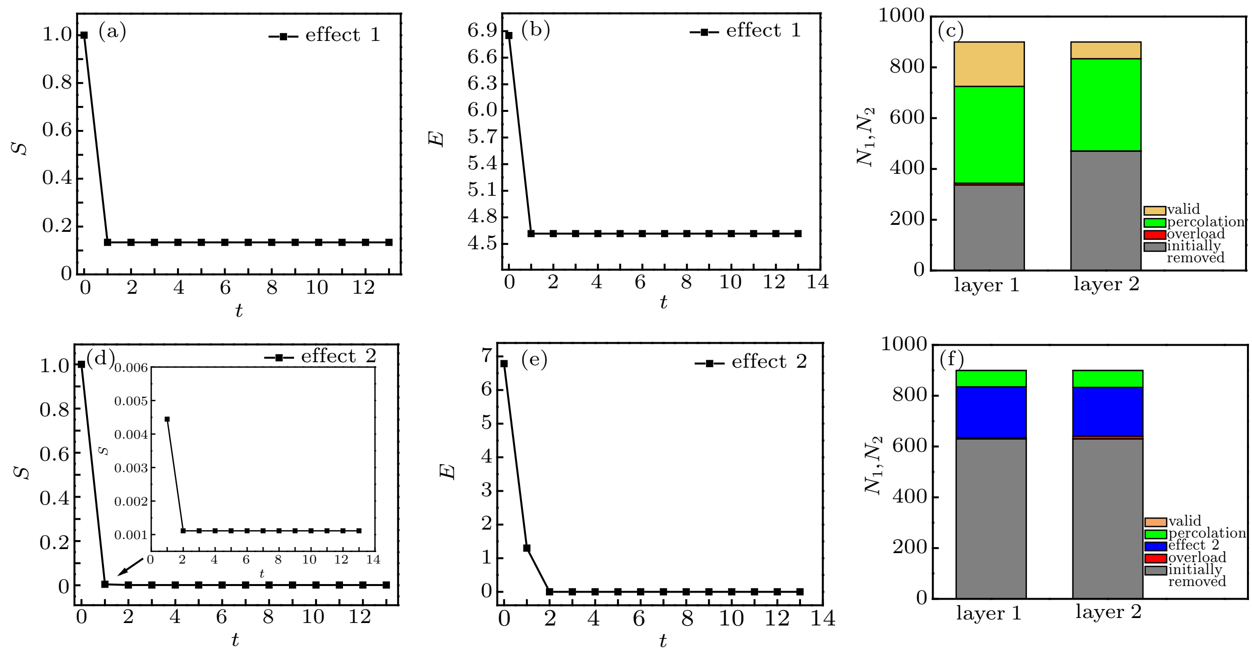

Fig. 7. The evolution processes of two double-layer spatial networks under effect 1 and effect 2 when $p = 0.7$. The parameters are $a = 5$, $b = 0.8$, $c = 0.5$, $r = 1$, $\zeta = 0.2$, $N_1 = N_2 = 900$, $\langle k\rangle_1 = \langle k\rangle_2 = 4$. N_x and $\langle k\rangle_x$ are the number of nodes and average degree of layer x individually, $x = 1$, 2. Each point on the figures is the average value of 20 experiments. Panels (a) and (b) are the changes of dynamic network size $S(t)$ and dynamic network entropy $E(t)$ under effect 1. Panel (c) shows the nodes state when the network is stable under effect 1. Panels (d)–(f) are similar to panels (a)–(c) but under effect 2.

From Figs. 6 and 7, it can be found that the number of network iterations under effect 2 is greater than that under effect 1. Owing to the influence of node recovery, the spatial network robustness under effect 1 is stronger than that under effect 2. It is worth noting that dynamic network size $S(t)$ is more suitable for describing the network evolution process than dynamic network entropy $E(t)$ in our model. As shown in Table 3, when t changes from 0 to 1, and 2 to 4, the changes of $E(t)$ are equal, *i.e.*, $E(0)-E(1)=E(2)-E(4)$. However, the changes of $S(t)$ are different, *i.e.*, $886(S(0)-S(1)) > 335(S(2)-S(4))$. Thus, in the next section, we will use the following metric to explain the network robustness:

$$S = \frac{S_1 + S_2 + \cdots + S_M}{N_1 + N_2 + \cdots + N_M}, \tag{10}$$

where S_x is the size of the largest connected component of layer x under the circumstance that the network is stable, N_x is the number of nodes of layer x, $x = 1, 2, \ldots, M$. We remark that the relation between Eqs. (8) and (10) is that S is the final state of $S(t)$. For example, as illustrated in Table 3, the network comes to be stable when $t \geq 6$. Consequently, $S(t) = S$, $t \geq 6$.

Table 3. The dataset of Figs. 6(d) and 6(e).

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$
$S(t)$	1800	914	645	534	310	140	118	118
$E(t)$	6.78	6.04	5.71	5.52	4.97	4.15	3.99	3.99

4.2. The affecting factors of the network robustness

In this subsection, we will analyze the influence factors of multilayer spatial network robustness. Two double-layer networks, two three-layer networks, two four-layer networks are constructed and equation (10) is utilized to describe the network robustness. It is generally known that the larger the values of $c \times C_i$, the stronger the network robustness. According to Eq. (5), the load capacity C_i of a node will increase by adjusting the values of a and b. As c increases, the network will perform better. However the load capacity of a node is limited in real life, such as the computer power of server is finite. Therefore, we need to investigate other factors affecting the network robustness.

Firstly, the effect of the spatial parameter ζ is brought forward. Inspired by Eq. (7), the values of ζ are chosen as 0.2, 2, 20 for simulation. As shown in Fig. 8, the network performs better as ζ decreases when p is small. That means the smaller the distance between nodes within the layer, the stronger the network robustness. When p is large, the network robustness grows to be stronger as ζ increase. In addition, as the number of layers M increases, the behaviors of networks are different under effect 1 and effect 2. In terms of effect 1, with the increase of M, the probability of node recovery becomes larger, which leads to a better performance of the network robustness. As for effect 2, the greater the number of layers M, the higher the probability that a node fails, followed by the worse performance of networks.

Secondly, the influence of length r is also considered. As r increases, the network robustness becomes worse, which can be easily seen from Fig. 9. That means the connection between layers based on the same coordinate will enhance the network robustness. The reason is that the larger the values of r, the more the nodes satisfying the condition Eq. (2). For this reason, the probability that a node fails will increase and the probability that a node recovers will decrease. What is

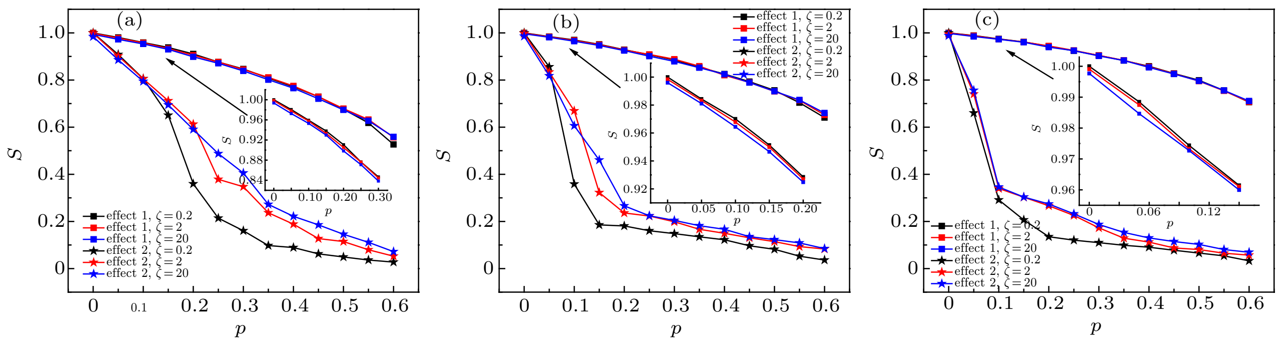

Fig. 8. The changes of S under the effect of ζ in multilayer spatial networks. When p is small, as ζ decreases, the network robustness becomes more reliable. When p is large, the network performs better as ζ increases. Each point on the figures is the average value of 20 experiments. (a) Two-layer networks. The parameters are $a = 6$, $b = 0.5$, $c = 1.5$, $r = 3$, $N_1 = 225$, $N_2 = 400$, $\langle k \rangle_1 = 5$, $\langle k \rangle_2 = 6$. N_x and $\langle k \rangle_x$ are the number of nodes and average degree of layer x individually, $x = 1, 2$. (b) Three-layer networks. The added parameters are $N_3 = 625$ and $\langle k \rangle_3 = 7$. (c) Four-layer networks. The added parameters are $N_4 = 900$ and $\langle k \rangle_4 = 8$.

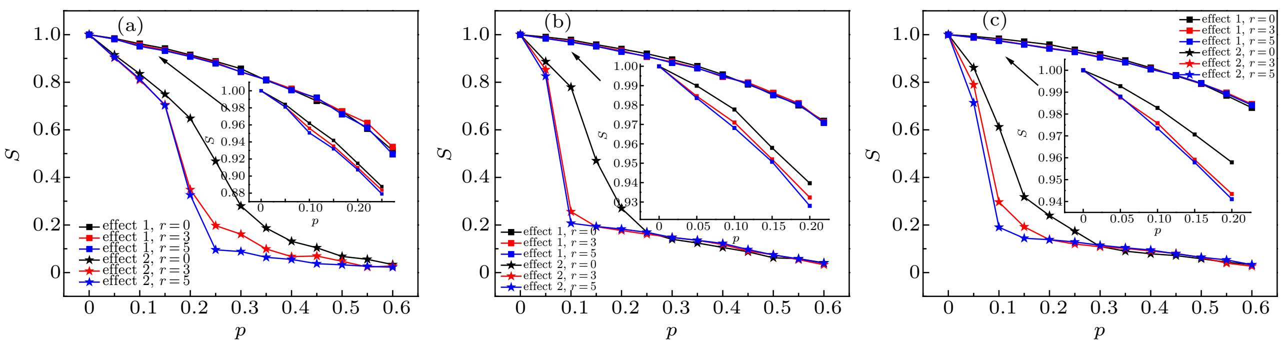

Fig. 9. The changes of S under the effect of r in multilayer spatial networks. As r decreases, the network robustness becomes stronger. Each point on the figures is the average value of 20 experiments. (a) Two-layer networks. The parameters are $a = 6$, $b = 0.5$, $c = 1.5$, $\zeta = 0.2$, $N_1 = 225$, $N_2 = 400$, $\langle k \rangle_1 = 5$, $\langle k \rangle_2 = 6$. N_x, and $\langle k \rangle_x$ are the number of nodes and average degree of layer x individually, $x = 1, 2$. (b) Three-layer networks. The added parameters are $N_3 = 625$ and $\langle k \rangle_3 = 7$. (c) Four-layer networks. The added parameters are $N_4 = 900$ and $\langle k \rangle_4 = 8$.

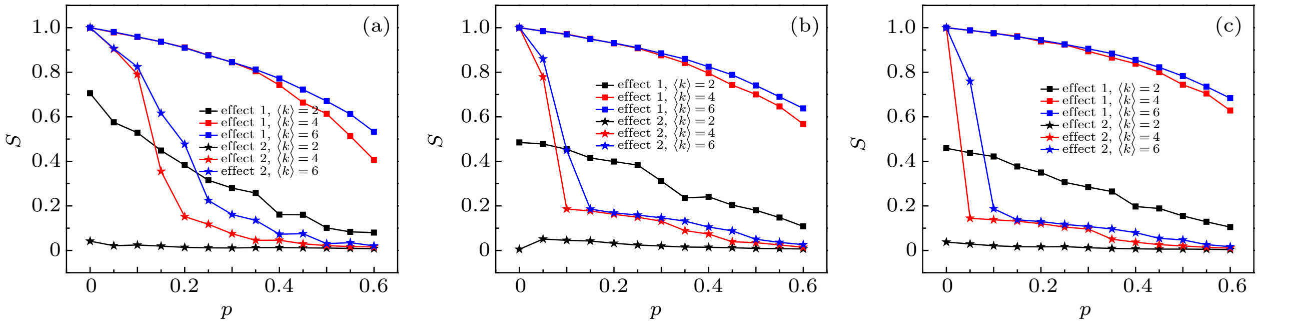

Fig. 10. The changes of S under the effect of $\langle k \rangle$ in multilayer spatial networks. As $\langle k \rangle$ increases, the network robustness becomes stronger. Each point on the figures is the average value of 20 experiments. (a) Two-layer networks. The parameters are $a = 6$, $b = 0.5$, $c = 1.5$, $\zeta = 0.2$, $r = 3$, $N_1 = 225$, $N_2 = 400$. N_x is the number of nodes of layer x, x=1,2. (b) Three-layer networks. The added parameter is $N_3 = 625$. (c) Four-layer networks. The added parameter is $N_4 = 900$.

more, the network with more layers performs better than that with fewer layers under effect 1. On the contrary, the network behaves worse as M increases under effect 2.

Thirdly, the effect of the average degree $\langle k \rangle$ is investigated. As illustrated in Fig. 10, the network robustness manifests better performance as $\langle k \rangle$ increases. The root cause is that the larger the values of $\langle k \rangle$, the more the link number of the network. Accordingly, the probability that a node becomes invalid due to overload will decrease and the probability that a node connects to the largest connected component will increase. Hence, the network robustness will become stronger.

4.3. The comparative analysis on affecting factors

In this subsection, we will make a comparative analysis of affecting factors discussed in Subsection 4.2. By increasing the same values of different parameters, we analyze the changes of network robustness.

Figure 11(c) is chosen as an example to explain the steps of the method. Firstly, we construct a four-layer spatial network and obtain the network robustness S under different probability p. The fixed parameters are $a = 6$, $b = 0.5$, $c = 1.5$, $N_1 = 225$, $N_2 = 400$, $N_3 = 625$, $N_4 = 900$. N_x is the number of nodes of layer x, x=1,2,3,4. The variable parameters are $\zeta = 0.2$, $r = 3$, $\langle k \rangle_1 = 5$, $\langle k \rangle_2 = 6$, $\langle k \rangle_3 = 7$, $\langle k \rangle_4 = 8$. $\langle k \rangle_x$ is average degree of layer x, x = 1, 2, 3, 4. Secondly, we change $\zeta = 0.2$ into $\zeta = 2.2$ and do not alter other parameters. Therefore, we obtain the values of S_ζ and the absolute values of network robustness changes $|\Delta S|_\zeta = |S - S_\zeta|$. Thirdly, we change $r = 3$ into $r = 5$. Other parameters are not changed, *i.e.*, $\zeta = 0.2$, $\langle k \rangle_1 = 5$, $\langle k \rangle_2 = 6$, $\langle k \rangle_3 = 7$, $\langle k \rangle_4 = 8$. The values of S_r and $|\Delta S|_r = |S - S_r|$ can be obtained. Fourthly, we use $\langle k \rangle_1 = 7$, $\langle k \rangle_2 = 8$, $\langle k \rangle_3 = 9$, $\langle k \rangle_4 = 10$ to obtain the values of $S_{\langle k \rangle}$ and $|\Delta S|_{\langle k \rangle} = |S - S_{\langle k \rangle}|$ under the conditions of $\zeta = 0.2$ and $r = 3$. At last, we compare $|\Delta S|_\zeta$, $|\Delta S|_r$, and $|\Delta S|_{\langle k \rangle}$ to analyze which factor has the greatest impact on network robustness.

The key factors are different under effect 1 and effect 2. As shown in Fig. 11, when $p < 0.55$, there is little difference between these factors under effect 1. The reason is that most nodes are effective when p is small. When $p > 0.55$, the spatial parameter ζ is the key factor. The root cause is that the larger the values of ζ, the higher the probability that node connects to the largest connected component. As illustrated in Fig. 12(a), when $p < 0.2$ and the layer number M is small, the average degree $\langle k \rangle$ is the leading factor under effect 2. The reason is that as $\langle k \rangle$ increases, the probability that node overloads will decrease. With the increase of M, the influence of length r becomes larger, as shown in Fig. 12(c). When $p > 0.2$, the effect of ζ is the largest. The fundamental cause is similar to the reason of effect 1. Therefore, the key factors are different in various circumstances under effect 2. We could change the parameters to alter the network robustness according to the actual situation. It is worth noting that the peaks of changes in network robustness are different under effect 1 and effect 2. The reason is that the network robustness performs better under effect 1 relative to effect 2, as shown in Figs. 8–10.

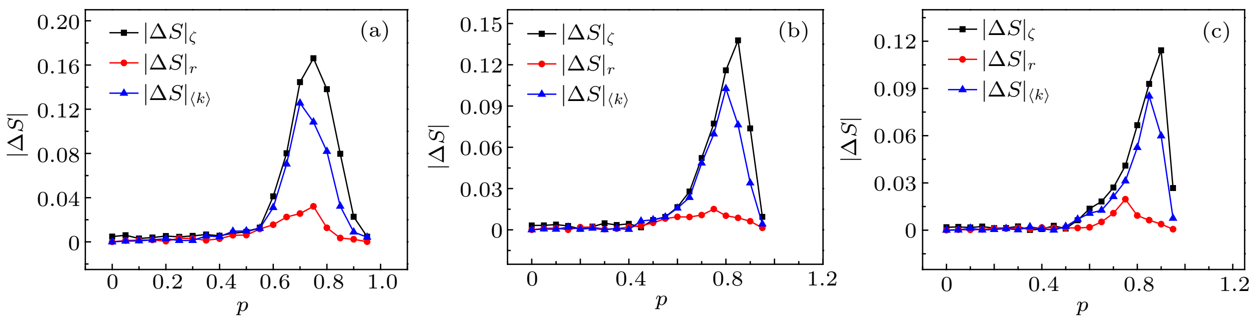

Fig. 11. $|\Delta S|$ as a function of p on the multilayer spatial networks under effect 1. The fixed parameters are $a = 6$, $b = 0.5$, $c = 1.5$, $N_1 = 225$, $N_2 = 400$, $N_3 = 625$, $N_4 = 900$. N_x is the number of nodes of layer x, x = 1, 2, 3, 4. Each point on the figures is the average value of 100 experiments. (a) Two-layer networks. (b) Three-layer networks. (c) Four-layer networks.

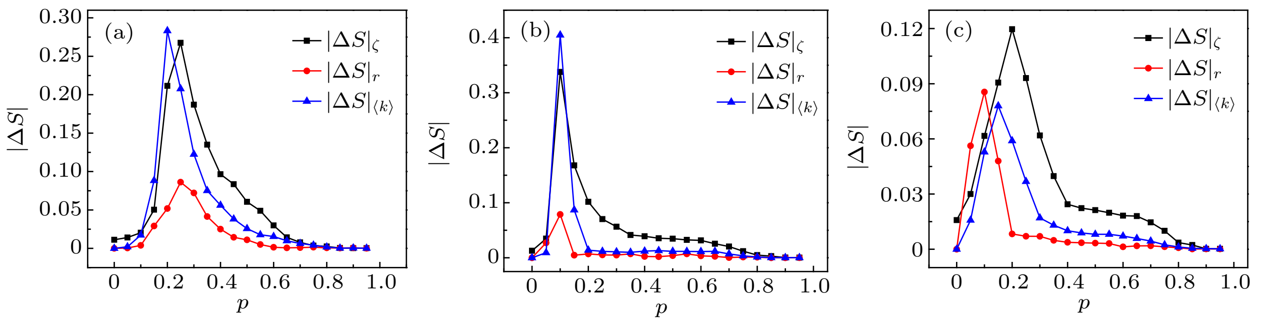

Fig. 12. $|\Delta S|$ as a function of p on the multilayer spatial networks under effect 2. The fixed parameters are $a = 6$, $b = 0.5$, $c = 1.5$, $N_1 = 225$, $N_2 = 400$, $N_3 = 625$, $N_4 = 900$. N_x is the number of nodes of layer x, $x = 1, 2, 3, 4$. Each point on the figures is the average value of 100 experiments. (a) Two-layer networks. (b) Three-layer networks. (c) Four-layer networks.

4.4. Discussion

There are many systems are embedded in space and the link lengths are exponentially distributed, such as infrastructure networks, computer networks, and social networks. The conclusions of this paper have extensive applicability and practical significance for the spatial systems.

In Section 3, the spatial argument ζ, node number N, and average degree $\langle k\rangle$ have an impact on the spatial network structure and the difference between nodes. Therefore, these results provide insight into maintaining the network structure stability and adjusting the difference degree between nodes of the embedded spatiality systems. For example, in the aviation network, flights between different cities correspond to various distances. To keep system stable and provide convenience for people, which flights need to open is important. In the server system, how many servers are at least equipped to maintain the system stability and reduce resource consumption. In some communication systems, nodes need to adopt the same communication equipment and architecture. Therefore, how to reduce the difference between nodes is necessary.

In Section 4, the affecting factors of the network robustness are discussed and compared, such as the spatial argument ζ, length r, and average degree $\langle k\rangle$. These results have contributed to enhancing the robustness and restoring a network in which cascading happens under various circumstances. Take the case of a cascaded multilayer system, the recovery of the system involves many factors such as the number of layers, the number of invalid nodes, and the interaction mechanism between layers. Therefore, this paper provides an idea on how to change the parameters to maximize the enhancement of network robustness and keep down costs.

5. Conclusion

In this paper, we construct two cascading failure models on multilayer spatial networks, taking into account the influences of overload, the effect between layers, as well as the percolation. As observed from our research, the distance between nodes within the layer obeys the exponential distribution and the connection between layers generalizes the one-to-one correspondence based on node position. Entropy is applied to reflect the spatial network structure and indicate the differences between nodes. With the decrease of the spatial parameter ζ or the increments of the node number N and the average degree $\langle k\rangle$, it can be seen from simulations that the spatial network structure tends to be more stable and the difference degree between nodes becomes larger. By virtue of the approaches of the dynamic network size $S(t)$ and dynamic network entropy $E(t)$, we describe the network stability and robustness. The number of failure nodes caused by different reasons is counted and analyzed respectively. Besides, the factors affecting network robustness are analyzed and compared as well. Simulations demonstrate that the behaviors of the spatial network perform better as $\langle k\rangle$ increases. As r decreases, the network robustness grows to be stronger. When p is small, as ζ decreases, the network robustness becomes more reliable. When p is large, the network performs better as ζ increases. Furthermore, as layer number M increases, the spatial network presents different behaviors under effect 1 and effect 2. If M increases, spatial networks will perform better under effect 1. In contrast, the network robustness gets worse under effect 2. At last, the practical significance of the results is also discussed.

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