Exact scattering states in one-dimensional Hermitian and non-Hermitian potentials*

Ruo-Lin Chai(柴若霖)[1], Qiong-Tao Xie(谢琼涛)[1,†], and Xiao-Liang Liu(刘小良)[2]

[1] *College of Physics and Electronic Engineering, Hainan Normal University, Haikou* 571158, *China*
[2] *School of Physics and Electronics, Central South University, Changsha* 410083, *China*

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The scattering states in one-dimensional Hermitian and non-Hermitian potentials are investigated. An analytical solution for the scattering states is presented in terms of Heun functions. It is shown that for some specially chosen parameter conditions, an infinite number of the exact scattering states is obtained. In the Hermitian potentials, they correspond to the reflectionless states. In the non-Hermitian complex potentials with parity-time symmetry, they are the unidirectionally reflectionless states.

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1. Introduction

In the past decades, the concept of reflectionless states has attracted extensive interest of research. On the one hand, several reflectionless potentials have been realized in engineered photonic lattices.[1–5] In the engineered photonic lattices, the temporal evolution of quantum systems can be mapped into the spatial propagation of light waves. These optical structures provide a controllable platform for studying the properties of the scattering states. By suitably modulating the coupling between the waveguides, reflectionless characteristics and supersymmetric scattering have been observed experimentally.[1,2]

On the other hand, the concept of the reflectionless states has found wide applications in non-Hermitian systems with parity–time (PT) symmetry.[6–11] It has been shown that these non-Hermitian systems exhibit a rich variety of unique scattering properties without Hermitian counterparts.[12–15] Many interesting phenomena have been observed in different non-Hermitian systems, such as nonreciprocal light transmission in microcavities,[16] unidirectional reflectionlessness in optical metamaterials,[17] and coherent perfect absorption in coupled resonators.[18]

In the present work, our main aim is to investigate the scattering states in a type of one-dimensional potential and its non-Hermitian extension. We present the analytical solution for the scattering states in terms of Heun functions. It is found that for specially chosen parameter conditions, there exist an infinite number of the exact analytical solutions for the scattering states. In the Hermitian situation, the exact scatting states have zero reflection coefficients. In the non-Hermitian extension, we give an analytical demonstration for the unidirectional reflectionlessness.

2. Exact scattering states in a Hermitian potential

We begin with the following Schrödinger equation ($2m = \hbar = 1$)

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)+V(x)\psi(x)=E\psi(x), \tag{1}$$

where the potential $V(x)$ is given as

$$V(x)=\frac{V_1}{g+\cosh x}+\frac{V_2}{(g+\cosh x)^2}+\frac{V_3\sinh x}{(g+\cosh x)^2}. \tag{2}$$

Here $V_{1,2,3}$ and $g>-1$ are real potential parameters. In the case of $V_1=V_3=g=0$, the resulting potential is the well-known sech-squared potential which allows the exact solutions for the reflectionless states for $V_2=-n(n+1)$ with integer $n>0$.[19] This reflectionless sech-squared potential has been realized in engineered photonic systems.[1,2] In the case of $g<0$ and $V_1=V_3=0$, an exact transmission state was found in a Bose–Einstein condensate which is described by a nonlinear Schrödinger equation.[20] In the case of $V_3=0$, the resulting potential is a double-well potential under certain parameter conditions, and exact bound states have been presented.[21–24] Thus this potential (2) can be used to study various double-well systems, and find applications in systems of Bose–Einstein condensates. In this work, we mainly discuss the scattering states.

To solve the Schrödinger equation (1), we make the transformations

$$z=\frac{\mathrm{e}^{-x}}{\mathrm{e}^{-x}-r_1},\quad \psi(x)=z^{\lambda_1 x}(z-1)^{\lambda_2}(z-a)^{\lambda_3}\phi(z), \tag{3}$$

with $r_1=\sqrt{g^2-1}-g$, $a=r_2/(r_2-r_1)$, $r_2=-\sqrt{g^2-1}-g$, $\lambda_1=-\lambda_2=-\sqrt{-E}$, and $\lambda_3=1/2-$

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†Corresponding author. E-mail: qiongtaoxie@yahoo.com
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$\sqrt{1+4V_2/(g^2-1)+4V_3/\sqrt{g^2-1}}/2$. After a simple calculation, it is found that $\phi(z)$ obeys the Heun equation[25,26]

$$\frac{d^2\phi}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-a}\right)\frac{d\phi}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}\phi=0, \quad (4)$$

where $\gamma=1+2\lambda_1$, $\delta=1+2\lambda_2$, $\alpha=\lambda_3+\lambda_4$, $\beta=1+\lambda_3-\lambda_4$, $\lambda_4=1/2-\sqrt{1+4V_2/(g^2-1)-4V_3/\sqrt{g^2-1}}/2$, $\varepsilon=\alpha+\beta-\delta-\gamma+1=2\lambda_3$, and $q=(V_1+V_3)/\sqrt{g^2-1}+(1+2\lambda_1)\lambda_3$. Mathematically, if $\gamma=1-2\sqrt{-E}$ is not an integer, the Heun equation will have two linearly independent solutions near $z=0$.[25,26] For the scattering states with $E>0$, $\gamma=1-2\sqrt{-E}$ is complex, thus we obtain the two linearly independent solutions $\phi_{1,2}(x)$ as follows:

$$\phi_1(z) = Hl(a,q;\alpha,\beta,\gamma,\delta;z), \quad (5)$$

$$\phi_2(z) = z^{1-\gamma}Hl(a,q+(\varepsilon+\delta a)(1-\gamma); \alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\delta;z). \quad (6)$$

Here $Hl(a,q;\alpha,\beta,\gamma,\delta;z)=\sum_{n=0}^{\infty}h_nz^n$ is an infinite series known as the Heun function. The coefficients h_n are determined by the three-term recurrence relation: $R_{n-1}h_{n-1}+P_nh_n+Q_{n+1}h_{n+1}=0$ with the initial conditions $h_0=1$ and $h_{-1}=0$. Here $R_n=(n+\alpha)(n+\beta)$, $P_n=-q-n(n-1+\gamma)(1+a)-n(a\delta+\varepsilon)$, and $Q_n=an(n-1+\gamma)$. In terms of the Heun functions, the analytical solutions for this potential are given as

$$\psi_1(x) = z^{\lambda_1}(z-1)^{\lambda_2}(z-a)^{\lambda_3}Hl(a,q;\alpha,\beta,\gamma,\delta;z), \quad (7)$$

$$\psi_2(x) = z^{-\lambda_1}(z-1)^{\lambda_2}(z-a)^{\lambda_3} \times Hl(a,q+(\varepsilon+\delta a)(1-\gamma); \alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\delta;z). \quad (8)$$

To discuss the scattering states with the energy $E=k^2>0$, we need to discuss the asymptotic behavior of $\psi_{1,2}(x)$. As $x\to\infty$, one has $z=e^{-x}/(e^{-x}-r_1)\to 0$, thus $\psi_1(x)=(-a)^{\lambda_3}/(r_1)^{\lambda_1}e^{ikx}$ and $\psi_2(x)=(r_1)^{\lambda_1}(-a)^{\lambda_3}e^{-ikx}$. We assume that a particle is incident from the left of the potential, thus $\psi_1(x)$ is used to construct the analytical solution for the scattering states

$$\psi_s(x)=Az^{\lambda_1}(z-1)^{\lambda_2}(z-a)^{\lambda_3}Hl(a,q;\alpha,\beta,\gamma,\delta;z), \quad (9)$$

where A is a constant to be determined. In order to obtain the transmission and reflection coefficients, we need to know the asymptotic behavior of $\psi_s(x)$ as $x\to-\infty$. However, this cannot be obtained easily. The reason is given as follows. The Heun function $Hl(a,q;\alpha,\beta,\gamma,\delta;z)$ is only analytical in the range of $|z|<\min\{a,1\}$. As $x\to-\infty$, we have $z=e^{-x}/(e^{-x}-r_1)\to 1$. $Hl(a,q;\alpha,\beta,\gamma,\delta;1)$ is thus not analytical.[25,26] This problem can be solved by constructing the solutions with good asymptotic behavior as $x\to-\infty$.[21]

To discuss the asymptotic behavior as $x\to-\infty$, we may make different transformations

$$y=\frac{-r_1}{e^{-x}-r_1}, \quad \psi(x)=y^{-\lambda_1}(y-1)^{-\lambda_2}(y-a_1)^{\lambda_3}\phi(y), \quad (10)$$

with $a_1=r_1/(r_1-r_2)$, and then obtain the Heun equation for $\phi(y)$ as follows:[25,26]

$$\frac{d^2\phi}{dy^2}+\left(\frac{\gamma_1}{y}+\frac{\delta_1}{y-1}+\frac{\varepsilon_1}{y-a_1}\right)\frac{d\phi}{dy}+\frac{\alpha_1\beta_1y-q_1}{y(y-1)(y-a_1)}\phi=0, \quad (11)$$

where $\gamma_1=1-2\lambda_1$, $\delta_1=-2\lambda_2+1$, $\varepsilon_1=2\lambda_3$, $\alpha_1=2\lambda_3$, $\beta_1=1$, and $q_1=-V_2/\sqrt{g^2-1}+(1-2\lambda_1)\lambda_3$. For the scattering states with $E>0$, $\gamma_1=1+2\sqrt{-E}$ is not an integer, and thus the Heun equation supports two linearly independent solutions $\phi_{3,4}(x)$ as follows:

$$\phi_3(y) = Hl(a_1,q_1;\alpha_1,\beta_1,\gamma_1,\delta_1;y), \quad (12)$$

$$\phi_4(y) = y^{1-\gamma_1}Hl(a_1,q_1+(\varepsilon_1+\delta_1a_1)(1-\gamma_1); \alpha_1-\gamma_1+1,\beta_1-\gamma_1+1,2-\gamma_1,\delta_1;y), \quad (13)$$

which leads to the analytical solutions

$$\psi_3(x) = y^{-\lambda_1}(y-1)^{-\lambda_2}(y-a_1)^{\lambda_3} \times Hl(a_1,q_1;\alpha_1,\beta_1,\gamma_1,\delta_1;y), \quad (14)$$

$$\psi_4(x) = y^{\lambda_1}(y-1)^{-\lambda_2}(y-a_1)^{\lambda_3} \times Hl(a_1,q_1+(\varepsilon_1+\delta_1a_1)(1-\gamma_1); \alpha_1-\gamma_1+1,\beta_1-\gamma_1+1,2-\gamma_1,\delta_1;y). \quad (15)$$

In terms of the analytical solutions $\psi_{3,4}$, the analytical solution for the scattering states is given as

$$\psi_s(x)=A(C_1\psi_3(x)+C_2\psi_4(x)), \quad (16)$$

where the coefficients $C_{1,2}$ are constants to be determined. Clearly, in the common valid regions of $\psi_{1,3,4}$, we have

$$\psi_1(x)=C_1\psi_3(x)+C_2\psi_4(x). \quad (17)$$

This leads to the explicit expressions for the coefficients $C_{1,2}$,

$$C_1=\frac{W(\psi_1,\psi_4)}{W(\psi_3,\psi_4)}, \quad C_2=-\frac{W(\psi_1,\psi_3)}{W(\psi_3,\psi_4)}. \quad (18)$$

Here $W(\phi,\varphi)=\phi(x)d\varphi(x)/dx-\varphi(x)d\phi(x)/dx$ is the Wronskian of two functions $\phi(x)$ and $\varphi(x)$. As $x\to-\infty$, we have the asymptotic behavior

$$\psi_3(x) = (-r_1)^{-\lambda_1}(-1)^{-\lambda_2}(-a_1)^{\lambda_3}e^{ikx}, \quad (19)$$

$$\psi_4(x) = (-r_1)^{\lambda_1}(-1)^{-\lambda_2}(-a_1)^{\lambda_3}e^{-ikx}. \quad (20)$$

If we set $A=1/(C_1(-r_1)^{-\lambda_1}(-1)^{-\lambda_2}(-a_1)^{\lambda_3})$, we have

$$\psi_s(x)=A(C_1\psi_3(x)+C_2\psi_4(x))=e^{ikx}+re^{-ikx}, \quad (21)$$

thereby resulting in the reflection and transmission coefficients

$$r=\frac{r_1^{2\lambda_1}C_2}{C_1}, \quad t=\frac{a^{\lambda_3}}{a_1^{\lambda_3}C_1}. \quad (22)$$

Therefore, the reflection and transmission coefficients are related to the Wronskians of the three analytical solutions $\psi_{1,3,4}$.

Besides the analytical solutions in terms of the Heun functions, there exist exact analytical solutions in terms of elementary functions, as studied in the following. It has been shown that for certain special conditions of

$$\alpha, \beta = -N, \quad N = 0, 1, 2, \ldots, \tag{23}$$

$$h_{N+1} = 0, \tag{24}$$

the Heun function $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$ can be terminated as a polynomial in z.[25,26] These terminated conditions allow us to obtain an infinite number of exact analytical solutions. From the condition $\alpha = \lambda_3 + \lambda_4 = -N$ in $\psi_s(x)$, we have the parameter relation

$$V_2 = -\frac{N}{2}\left(\frac{N}{2} + 1\right)(1 - g^2) + \frac{V_3^2}{(N+1)^2}, \tag{25}$$

with $N \geq 1$. To give a simple form of the exact solutions, we first use the relation $Hl(a, q; \alpha, \beta, \gamma, \delta; z) = (1 - z)^{-\alpha} Hl(a/(a-1), (\gamma\alpha a - q)/(a-1); \alpha, \alpha - \delta + 1, \gamma, \alpha - \beta + 1; z/(z-1))$ in $\psi_s(x)$,[26] and then set $A = 1$. Finally, the exact analytical solutions for the scattering states are given as

$$\psi_s^N(x) = \frac{\mathrm{e}^{\,\mathrm{i}kx}}{\sqrt{(\mathrm{e}^{-x} - r_1)^N (\mathrm{e}^{-x} - r_2)^N}} \sum_{n=0}^{N} h_n(E_{\mathrm{ex}}^N) \frac{\mathrm{e}^{-nx}}{r_1^n}, \tag{26}$$

where the exact energies E_{ex}^N are determined by $h_{N+1} = 0$. We now discuss the asymptotic behavior of $\psi_s^N(x)$ as $x \to \pm\infty$. After a simple analysis, we have $\psi_s^N(x) = \mathrm{e}^{\,\mathrm{i}kx}$ as $x \to \infty$, and $\psi_s^N(x) = (h_N/r_1^N)\,\mathrm{e}^{\,\mathrm{i}kx}$ as $x \to -\infty$. These results indicate that the exact analytical solutions $\psi_s^N(x)$ correspond to the scattering states with the zero reflection coefficient. In the following, we present the exact results in the case of $N = 1$ as an example.

To give a simple explicit expression, we consider the case of $V_3 = 0$. Under the condition $h_1 = 0$, we obtain the simple parametric dependence of the exact energy

$$E_{\mathrm{ex}}^1 = k^2 = -\frac{1}{4}\frac{(2V_1 + g)^2 - 1}{g^2 - 1}. \tag{27}$$

It is valid for $g > -1$ and $g \neq 1$. Depending on the parameter value of g, we have different parameter regions for V_1, $-(1+g)/2 < V_1 < (1-g)/2$ for $g > 1$, and $V_1 > (1-g)/2$ or $V_1 < -(1+g)/2$ for $-1 < g < 1$. The corresponding scattering state takes the form

$$\psi_s^1(x) = \frac{\mathrm{e}^{\,\mathrm{i}kx}}{\sqrt{(\mathrm{e}^{-x} - r_1)(\mathrm{e}^{-x} - r_2)}} \times \left(1 + \left(g + \frac{2V_1}{1 - 2\mathrm{i}|k|}\right)\mathrm{e}^{-x}\right). \tag{28}$$

It follows that we have $\psi_s^1(x) = \mathrm{e}^{\,\mathrm{i}kx}$ as $x \to \infty$, and $\psi_s^1(x) = (g + \frac{2V_1}{1-2\mathrm{i}|k|})\,\mathrm{e}^{\,\mathrm{i}kx}$ as $x \to -\infty$. We can verify the result of $|(g + \frac{2V_1}{1-2\mathrm{i}|k|})| = 1$ easily. Therefore, the solution $\psi_s^1(x)$ represents the reflectionless state.

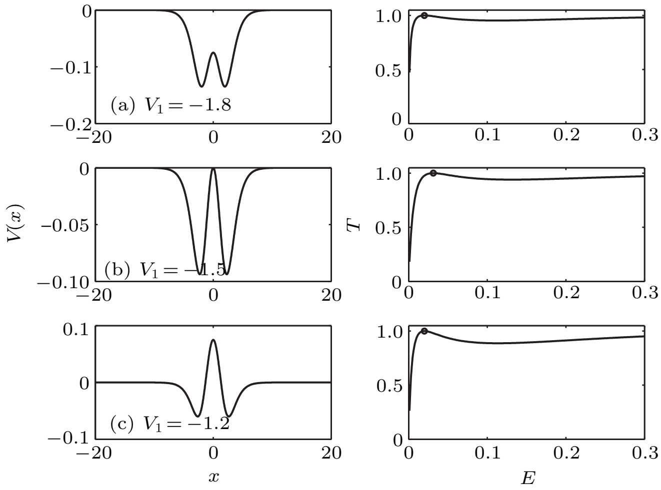

Fig. 1. Profiles of the potential $V(x)$ for (a) $V_1 = -1.8$, (b) $V_1 = -1.5$, (c) $V_1 = -1.2$ with $g = 3$, $V_2 = 6$, and $V_3 = 0$. In the right column, we plot the corresponding numerical results for the transmission probabilities T as a function of the incident energies E. The circles are for the exact results $T = 1$ at $E = E_{\mathrm{ex}}^1$.

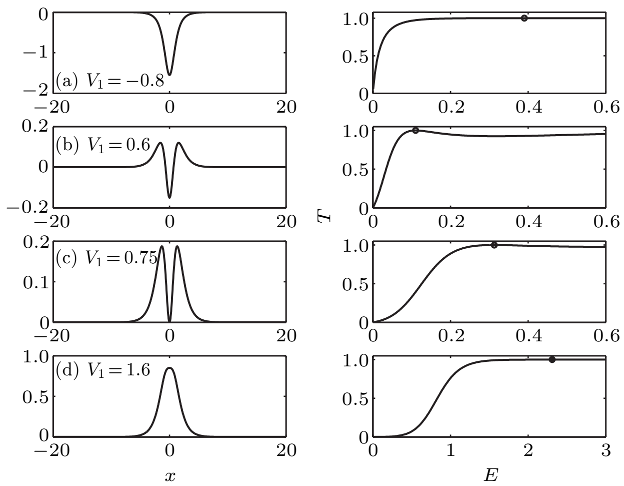

Fig. 2. Profiles of the potential $V(x)$ for (a) $V_1 = -0.8$, (b) $V_1 = 0.6$, (c) $V_1 = 0.75$, and (d) $V_1 = 1.6$ with $g = V_3 = 0$ and $V_2 = -3/4$. The corresponding transmission probabilities as a function of the incident energies E are plotted in the right column. The circles are for the exact results $T = 1$ at $E = E_{\mathrm{ex}}^1$.

In Fig. 1, we show the profiles of $V(x)$ for this case of $N = 1$. We set $g = 3$, and then have the parameter range of V_1, $-2 < V_1 < -1$. In this parameter region, we take three different values of V_1, $V_1 = -1.8, -1.5, -1.2$. It is observed that the resulting potentials show double-well structures. In the right column of Fig. 1, we plot the numerical results for the transmission probabilities as a function of the incident energies E. The circles are for the exact results $T = 1$ at $E = E_{\mathrm{ex}}^1$. It is observed that as the incident energies E are increased from $E = 0$ to E_{ex}^1, the transmission probabilities are increased from $T = 0$ to $T = 1$. The analytical and numerical results agree very well. If the incident energies E are increased from E_{ex}^1, the transmission probabilities are first decreased, and then increased. In Fig. 2, we study the case of $g = V_3 = 0$. For such chosen parameter values, we have $V_2 = -3/4$. The resulting exact reflectionless states exist in the parameter range of $V_1 < -1/2$ or $V_1 > 1/2$. For $V_1 < -1/2$, the resulting potentials have single-well structures, as shown in Fig. 2(a). In the range of $V_1 > 1/2$, as V_1 is increased gradually, the resulting potentials change from double barrier to single barrier, as shown in Figs. 2(b)–2(d). Therefore, we also obtain the exact reflectionless states in the single or double barrier structures.

3. Exact scattering states in a complex potential

In this section, we study the case where the potential parameter V_3 is replaced by an imaginary parameter iV_3. The resulting one-dimensional PT symmetric complex potential is given as

$$V(x) = V_r(x) + iV_i(x) = \frac{V_1}{g+\cosh x} + \frac{V_2}{(g+\cosh x)^2} + \frac{iV_3 \sinh x}{(g+\cosh x)^2}. \tag{29}$$

Here V_3 are real potential parameter. $V_r(x) = V_1/(g+\cosh x) + V_2/(g+\cosh x)^2$ and $V_i(x) = V_3 \sinh x/(g+\cosh x)^2$ are the real and imaginary parts of the complex potential. In general, the complex potential provides an effective description for open quantum systems. In atomic systems, the complex potential may emerge from the interaction of near resonant light with open atomic systems.[27] In classical photonic systems, the complex potential may arise due to the complex index of refraction.[28] The case of $g = V_1 = 0$ corresponds to the PT symmetric Scarf II potential which is exactly solvable.[12–14] In the following, we shall show that there exist exact scattering states in this modified complex Scarf II potential.

For brevity, we discuss the specific case of $V_1 = -2kA\sqrt{1-g^2}$, $V_2 = A^2(g^2-1)$, and $V_3 = -A\sqrt{1-g^2}$, where A is a real parameter. With this choice of $V_{1,2,3}$, we take $\lambda_3 = -\lambda_4 = A$, thus $\alpha = 0$. Due to $\alpha = 0$, the termination condition (23) is satisfied. From the condition $h_1 = 0$ in $\phi_1(z)$, we obtain the energy $E = k^2$, and thus the corresponding eigenfunction

$$\psi(x) = e^{ikx}\left(\frac{e^{-x}-r_2}{e^{-x}-r_1}\right)^A. \tag{30}$$

After analyzing asymptotic behavior of $\psi(x)$, we have $\psi(x) = e^{ikx}$ as $x \to -\infty$, and $\psi(x) = (r_2/r_1)^A e^{ikx}$ as $x \to \infty$. It is seen clearly that the reflection probability $R = 0$. Therefore, the obtained eigenfunction represents the reflectionless state.

On the other hand, with $\alpha = 0$ and $E = k^2$, the Heun function in $\phi_2(z)$ cannot be terminated as a polynomial. However, it is found that for $g = 0$ and $A = n/2$ with integer n, $\phi_2(z)$ may be related to a hypergeometric function. For example, for the choice of $A = 1/2$ and $g = 0$, we obtain the eigenfunction of the form

$$\psi(x) = e^{-ikx}\left(\frac{e^{-x}+i}{e^{-x}-i}\right)^A \times (-1 + 2F(1, -2ik, 1-2ik, -i e^x)). \tag{31}$$

Here $F(\alpha, \beta, \gamma, z)$ is the hypergeometric function.[26] From the asymptotic behavior for the hypergeometric function,[26] it follows that as $x \to -\infty$, $\psi(x) = -e^{-ikx}$, and as $x \to \infty$, $\psi(x) = -i e^{-ikx} + i(2(i)^{2ik}\pi k/\sinh(2\pi k))e^{ikx}$. In this situation, the reflection probability $R = 4\pi^2 k^2/\sinh^2(2\pi k)$ is not zero. Therefore, our exact results reveal that with the same incident energy $E = k^2$, the reflection probability is sensitive to the direction of incidence. If a particle is incident from the left, its reflection probability is zero, and if it is incident from the right, its reflection probability is nonzero. This phenomena is called the unidirectional reflectionlessness which has been observed experimentally.[17] In Fig. 3, we show the profiles of $V_{r,i}(x)$ for four different values of k with $A = 2$ and $g = 0$. It is observed that as k is increased, $V_r(x)$ is changed from the barrier to the well.

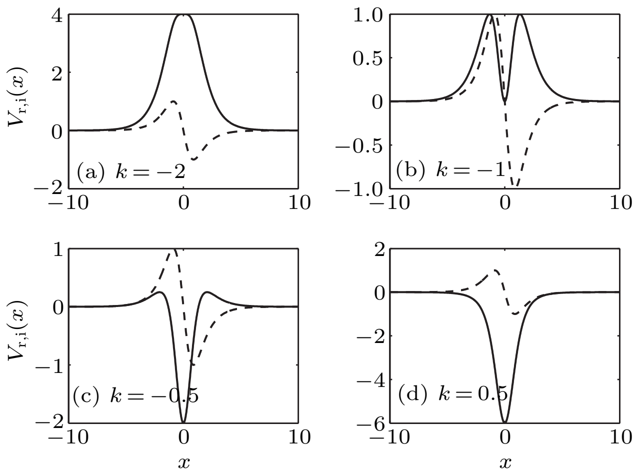

Fig. 3. Profiles of the real (solid lines) and imaginary (dashed lines) parts of the generalized PT symmetric Scarf II potential for (a) $k = -2$, (b) $k = -1$, (c) $k = -0.5$, and (d) $k = 0.5$. The potential parameters are given as $V_1 = -2kA$, $V_2 = -A^2$, and $V_3 = -A$ with $A = 2$ and $g = 0$.

Finally, we discuss the case of $g = 1$ where the above exact analytical results are invalid. In fact, in the case of $g = 1$, we have $r_1 = r_2 = -1$ and thus $a \to \infty$. The confluent Heun equation is obtained from the Heun equation through a confluence process.[25,26] The analytical solutions are then given by the confluent Heun functions. This will be discussed in detail in future. In the case of $g = 1$, we also obtain the exact scattering states. An example is given as follows. Here for brevity, we take $V_1 = -2kA$, $V_2 = -A^2$, and $V_3 = -A$, where A is a real parameter. We obtain an exact scattering state

$$\psi(x) = e^{ikx} e^{i\frac{2A}{1+e^x}}. \tag{32}$$

This result indicates that as a particle is incident from the left-hand side of this complex potential, its reflection probability is zero. However, for the same parameter conditions, we cannot obtain the exact scattering states incident from the right-hand side of this complex potential.

4. Conclusions

In summary, we have presented an analytical solution for the scattering states in a type of one-dimensional potential in terms of Heun functions. The parameter conditions for the existence of the exact scattering states are derived analytically and confirmed numerically. In the Hermitian situation, it is shown that the obtained exact scattering states correspond to the reflectionless states. Depending on the chosen parameters, the potential may vary from the double well to the single well, the single barrier, and the double barrier. The exact reflectionless states also appear in these structures. In addition, we obtain some exact unidirectionally reflectionless states in a generalized PT-symmetric non-Hermitian Scarf II potential. Our analytical results may find applications in engineered photonic lattices.

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