Simulation of anyons by cold atoms with induced electric dipole moment*

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We show that it is possible to simulate an anyon by a trapped atom which possesses an induced electric dipole moment in the background of electric and magnetic fields in a specific configuration. The electric and magnetic fields we applied contain a magnetic and two electric fields. We find that when the atom is cooled down to the limit of the negligibly small kinetic energy, the atom behaves like an anyon because its angular momentum takes fractional values. The fractional part of the angular momentum is determined by both the magnetic and one of the electric fields. Roles electric and magnetic fields played are analyzed.

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1. Introduction

Simulation of physical phenomena which occur originally in charged particles by neutral ones is an interesting subject. An example is the simulation of Aharononv–Bohm (AB) effect by neutral particles. The AB effect predicts that a charged particle will accumulate a geometrical phase when it moves around a long-thin magnetic-flux carried solenoid.[1]

The simulation of AB effect by a neutral particle which possesses a permanent magnetic dipole moment was proposed by Aharonov and Casher. In Ref. [2], they predicted that a neutral particle with a permanent magnetic dipole moment would acquire a geometrical phase if it moved around a uniformly electric charged long filament with the direction of the magnetic dipole moment parallel to the filament. It is the Aharonov–Casher (AC) effect.

The simulation of AB effect by a neutral particle with permanent electric dipole moment was proposed in Refs. [3,4]. It was predicted that a neutral particle with a permanent electric dipole moment would receive a geometrical phase if it circled around a uniformly magnetic charged long filament. It is named He–Mckellar–Wilkens (HMW) effect. The observation of HMW effect in experiments is difficult since the magnetic field in HMW effect is produced by magnetic monopoles.[5,6]

In order to avoid this difficulty, the authors in Ref. [7] proposed an alternative method to observe the HMW effect. Instead of using a neutral particle which possesses a permanent electric dipole moment, they proposed to use a neutral particle with an induced electric dipole moment interacting with an electric field and a magnetic field. Compared with HMW effect, the magnetic field in the proposal[7] is easily prepared in experiments.

Another example is Landau levels. Landau levels are eigenvalues of a charged planar particle interacting with a uniform perpendicular magnetic field. In Ref. [8], the authors showed that Landau levels could be simulated by an atom which possesses a permanent magnetic dipole moment in the background of an electric field. Since then, there are many research works concerning the analogy between Landau levels and spectra of neutral particles which possess permanent electric or magnetic dipoles interacting with electric and magnetic fields.[9–21]

We shall show that anyons,[22,23] which were mostly realized by charged particles before, can also be simulated by a neutral particle with an induced electric dipole moment. As is known, eigenvalues of the canonical angular momentum must be quantized in the three-dimensional space.[24,25] However, in the two-dimensional space, eigenvalues of the canonical angular momentum can take fractional values.[26,27] The reason is that the rotation group in three-dimensional space is a non-Abelian one while it is Abelian in the two-dimensional space. Particles which have the fractional angular momentum (FAM) are named anyons.[22,23] Anyons play important roles in understanding quantum Hall effects[28] and high T_c superconductivity.[29] There are several ways to realize anyons. Because of the dynamical properties of the Chern–Simons gauge field, in the absence of the Maxwell term, one

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can realize anyons by coupling charged particles to the Chern–Simons gauge field in $(2+1)$-dimensional space–time.[30–33] Recently, anyons receive renewed interests.[34–36]

Reference [37] proposed an alternative approach to realize anyons. The author coupled an ion to two magnetic fields. One is a uniform magnetic field and the other is generated by a long-thin magnetic solenoid. Provided the kinetic energy of this ion is cooled down to its lowest level by using the cold atomic technologies, the author found that eigenvalues of the canonical angular momentum of this charged particle can take fractional values. The fractional part is determined by the magnetic flux inside the magnetic solenoid.

In this paper, we propose to simulate anyons by coupling neutral particles, for example, atoms, which possess an induced electric dipole moment to electric and magnetic fields. The electric and magnetic fields we applied contain a magnetic field and two electric fields. The organization of this paper is as follows: in the next section, we introduce our model. Then, we quantize the model canonically and pay attention to its rotation property. Although the canonical angular momentum of this model only can take integer values, we show that the canonical angular momentum of the reduced model, which is obtained by cooling down the kinetic energy of the atom to the negligibly small, takes fractional values. The fractional part of the angular momentum depends on the intensity of the magnetic and only one of the electric fields explicitly. In Section 3, we analyze the roles two electric fields played in the simulation of anyons. We prove that both of the electric fields are necessary to simulate anyons. Summations and conclusions will be given in the last section.

2. Fractional angular momentum

The model we considered is an atom which possesses an induced electric dipole moment (with no permanent electric or magnetic dipole moments) interacting with electric and magnetic fields. The electric and magnetic fields we applied consist of a pair of electric fields $\boldsymbol{E}^{(1)}$, $\boldsymbol{E}^{(2)}$ and a uniform magnetic field $\boldsymbol{B}$.

The atom moves in a cylinder in which a uniform volume charge density ρ is distributed. A long filament with uniform electric charge per length is along the symmetry axis of the cylinder. The magnetic field is uniform and parallel to the symmetry axis of the cylinder which we take to be the z-axis.

The electric fields $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ are produced by the long electric charged filament and the uniformly distributed electric charges interior the cylinder, respectively. As a result, these two electric fields are in the radial direction. Explicitly, the electric and magnetic fields we considered are

$$\boldsymbol{E}^{(1)} = \frac{k}{r}\boldsymbol{e}_r, \quad \boldsymbol{E}^{(2)} = \frac{\rho}{2} r \boldsymbol{e}_r, \tag{1}$$

$$\boldsymbol{B} = B\boldsymbol{e}_z, \tag{2}$$

where k and ρ are parameters which are characters of these two electric fields, and $\boldsymbol{e}_r$ is the unit vector along the radial direction on the plane. Besides the electric and magnetic fields (1) and (2), the atom is trapped by a harmonic potential simultaneously. The harmonic potential and electric as well as magnetic fields are designed to make the motion of the atom be rotationally symmetric.

In Ref. [7], the authors showed that a neutral atom with an induced electric dipole moment would receive a topological phase if it moves around this uniformly electric charged filament in the presence of the magnetic field (2). By confining the atom on a rigid circle, the authors of Ref. [38] investigated the eigenvalue problem of the model considered here.

Due to the electric fields, the atom will be polarized, i.e., it will induce an electric dipole moment

$$\boldsymbol{d} = \alpha\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right), \tag{3}$$

where α, $\boldsymbol{v}$, and c are the dielectric polarizability, velocity of the atom, and speed of light in the vacuum, respectively, and $\boldsymbol{E}$ is the summation of the two electric fields, $\boldsymbol{E} = \boldsymbol{E}^{(1)} + \boldsymbol{E}^{(2)}$. The second term on the right-hand side of the above equation actually is the relativistic effect, it reflects the fact that a moving particle in a magnetic field will feel an electric field $\sim \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$.[39]

Taking the electric and magnetic fields (1) and (2) into account and trapping the atom by a harmonic potential, we get the Lagrangian which describes the dynamics of the atom. It is

$$L = \frac{1}{2}m\boldsymbol{v}^2 + \frac{1}{2}\boldsymbol{d}\cdot\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) - \frac{1}{2}K\boldsymbol{r}^2, \tag{4}$$

where the last term is the harmonic potential provided by a trap.

Substituting the expression $\boldsymbol{d} = \alpha(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B})$ into the above Lagrangian and confining the motion of the atom in the plane perpendicular to the magnetic field inside the cylinder, we simplify the Lagrangian (4) to the form (the Latin indexes i, j take values 1, 2 and the summation convention is applied throughout the present paper)

$$L = \frac{1}{2}M\dot{x}_i^2 - \frac{\alpha B}{c}\varepsilon_{ij}\dot{x}_i E_j + \frac{1}{2}\alpha E_i^2 - \frac{1}{2}Kx_i^2, \tag{5}$$

where $M = m + \alpha B^2/c^2$ is the effective mass.

We should quantize the model (5) before studying its quantum properties. To this end, we define the canonical momenta with respect to variables x_i

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = M\dot{x}_i - \frac{\alpha B}{c}\varepsilon_{ij}E_j. \tag{6}$$

The classical Poisson brackets among canonical variables x_i, p_i are

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}. \tag{7}$$

Then the canonical Hamiltonian is achieved by the Legendre

transformation

$$H=\frac{1}{2M}\left(p_i+\frac{\alpha B}{c}\varepsilon_{ij}E_j\right)^2-\frac{1}{2}\alpha E_i^2+\frac{1}{2}Kx_i^2. \tag{8}$$

The canonical quantization is accomplished when the replacements

$$x_i\to x_i,\quad p_i\to -\mathrm{i}\hbar\frac{\partial}{\partial x_i},\quad \{\ ,\ \}\to\frac{1}{\mathrm{i}\hbar}[\ ,\ ]$$

in the classical Hamiltonian (8) and the Poisson brackets (7) are complete.

The canonical angular momentum is

$$J=\varepsilon_{ij}x_ip_j, \tag{9}$$

which is proved to be conserved, i.e., $[J,\ H]=0$. It can also be written as $J=-\mathrm{i}\hbar\partial/\partial\varphi$, where φ is the azimuth angle. Obviously, eigenvalues of this canonical angular momentum must be quantized,

$$J_n=n\hbar,\ \ n=0,\pm1,\pm2,\ldots. \tag{10}$$

Besides this, the rotation symmetry of the model (5) on the plane can also be verified since $[J,\ x_i]=\mathrm{i}\hbar\varepsilon_{ij}x_j$.

Now, we consider the reduced model which is the limit of taking the kinetic energy in Eq. (5) to be negligibly small. This may be realized in experiments by cooling down the atom to a slower velocity so that the effective kinetic energy can be neglected (in an experiment carried out in the early of 1990s, the velocity of atoms can be cooled down to $\sim 1\ \mathrm{m\cdot s^{-1}}$ [40]). This kind of reduction is first considered in Ref. [41] during the studies of the Chern–Simons quantum mechanics.

The reduced model is described by the Lagrangian

$$L_{\mathrm{r}}=-\frac{\alpha B}{c}\varepsilon_{ij}\dot{x}_iE_j+\frac{1}{2}\alpha E_i^2-\frac{1}{2}Kx_i^2, \tag{11}$$

from which we get canonical momenta with respect to variables x_i. They are

$$p_i=\frac{\partial L_{\mathrm{r}}}{\partial\dot{x}_i}=-\frac{\alpha B}{c}\varepsilon_{ij}E_j. \tag{12}$$

The Hamiltonian of the reduced model can be read directly from the Lagrangian (11). It is

$$H_{\mathrm{r}}=\frac{K}{2}x_i^2-\frac{1}{2}\alpha E_i^2. \tag{13}$$

The righthand side terms of Eq. (12) do not contain velocities, thus, they are in fact the primary constraints in the terminology of Dirac.[42] We label them as

$$\phi_i^{(0)}=p_i+\frac{\alpha B}{c}\varepsilon_{ij}E_j\approx 0, \tag{14}$$

in which '$\approx$' means equivalent on the constraint surface. The existence of primary constraints shows that there are dependent degrees of freedom in the reduced model (11). The classical Poisson brackets among these two primary constraints can be obtained by a straightforward calculation. They are

$$\left\{\phi_i^{(0)},\ \phi_j^{(0)}\right\}=\frac{\alpha\rho B}{c}\varepsilon_{ij}. \tag{15}$$

Since $\{\phi_i^{(0)},\ \phi_j^{(0)}\}\neq 0$, the primary constraints $\phi_i^{(0)}$ belong to the second class and there are no secondary constraints. Therefore, the constraints $\phi_i^{(0)}$ can be used to eliminate the dependent degrees of freedom in the reduced model (11).

The canonical angular momentum in this reduced model has the same expression as Eq. (9), i.e., $J=\varepsilon_{ij}x_ip_j$. Since there are constraints $\phi_i^{(0)}\approx 0$ which lead to the dependence among canonical variables x_i, p_i, we rewrite the canonical angular momentum by substituting the constraints (14) into $J=\varepsilon_{ij}x_ip_j$, and obtain

$$J_{\mathrm{r}}=\varepsilon_{ij}x_ip_j=\frac{\alpha B}{c}x_iE_i. \tag{16}$$

Considering the explicit form of electric field (1), we get the canonical angular momentum of the reduced model. It is

$$J_{\mathrm{r}}=\frac{\alpha B}{c}x_i\left(E_i^{(1)}+E_i^{(2)}\right)=\frac{\alpha B}{c}\left(k+\frac{\rho}{2}x_i^2\right). \tag{17}$$

It is more convenient to get eigenvalues of the angular momentum (17) by algebraic method. In doing so, we must determine the commutator between x_i before further proceeding. The classical version of the commutator, i.e., the Dirac bracket, can be calculated by the definition[42]

$$\{x_i,\ x_j\}_{\mathrm{D}}=\{x_i,\ x_j\}-\{x_i,\ \phi_k^{(0)}\}\{\phi_k^{(0)},\ \phi_l^{(0)}\}^{-1}\{\phi_l^{(0)},\ x_j\}. \tag{18}$$

Upon straightforward algebraic calculation, we arrive at

$$\{x_i,\ x_j\}_{\mathrm{D}}=-\frac{c\varepsilon_{ij}}{\alpha\rho B}. \tag{19}$$

Thus, the commutators between x_i are

$$[x_i,\ x_j]=-\frac{\mathrm{i}\hbar c\varepsilon_{ij}}{\alpha\rho B}. \tag{20}$$

In view of these commutators and the angular momentum as well as the Hamiltonian of the reduced model (13), (17), we can get

$$[J_{\mathrm{r}},\ H_{\mathrm{r}}]=0,\qquad [J_{\mathrm{r}},\ x_i]=\mathrm{i}\hbar\varepsilon_{ij}x_j, \tag{21}$$

which show that the angular momentum of the reduced model J_{r} is still conserved and the rotation symmetry of the reduced model is still held.

Taking into account the above commutator, it is clear to see that apart from the term αBk, the canonical angular momentum (17) is equivalent to a one-dimensional harmonic oscillator. With the help of the commutators (20), one can write down the eigenvalues of the canonical angular momentum (12) immediately. They are

$$J_{\mathrm{r}n}=\frac{\alpha Bk}{c}+\left(n+\frac{1}{2}\right)\hbar. \tag{22}$$

Therefore, it shows that eigenvalues of the canonical angular momentum will take fractional values when its kinetic energy is cooled down to the negligibly small. The fractional part is determined by both the intensity of the applied magnetic field and the electric field $\boldsymbol{E}^{(1)}$.

From the eigenvalues of the canonical angular momentum (22), it seems that the electric field $\boldsymbol{E}^{(2)}$ does not have any influences on the FAM since the parameter ρ does not appear in Eq. (22) explicitly. In fact, the electric field $\boldsymbol{E}^{(2)}$ also plays important roles in producing the FAM. In the next section, we analyze the roles that two electric fields $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ played.

3. Roles two electric fields played

As we showed that besides the intensity of the magnetic field, the fractional part of the canonical angular momentum only contains the parameter k. Thus it seems that only the electric field $\boldsymbol{E}^{(1)}$ contributes to the FAM. In the following, we show that the electric field $\boldsymbol{E}^{(2)}$ also plays important roles in producing the FAM since the FAM will not appear in the absence of either of the electric fields.

First of all, we consider the case that the electric field $\boldsymbol{E}^{(1)}$ is turned off. In this case, the dynamics is determined by the Lagrangian

$$\bar{L}=\frac{1}{2}M\dot{x}_i^2-\frac{\alpha B}{c}\varepsilon_{ij}\dot{x}_iE_j^{(2)}+\frac{1}{2}\alpha\left(E_i^{(2)}\right)^2-\frac{1}{2}Kx_i^2. \tag{23}$$

Compared with Lagrangian (5) in which both of the electric fields are present, we find that the only difference is that the term $E_i=E_i^{(1)}+E_i^{(2)}$ is replaced by $E_i^{(2)}$.

The canonical momenta with respective to x_i are given by

$$p_i=\frac{\partial\bar{L}}{\partial\dot{x}_i}=M\dot{x}_i-\frac{\alpha B}{c}\varepsilon_{ij}E_j^{(2)}. \tag{24}$$

The model (23) can be quantized directly. The canonical angular momentum is defined as usual $\bar{J}=\varepsilon_{ij}x_ip_j=-\mathrm{i}\hbar\partial/\partial\varphi$ and its eigenvalues are $J_n=n\hbar,\ 0,\pm1,\pm2,\ldots$. It seems that as far as the rotation property is concerned, there is no difference between the models (5) and (23).

However, when the atom is cooled down to the negligibly small kinetic energy, their difference appears. To see it clearly, we set the effective kinetic energy term to zero in Lagrangian (23) in this limit. Therefore, the Lagrangian (23) reduces to

$$\bar{L}_{\mathrm{r}}=-\frac{\alpha B}{c}\varepsilon_{ij}\dot{x}_iE_j^{(2)}+\frac{1}{2}\alpha\left(E_i^{(2)}\right)^2-\frac{1}{2}Kx_i^2. \tag{25}$$

Introducing the canonical momenta with respective to x_i, we get two primary constraints as

$$\bar{\phi}_i^{(0)}=p_i+\frac{\alpha B}{c}\varepsilon_{ij}E_j^{(2)}\approx0. \tag{26}$$

The Poisson brackets between constraints (26) are

$$\left\{\bar{\phi}_i^{(0)},\ \bar{\phi}_j^{(0)}\right\}=\frac{\alpha B}{c}\rho\varepsilon_{ij}, \tag{27}$$

which are equivalent to Eq. (15). Therefore, they are the second class and can be used to eliminate the dependent degrees of freedom. Substituting the constraints (26) into canonical angular momentum $\bar{J}=\varepsilon_{ij}x_ip_j$, we find that the canonical angular momentum takes the from

$$\bar{J}=\frac{\alpha B}{c}x_iE_i^{(2)}=\frac{\alpha\rho B}{2c}x_i^2 \tag{28}$$

in this limit. Its eigenvalues can be obtained once we get the commutators between x_i. It can be checked that the Dirac brackets between x_i are nothing but Eq. (19). Thus, eigenvalues of the angular momentum are $\bar{J}_n=(n+\frac{1}{2})\hbar,\ n=0,\pm1,\pm2,\ldots$. Therefore, the electric field $\boldsymbol{E}^{(2)}$ alone can not produce the FAM.

On the contrary, if we turn off the electric field $\boldsymbol{E}^{(2)}$ and let $\boldsymbol{E}^{(1)}$ alone, the Lagrangian (5) becomes

$$\tilde{L}=\frac{1}{2}M\dot{x}_i^2-\frac{\alpha B}{c}\varepsilon_{ij}\dot{x}_iE_j^{(1)}+\frac{1}{2}\alpha\left(E_i^{(1)}\right)^2-\frac{1}{2}Kx_i^2. \tag{29}$$

We introduce the canonical momentum $p_i=\partial\tilde{L}/\partial\dot{x}_i=M\dot{x}_i-\frac{\alpha B}{c}\varepsilon_{ij}E_j^{(1)}$ and quantize the model (29) canonically. Then eigenvalues of the canonical angular momentum $\tilde{J}=\varepsilon_{ij}x_ip_j=-\mathrm{i}\hbar\partial/\partial\varphi$ must be quantized as $\tilde{J}_n=n\hbar,\ n=0,\ \pm1,\ \pm2,\ldots$.

The reduced model of the Lagrangian (29) turns out to be

$$\tilde{L}_{\mathrm{r}}=-\frac{\alpha B}{c}\varepsilon_{ij}\dot{x}_iE_j^{(1)}+\frac{1}{2}\alpha\left(E_i^{(1)}\right)^2-\frac{1}{2}Kx_i^2. \tag{30}$$

The Hamiltonian corresponding to this Lagrangian can be read directly from the above Lagrangian.[43] It is

$$\tilde{H}_{\mathrm{r}}=-\frac{1}{2}\alpha\left(E_i^{(1)}\right)^2+\frac{1}{2}Kx_i^2. \tag{31}$$

We define canonical momenta from the Lagrangian (30). They are

$$p_i=\frac{\partial\tilde{L}_{\mathrm{r}}}{\partial\dot{x}_i}=-\frac{\alpha B}{c}\varepsilon_{ij}E_j^{(1)}. \tag{32}$$

Once again, the introduction of canonical momenta leads to two primary constraints

$$\tilde{\phi}_i^{(0)}=p_i+\frac{\alpha B}{c}\varepsilon_{ij}E_j^{(1)}\approx0. \tag{33}$$

Different from Eqs. (15) and (27), the Poisson brackets between constraints $\tilde{\phi}_i^{(0)}\approx0$ in the present case are vanishing, i.e., $\{\tilde{\phi}_i^{(0)},\ \tilde{\phi}_j^{(0)}\}=0$. It means that there are secondary constraints. Each of the primary constraints (33) will lead to secondary constraints.

By applying the consistency condition to the primary constraints $\tilde{\phi}_i^{(0)}\approx0$, we get

$$\tilde{\phi}_i^{(1)}=\left\{\tilde{\phi}_i^{(0)},\ H\right\}=-\frac{\alpha k}{r^2}E_i^{(1)}-Kx_i\approx0. \tag{34}$$

We label the primary constraints (33) and the secondary constraints (34) in a unified way as $\Phi_I=(\tilde{\phi}_i^{(0)},\ \tilde{\phi}_i^{(1)}),\ I=1,2,3,4$. It can be verified that $\mathrm{Det}\{\Phi_I,\ \Phi_J\}\neq0$. Thus, there are no further constraints and all the constraints Φ_I are second class.

It means that when we turn off the electric field $\boldsymbol{E}^{(2)}$, the reduced model (29) does not have dynamical degrees of freedom. Thus, the electric field $\boldsymbol{E}^{(2)}$ plays important roles in producing the FAM: although it does not contribute to the frac-

tional part of the angular momentum directly, the FAM will not appear in its absence.

4. Conclusions and remarks

In this paper, we propose to simulate an anyon which was usually realized by a charged particle originally by using a trapped cold atom. This atom possesses an induced electric dipole moment interacting with electric and magnetic fields. The electromagnetic fields we applied contain a uniform magnetic field and two electric fields.

We prove that the canonical angular momentum of the model (4) can only take integer values. However, its reduced model which is obtained by cooling down the atom to the limit of the negligibly small kinetic energy produces the FAM. The magnitude of the FAM can be modulated by two parameters, i.e., the intensity of the applied magnetic field and the electric field $\boldsymbol{E}^{(1)}$. Apart from the fractional part, it is also interesting to observe that the differences between eigenvalues of canonical angular momentum are half integers. It is one of the characteristics of Chern–Simons quantum mechanics. In Ref. [44] the author proposed to realize the Chern–Simons quantum mechanics model by a cold Rydberg atom.

All the electric and magnetic fields play important roles in the simulation of FAM. The effect of the electric field $\boldsymbol{E}^{(1)}$ is evident since the magnitude of the FAM is proportional to the parameter k, which is the strength of electric field $\boldsymbol{E}^{(1)}$. Roles the electric field $\boldsymbol{E}^{(2)}$ played are subtle. At the first glance, the electric field $\boldsymbol{E}^{(2)}$ does not contribute to the fractional part of the angular momentum. However, it does influence the results since the FAM will not appear in its absence.

Besides the contribution to the effective mass, roles the magnetic field played will be more transparent if we introduce the effective gauge potentials $A_i^{\rm eff} = \varepsilon_{ij}E_j$ and rewrite the Lagrangian (5) as

$$L = \frac{1}{2}M\dot{x}_i^2 - \frac{\alpha B}{c}A_i^{\rm eff}\dot{x}_i + \frac{1}{2}\alpha E_i^2 - \frac{1}{2}Kx_i^2.$$

The interaction term is similar with a charged particle minimally coupling a gauge field. The magnetic field (2) acts as the coupling strength. Therefore, the magnetic field not only contributes to the mass of the atom, but also is the coupling strength of the interaction between the atom and the electric fields which is of fundamental importance in producing the FAM.

References

[1] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[2] Aharonov Y and Casher A 1994 *Phys. Rev. Lett.* **53** 319
[3] He X G and Mckellar B H J 1993 *Phys. Rev. A* **47** 3424
[4] Wilkens M 1994 *Phys. Rev. Lett.* **72** 5
[5] Dowling J P, Williams C P and Franson J D 1999 *Phys. Rev. Lett.* **83** 2486
[6] Wilkens M 1998 *Phys. Rev. Lett.* **81** 1534
[7] Wei H, Han R and Wei X 1995 *Phys. Rev. Lett.* **75** 2071
[8] Ericsson M and Sjöqvist E 2001 *Phys. Rev. A* **65** 013607
[9] Ribeiro L R, Furtado C and Nascimento J R 2006 *Phys. Lett. A* **348** 135
[10] Furtado C, Nascimento J R and Ribeiro L R 2006 *Phys. Lett. A* **358** 336
[11] Banerjee S, Agren H and Balatsky A V 2016 *Phys. Rev. B* **93** 235134
[12] Bakke K and Furtado C 2009 *Phys. Rev. A* **80** 032106
[13] Bakke K 2010 *Phys. Rev. A* **81** 052117
[14] Bakke K, Ribeiro L R, Furtado C and Nascimento J R 2009 *Phys. Rev. D* **79** 024008
[15] Bakke K, Ribeiro L R and Furtado C 2010 *Cent. Eur. J. Phys.* **8** 893
[16] Oliveira A B and Bakke K 2016 *Proc. R. Soc. A* **472** 20150858
[17] Oliveira A B and Bakke K 2016 *Int. J. Mod. Phys. A* **31** 1650019
[18] Oliveira A B and Bakke K 2016 *Eur. Phys. J. Plus* **131** 266
[19] Oliveira A B and Bakke K 2016 *Ann. Phys.* **365** 66
[20] Oliveira A B and Bakke K 2017 *R. Soc. Open Sci.* **4** 170541
[21] Basu B, Dhar D and Chatterjee S 2008 *Phys. Lett. A* **372** 4319
[22] Lerda A 1992 *Anyons, Quantum Mechanics of Particles with Fractional Statistics, Lecture Notes in Physics*, New Series m: Monographs (Berlin: Springer-Verlag) p. 39
[23] Khare A 2005 *Fractional Statistics and Quantum Theory* 2nd Edn. (Singapore: World Scientific) p. 32
[24] Dirac P A M 1958 *The Principles of Quantum Mechanics* 4th Edn. (OXford: Oxford University Press) p. 144
[25] Sakurai J J 2017 *Modern Quantum Mechanics* (Cambridge University Press) p. 157
[26] Wilcezk F 1982 *Phys. Rev. Lett.* **48** 1144
[27] Liang J Q and Ding X X 1988 *Phys. Rev. Lett.* **60** 836
[28] Chakraborty T and Pietiläinen P 1995 *The Quantum Hall Effect, fractional and Integral* 2nd Edn. (Berlin: Springer-Verlag) p. 120
[29] Wilczek F 1990 *Fractional Statistics and Anyon Superconductivity* (Singapore: World Scientific) p. 65
[30] Deser S, Jackiw R and Templeton S 1982 *Ann. Phys.* **140** 372
[31] Zhang S, Hanson T and Kivelson S 1989 *Phys. Rev. Lett.* **62** 82
[32] Jackiw R and Pi S Y 1991 *Phys. Rev. D* **44** 2524
[33] Forte S 1992 *Rev. Mod. Phys.* **64** 193
[34] Zhang Y, Sreejith G J, Gemelke N D and Jain J K 2014 *Phys. Rev. Lett.* **113** 160404
[35] Zhang Y, Sreejith G J and Jain J K 2015 *Phys. Rev. B* **92** 075116
[36] Lundholm D and Rougerie N 2016 *Phys. Rev. Lett.* **116** 170401
[37] Zhang J Z 2008 *Phys. Lett. B* **670** 205
[38] Azevedo F S, Silva E O, Castro L B, Filgueiras C and Cogollo D 2015 *Ann. Phys.* **362** 196
[39] Jackson J D 1999 *Classical electrodynamics* 3 Edn. (New York: Wiley) p. 583
[40] Shimizu F, Shimizu K and Takuma H 1991 *Opt. Lett.* **16** 339
[41] Dunne G V, Jackiw R and Trugenberger C A 1990 *Phys. Rev. D* **41** 661
[42] Dirac P A M 1964 *Lecture notes on quantum mechanics* (New York: Yeshiva University) p. 41
[43] Faddeev L D and Jackiw R 1988 *Phys. Rev. Lett.* **60** 1691
[44] Baxter C 1995 *Phys. Rev. Lett.* **74** 514