Soliton molecules and dynamics of the smooth positon for the Gerdjikov–Ivanov equation*

Xiangyu Yang(杨翔宇), Zhao Zhang(张钊), and Biao Li(李彪)†

School of Mathematics and Statistics, Ningbo University, Ningbo 315211, *China*

(Received 13 February 2020; revised manuscript received 5 June 2020; accepted manuscript online 18 June 2020)

Soliton molecules are firstly obtained by velocity resonance for the Gerdjikov–Ivanov equation, and n-order smooth positon solutions for the Gerdjikov–Ivanov equation are generated by means of the general determinant expression of n-soliton solution. The dynamics of the smooth positons of the Gerdjikov–Ivanov equation are discussed using the decomposition of the modulus square, the trajectories and time-dependent "phase shifts" of positons after the collision can be described approximately. Additionally, some novel hybrid solutions consisting solitons and positons are presented and their rather complicated dynamics are revealed.

Keywords: soliton molecules, degenerate Darboux transformation, positons, phase shift, Gerdjikov–Ivanov equation

PACS: 05.45.Yv, 02.30.Ik **DOI:** 10.1088/1674-1056/ab9de0

1. Introduction

The derivative nonlinear Schrödinger (DNLS) equation has been regarded as a model in a wide variety of fields such as weakly nonlinear dispersive water waves,[1] nonlinear optics fibers,[2] quantum field theory[3] and plasmas.[4] The DNLS equations have three generic deformations, the DNLS (also named as the Kaup–Newell equation),[5] the DNLS equation (also called the Chen–Lee–Liu equation)[6] and the DNLS equation, also known as the Gerdjikov–Ivanov (GI) equation[7]

$$\mathrm{i}q_t + q_{xx} - \mathrm{i}q^2 q_x^* + \frac{1}{2}q^3(q^*)^2 = 0, \tag{1}$$

where $\mathrm{i}^2 = -1$, the asterisk represents the complex conjugation, the subscripts denote the partial derivatives, x and t represent the space and time coordinates. The GI equation can also be regarded as an extension of the NLS when certain higher-order nonlinear effects are taken into account. The GI equation appears in quanta field theory, nonlinear optics, weak nonlinear dispersive water wave,[8] and it is a model for Alfvén waves propagating parallel to the ambient magnetic field, with q being the transverse magnetic field perturbation and x and t being space and time coordinates, respectively.[9] The Darboux transformation and its determinant expression for the GI equations have been provided in Refs. [10,11]. Breather solution is obtained by nonzero seed solution. Rouge wave and higher-order rouge solutions have been shown in Refs. [11,12].

Soliton molecules, constructed from a number of "atoms" each being a fundamental soliton, have been become one of the most challenging open frontiers of the field.[13] Solitons can form bound states that are frequently referred to as soliton molecules as they exhibit molecule-like dynamics. Investigation on soliton molecules provides an effective way to study soliton interactions, and the formation and dissociation of soliton molecules are closely linked to soliton collision, soliton splashing and the trapping of solitons. Soliton molecules also present the possibility of transferring optical data surpassing the limitation of binary coding.[14] Recently, a conceptually different soliton molecule, the intermittent-vibration soliton molecule, is discovered and characterized.[15] By means of an emerging time-stretch technique, Herink *et al.*[16] resolved the evolution of femtosecond soliton molecules in the cavity of a few-cycle mode-locked laser. Liu[17] have reported the first observation of the entire buildup process of soliton molecules in a mode-locked laser. Peng *et al.*[18] unveiled the build-up of dissipative soliton in mode-locked fibre lasers and employed autocorrelation analysis to investigate temporal evolution. In Ref. [19], experimental investigations on the dynamics of soliton molecules in the normal-dispersion regime were firstly presented. In addition, the authors generated breathers in a mode-locked laser.[20] Furthermore, the velocity resonance is a new possible mechanism to form soliton molecules in a fluid model theoretically.[21–24]

Positon solution can be regarded as a special soliton solution, which has a strong correlation potential of super transparency in quantum physical field. Matveev first discovered positon solution when considering the Korteweg-de Vries (KdV) equation.[25] He showed a new family of solutions of the KdV equation in elementary functions with the help of the limiting procedure $\lambda_i \to \lambda_j$ and called them "positons".

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11775121 and 11435005), and the K. C. Wong Magna Fund in Ningbo University.

†Corresponding author. E-mail: libiao@nbu.edu.cn

 http://iopscience.iop.org/cpb http://cpb.iphy.ac.cn

Dynamic properties of positons of the KdV equation was discussed in detail, the new features of positons of higher order were shown,[26] and extension of the analysis to positons for other nonlinear evolution equations was indicated. In Ref. [27] positons have been established as a singular limit of a two-soliton expression, and extended to (2+1)-dimensional phenomena. The dark soliton, bright soliton and positon solutions to NLS equations with self-consistent sources and their properties were analyzed.[28] Moreover, Dubard *et al.*[29] described some basic properties of multi-positon and positon-soliton solutions to the KdV equations and speculated about their possible links with freak waves.

In recent years, adopting different representations of a general solution through zero background "seed" solution of the corresponding equations to obtain solutions is an important method to obtain positons, which is different from rouge waves.[30–32] Also, smooth positons of other systems were studied, such as the complex modified KdV equation, the derivative nonlinear Schrödinger equation, and the Kundu–Eckhaus equation.[33–35] To our knowledge, the soliton molecules, the dynamic of positon solution and hybrid solution for Eq. (1) have not been studied.

The organization of the paper is as follows. In Section 2, we present the velocity resonance method to generate soliton molecules of the GI equation. In Section 3, we present the determinant expression of n-order positon solution to the GI equation by degeneration DT. As the application of the formula, we give the explicit expression of two-positon solution and plot the two-, three- and four-positon solutions. In Section 4, the trajectory of positon solutions and and time-varying "phase shift" of two- and three-positon solutions after collisions are discussed in detail. In Section 5, hybrid solutions consisting of solitons and positons are investigated. The conclusion is provided in Section 6.

2. Soliton molecules of the Gerdjikov–Ivanov equation

Considering the spectral problem

$$\begin{cases} \Psi_x = (J\lambda^2 + Q_1\lambda + Q_0)\Psi = U\Psi, \\ \Psi_t = (2J\lambda^4 + V_3\lambda^3 + V_2\lambda^2 + V_1\lambda + V_0)\Psi = V\Psi, \end{cases} \tag{2}$$

with

$$\Psi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad J = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} -\frac{1}{2}\mathrm{i}qr & 0 \\ 0 & \frac{1}{2}\mathrm{i}qr \end{pmatrix},$$

$$V_3 = 2Q_1, \quad V_2 = Jqr, \quad V_1 = \begin{pmatrix} 0 & \mathrm{i}q_x \\ -\mathrm{i}r_x & 0 \end{pmatrix},$$

$$V_0 = \begin{pmatrix} \frac{1}{2}(rq_x - qr_x) + \frac{1}{4}\mathrm{i}q^2r^2 & 0 \\ 0 & -\frac{1}{2}(rq_x - qr_x) - \frac{1}{4}\mathrm{i}q^2r^2 \end{pmatrix},$$

where q and r are two potentials, Ψ is the eigenfunction of Eq. (2) corresponding to the complex spectral parameter λ. The zero curvature equation $U_t - V_x + [U,V] = 0$ infers the following system

$$\begin{cases} \mathrm{i}q_t + q_{xx} + \mathrm{i}q^2r_x + \frac{1}{2}q^3r^2 = 0, \\ \mathrm{i}r_t - r_{xx} + \mathrm{i}r^2q_x - \frac{1}{2}q^2r^3 = 0. \end{cases} \tag{3}$$

This system admits the reduction $r = -q^*$ and reduces to the GI equation (1). The determinant representation of the n-fold DT for the GI equation has been given in Ref. [11]. We cite the main theorem as follows.

Theorem 1 Let $\begin{pmatrix} \phi_i \\ \psi_i \end{pmatrix}$ $(i = 1, 2, \ldots, 2n)$ be distinct solutions related to λ_i of the spectral problem, then $(q^{[n]}, r^{[n]})$ given by the following formulae are new solutions to the GI equation:

$$q^{[n]} = q + 2\mathrm{i}\frac{\Omega_{11}}{\Omega_{12}}, \quad r^{[n]} = r - 2\mathrm{i}\frac{\Omega_{21}}{\Omega_{22}}. \tag{4}$$

Here

$$\Omega_{11} = \begin{vmatrix} \phi_1 & \lambda_1\psi_1 & \cdots & \lambda_1^{2n-3}\psi_1 & \lambda_1^{2n-2}\phi_1 & -\lambda_1^{2n}\phi_1 \\ \phi_2 & \lambda_2\psi_2 & \cdots & \lambda_2^{2n-3}\psi_2 & \lambda_2^{2n-2}\phi_2 & -\lambda_2^{2n}\phi_2 \\ \phi_3 & \lambda_3\psi_3 & \cdots & \lambda_3^{2n-3}\psi_3 & \lambda_3^{2n-2}\phi_3 & -\lambda_3^{2n}\phi_3 \\ \phi_4 & \lambda_4\psi_4 & \cdots & \lambda_4^{2n-3}\psi_4 & \lambda_4^{2n-2}\phi_4 & -\lambda_4^{2n}\phi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{2n} & \lambda_{2n}\psi_{2n} & \cdots & \lambda_{2n}^{2n-3}\psi_{2n} & \lambda_{2n}^{2n-2}\phi_{2n} & -\lambda_{2n}^{2n}\phi_{2n} \end{vmatrix},$$

$$\Omega_{12} = \begin{vmatrix} \phi_1 & \lambda_1\psi_1 & \cdots & \lambda_1^{2n-3}\psi_1 & \lambda_1^{2n-2}\phi_1 & \lambda_1^{2n-1}\psi_1 \\ \phi_2 & \lambda_2\psi_2 & \cdots & \lambda_2^{2n-3}\psi_2 & \lambda_2^{2n-2}\phi_2 & \lambda_2^{2n-1}\psi_2 \\ \phi_3 & \lambda_3\psi_3 & \cdots & \lambda_3^{2n-3}\psi_3 & \lambda_3^{2n-2}\phi_3 & \lambda_3^{2n-1}\psi_3 \\ \phi_1 & \lambda_4\psi_4 & \cdots & \lambda_4^{2n-3}\psi_4 & \lambda_4^{2n-2}\phi_4 & \lambda_4^{2n-1}\psi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{2n} & \lambda_{2n}\psi_{2n} & \cdots & \lambda_{2n}^{2n-3}\psi_{2n} & \lambda_{2n}^{2n-2}\phi_{2n} & \lambda_{2n}^{2n-1}\psi_{2n} \end{vmatrix},$$

$$\Omega_{21} = \begin{vmatrix} \psi_1 & \lambda_1\phi_1 & \cdots & \lambda_1^{n-3}\phi_1 & \lambda_1^{n-2}\psi_1 & -\lambda_1^{n}\psi_1 \\ \psi_2 & \lambda_2\phi_2 & \cdots & \lambda_2^{n-3}\phi_2 & \lambda_2^{n-2}\psi_2 & -\lambda_2^{n}\psi_2 \\ \psi_3 & \lambda_3\phi_3 & \cdots & \lambda_3^{n-3}\phi_3 & \lambda_3^{n-2}\psi_3 & -\lambda_3^{n}\psi_3 \\ \psi_4 & \lambda_4\phi_4 & \cdots & \lambda_4^{n-3}\phi_4 & \lambda_4^{n-2}\psi_4 & -\lambda_4^{n}\psi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{2n} & \lambda_{2n}\phi_{2n} & \cdots & \lambda_{2n}^{2n-3}\phi_{2n} & \lambda_{2n}^{2n-2}\psi_{2n} & -\lambda_{2n}^{2n}\psi_{2n} \end{vmatrix},$$

$$\Omega_{22} = \begin{vmatrix} \psi_1 & \lambda_1\phi_1 & \cdots & \lambda_1^{n-3}\phi_1 & \lambda_1^{n-2}\psi_1 & \lambda_1^{n-1}\phi_1 \\ \psi_2 & \lambda_2\phi_2 & \cdots & \lambda_2^{n-3}\phi_2 & \lambda_2^{n-2}\psi_2 & \lambda_2^{n-1}\phi_2 \\ \psi_3 & \lambda_3\phi_3 & \cdots & \lambda_3^{n-3}\phi_3 & \lambda_1^{n-2}\psi_3 & \lambda_3^{n-1}\phi_3 \\ \psi_4 & \lambda_4\phi_4 & \cdots & \lambda_4^{n-3}\phi_4 & \lambda_1^{n-2}\psi_4 & \lambda_4^{n-1}\phi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{2n} & \lambda_{2n}\phi_{2n} & \cdots & \lambda_{2n}^{2n-3}\phi_{2n} & \lambda_{2n}^{2n-2}\psi_{2n} & \lambda_{2n}^{2n-1}\phi_{2n} \end{vmatrix}. \tag{5}$$

We start with the soliton solutions by Eq. (4) and assume hereafter that eigenfunctions and eigenvalues are as follows:

$$\lambda_{2n} = \lambda_{2n-1}^*, \quad \Psi_{2n} = \begin{pmatrix} \phi_{2n} \\ \psi_{2n} \end{pmatrix} = \begin{pmatrix} -\psi_{2n-1}^* \\ \phi_{2n-1}^* \end{pmatrix}. \tag{6}$$

The eigenfunctions of the spectral problem (2) by setting seed solution $q = 0$ with eigenvalues $\lambda_j = \alpha_j + \mathrm{i}\beta_j$ are solved as

$$\Psi_j = \Psi(\lambda_j) = \begin{pmatrix} \phi_j(\lambda_j) \\ \psi_j(\lambda_j) \end{pmatrix} = \begin{pmatrix} \exp(-\mathrm{i}\lambda_j^2(2\lambda_j^2 t + x)) \\ \exp(\mathrm{i}\lambda_j^2(2\lambda_j^2 t + x)) \end{pmatrix}. \tag{7}$$

When $n=1$, let $\lambda_2=\lambda_1^*$, and $\Psi_1=\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}$, $\Psi_2=\begin{pmatrix}-\psi_1^*\\ \phi_1^*\end{pmatrix}$, the explicit formula of one-soliton solution is

$$q_{1-s}=\frac{16\mathrm{e}^{-2\mathrm{i}h}\alpha_1\beta_1}{\alpha_1\cosh(4\alpha_1\beta_1 H)-\mathrm{i}\beta_1\sinh(4\alpha_1\beta_1 H)},\tag{8}$$

with $h=\alpha_1^2x+2\alpha_1^4t-12\alpha_1^2\beta_1^2\,t-\beta_1^2x+2\beta_1^4\,t$, $H=4\,t\alpha_1^2-4\,t\beta_1^2+x$. The trajectory of wave crest is

$$x=-4t(\alpha_1^2-\beta_1^2).\tag{9}$$

The key to generating soliton molecules is to start with eigenfunctions with a "phase shift" by choosing zero "seed solution" $q=0$, the general form of eigenfunctions is as follows:

$$\Psi_j=\Psi(\lambda_j)=\begin{pmatrix}\phi_j(\lambda_j)\\ \psi_j(\lambda_j)\end{pmatrix}=\begin{pmatrix}\exp(-\mathrm{i}\lambda_j^2(2\lambda_j^2t+x)-\xi)\\ \exp(\mathrm{i}\lambda_j^2(2\lambda_j^2t+x)+\xi)\end{pmatrix},\tag{10}$$

where ξ is a real constant and $\lambda_j=\alpha_j+i\beta_j$.

Then we can reach the following velocity resonance condition. In order to obtain a molecule consisting of n solitons, we can constrain the parameters in Eq. (4) as follows:

$$\begin{aligned}&\alpha_{2j-1}^2-\beta_{2j-1}^2=v_0,\quad \lambda_1\neq\lambda_3\neq\cdots\neq\lambda_{2j-1},\\ &j=1,2,\ldots,n,\end{aligned}\tag{11}$$

where v_0 is a real constant. Here we choose $v_0=1/4$, and soliton molecules consisting different solitons are shown in Fig. 1.

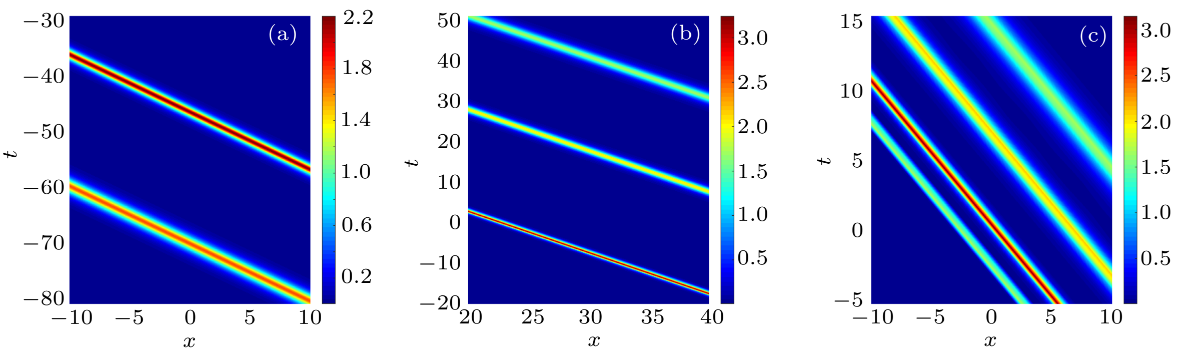

Fig. 1. (a) Soliton molecule consisting of two solitons with parameter selections $\lambda_1=\frac{2}{3}+\frac{\sqrt{7}\mathrm{i}}{6}$, $\lambda_3=\frac{3}{4}+\frac{\sqrt{5}\mathrm{i}}{4}$, $\xi=40$. (b) Soliton molecule consisting of three solitons with parameter selections $\lambda_1=\frac{2}{3}+\frac{\sqrt{7}\mathrm{i}}{6}$, $\lambda_3=1+\frac{\sqrt{3}\mathrm{i}}{2}$, $\lambda_5=\frac{3}{4}+\frac{\sqrt{5}\mathrm{i}}{4}$, $\xi=40$. (c) Soliton molecule consisting of four solitons with parameter selections $\lambda_1=\frac{2}{3}+\frac{\sqrt{7}\mathrm{i}}{6}$, $\lambda_3=1+\frac{\sqrt{3}\mathrm{i}}{2}$, $\lambda_5=\frac{3}{4}+\frac{\sqrt{5}\mathrm{i}}{4}$, $\lambda_7=\frac{4}{3}+\frac{\sqrt{55}\mathrm{i}}{6}$, $\xi=10$.

3. Smooth positons of the Gerdjikov–Ivanov equation

When $\lambda_1=\lambda_3$ and letting $n=2$ in Eq. (4), the denominator of the fraction is zero. Then the DT cannot carry on with the same eigenvalue. In order to overcome this deficiency, degenerate DT for the GI equation can be obtained by setting $\lambda_{2j-1}=\lambda_1+\varepsilon$ $(j=2,3,\ldots,n)$, the expression of $q^{[n]}$ will not be zero. In this section, smooth positons are obtained using degenerate DT and higher-order Taylor expansion from a zero seed solution to the GI equation. Substituting Eq. (7), $q=0$, and $\lambda_{2j-1}=\lambda_1+\varepsilon$ $(j=2,3,\ldots,n)$ into Eq. (12), using Taylor expansion to ε, we can obtain the smooth positon solutions to the GI equation. The determinant expression of the n-positive definite solution to the GI equation can be obtained from the following proposition.

Proposition Based on the degenerate limit $\lambda_{2j-1}\to\lambda_1$, n-order soliton solution from zero seed solution $q=0$ produces n-order positon solution q_{n-p} of the GI equation

$$q_{n-p}=2\mathrm{i}\frac{N'_{2n}}{W'_{2n}},\tag{12}$$

where

$$N'_{2n}=\left(\frac{\partial^{n_i-1}}{\partial\varepsilon^{n_i-1}}\bigg|_{\varepsilon=0}(\Omega_{11})_{ij}(\lambda_1+\varepsilon)\right)_{2n\times 2n},$$

$$W'_{2n}=\left(\frac{\partial^{n_i-1}}{\partial\varepsilon^{n_i-1}}\bigg|_{\varepsilon=0}(\Omega_{12})_{ij}(\lambda_1+\varepsilon)\right)_{2n\times 2n},$$

and $n_i=\left[\frac{i+1}{2}\right]$, $[i]$ is the floor function of i.

The reduction conditions are $\lambda_{2j}=\lambda_{2j-1}^*\phi_{2j}=-\psi_{2j-1}^*$, $\psi_{2j}=\phi_{2j-1}^*$ $(1,2,3,\ldots,n)$ in the above proposition.

Based on proposition 1, we can reach the exact formula of positon solutions conveniently, because the complex process of limits $\lambda_{2j-1}\to\lambda_1$ is avoided, we can construct higher-order positon solution easily. Higher-order positon solutions' expression is too tedious, we only present the explicit expression of the two-positon solution by setting $n=2$ in proposition 1. The explicit formula of the two-positon solution is

$$\begin{aligned}q_{2-p}&=\frac{A_1\mathrm{e}^{12\alpha_1\beta_1H-2\mathrm{i}h}+A_2\mathrm{e}^{4\alpha_1\beta_1H-2\mathrm{i}h}}{B_1\mathrm{e}^{8\alpha_1\beta_1H}+B_2},\\ A_1&=-16\alpha_1\beta_1(48\mathrm{i}\alpha_1^5\beta_1^2t+32\mathrm{i}\alpha_1^3\beta_1^4t-16\mathrm{i}\alpha_1\beta_1^6t\\ &\quad-16\alpha_1^6\beta_1t+32\alpha_1^4\beta_1^3t+48\alpha_1^2\beta_1^5t+4\mathrm{i}\alpha_1^3\beta_1^2x\\ &\quad+4\mathrm{i}\alpha_1\beta_1^4x-4\alpha_1^4\beta_1x-4\alpha_1^2\beta_1^3x-\mathrm{i}\beta_1^3+\alpha_1^3),\end{aligned}$$

$$
\begin{aligned}
A_2 =& -16\alpha_1\beta_1(48\mathrm{i}\alpha_1^5\beta_1^2t+32\mathrm{i}\alpha_1^3\beta_1^4t-16\mathrm{i}\alpha_1\beta_1^6t\\
&+16\alpha_1^6\beta_1t-32\alpha_1^4\beta_1^3t-48\alpha_1^2\beta_1^5t+4\mathrm{i}\alpha_1^3\beta_1^2x\\
&+4\mathrm{i}\alpha_1\beta_1^4x+4\alpha_1^4\beta_1x+4\alpha_1^2\beta_1^2x+\mathrm{i}\beta_1^3+\alpha_1^3),\\
B_1 =& -1024\alpha_1^{10}\beta_1^2t^2-4096\alpha_1^8\beta_1^4t^2-512\alpha_1^8\beta_1^2tx\\
&-6144\alpha_1^6\beta_1^6t^2-512\alpha_1^6\beta_1^4tx+128\mathrm{i}\alpha_1^6\beta_1^2t\\
&-64\alpha_1^6\beta_1^2x^2-4096\alpha_1^4\beta_1^8t^2+512\alpha_1^4\beta_1^6tx\\
&+32\mathrm{i}\alpha_1^4\beta_1^2x-128\alpha_1^4\beta_1^4x^2-32\mathrm{i}\alpha_1^2\beta_1^4x\\
&-1024\alpha_1^2\beta_1^{10}t^2+512\alpha_1^2\beta_1^8tx+128\mathrm{i}\alpha_1^2\beta_1^6t\\
&-64\alpha_1^2\beta_1^6x^2-768\mathrm{i}\alpha_1^4\beta_1^4t-2\alpha^4-2\beta^4,\\
B_2 =& (2\mathrm{i}\mathrm{e}^{16\alpha\beta H}\alpha\beta-\mathrm{e}^{16\alpha\beta H}\alpha^2+\mathrm{e}^{16\alpha\beta H}\beta^2\\
&-2\mathrm{i}\alpha\beta-\alpha^2+\beta^2)(\alpha^2+\beta^2),\\
h =& \alpha_1^2x+2\alpha_1^4t-12\alpha_1^2\beta_1^2t-\beta_1^2x+2\beta_1^4t,\\
H =& 4t\alpha_1^2-4t\beta_1^2+x.
\end{aligned}
\tag{13}
$$

We provide the three-dimensional plot and density plot for two-positon, three-positon and four-positon cases, see Figs. 2(a), 2(b), 3(a), 3(b), and 4.

4. Dynamics of the positons of the Gerdjikov–Ivanov equation

In order to find out positon's trajectories clearly, their dynamics properties must be analyzed. Obviously, neither q_{2-p} is a traveling wave with a constant profile, nor the trajectory of two positions is a straight line, but a slowly changing curve. The most well-known idea that soliton is two-soliton can be viewed as the decomposition of two solitons when $t\to\infty$. The two-soliton will become to two-positon after performing a limit method, which should be decomposed asymptotically into two solitons.

Proposition 2 When $|t|\to\infty$, two-positon solution of the GI equation has the decomposition

$$
\begin{aligned}
|q_{2-p}|^2 \approx & \left|q_{1-s}\Big(H+\frac{\ln(4096\alpha_1^4\beta_1^4t^2)}{8\alpha_1\beta_1}\Big)\right|^2\\
&+\left|q_{1-s}\Big(H-\frac{\ln(4096\alpha_1^4\beta_1^4t^2)}{8\alpha_1\beta_1}\Big)\right|^2,
\end{aligned}
\tag{14}
$$

two approximate trajectory equations are $H\pm\frac{\ln(4096\alpha_1^4\beta_1^4t^2)}{8\alpha_1\beta_1}=0$, where $H=4t\alpha_1^2-4t\beta_1^2+x$.

Proof 1 We assume that the $|q_{2-p}|^2$ has the following decomposition:

$$
|q_{2-p}|^2\approx|q_{1-s}(H+c_1)|^2+|q_{1-s}(H-c_1)|^2,\ t\to\infty, \tag{15}
$$

with

$$
q_{1-s}(\theta)=\frac{16\mathrm{e}^{-2\mathrm{i}h}\alpha_1\beta_1}{\alpha_1\cosh(4\alpha_1\beta_1\theta)-\mathrm{i}\beta_1\sinh(4\alpha_1\beta_1\theta)}, \tag{16}
$$

where $\theta=H\pm c_1$. Substituting q_{2-p} and Eq. (16) into Eq. (15), and considering the approximation in the neighborhood of $H=0$ for Eq. (15), we can obtain $c_1\approx\frac{\ln(4096\alpha_1^4\beta_1^4t^2)}{8\alpha_1\beta_1}$.

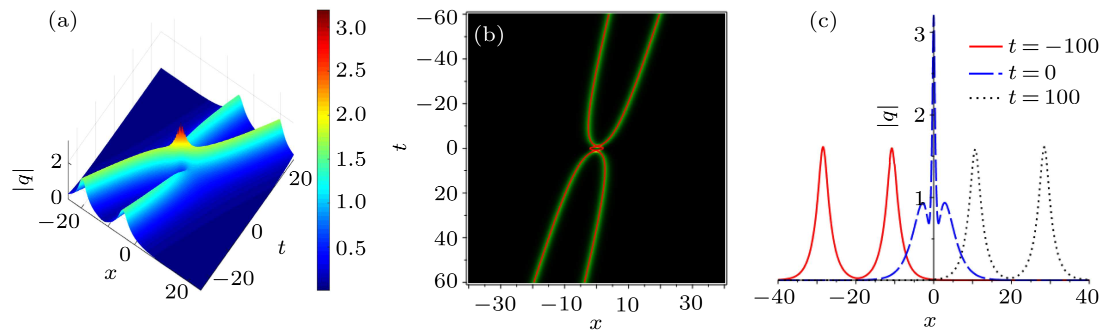

Fig. 2. The evolution of a two-positon $|q_{2-p}|$ with $\alpha_1=1/3$, $\beta_1=2/5$ of the GI equation: (a) 3D plot, (b) density plot, where two red curves are approximate trajectories defined by $H\pm\frac{\ln(4096\alpha_1^4\beta_1^4t^2)}{8\alpha_1\beta_1}=0$, which compared with density plot are shown consistence; (c) 2D plot of two-positon solution $|q_{2-p}|$ at $t=-100$, $t=0$, $t=100$.

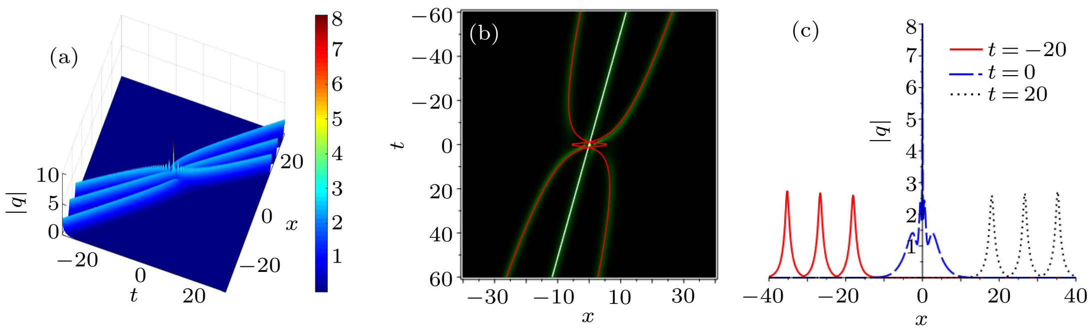

Fig. 3. The evolution of a three-positon $|q_{3-p}|$ with $\alpha_1=1/3$, $\beta_1=2/5$ of the GI equation: (a) 3D plot, (b) density plot, where two red curves are approximate trajectories defined by $H\pm\frac{\ln(4194304\alpha_1^8\beta_1^8t^4)}{8\alpha_1\beta_1}$ and the middle white curve is trajectory without phase shift, which compared with density plot are shown consistence; (c) 2D plot of three-positon solution $|q_{3-p}|$ at $t=-20$, $t=0$, $t=20$.

Proposition 3 When $|t| \to \infty$, the three-positon solution of the GI equation has the decomposition

$$|q_{3-p}|^2 \approx \left|q_{1-s}\left(H+\frac{\ln(4194304\alpha_1^8\beta_1^8t^4)}{8\alpha_1\beta_1}\right)\right|^2+|q_{1-s}(H)|^2 + \left|q_{1-s}\left(H-\frac{\ln(4194304\alpha_1^8\beta_1^8t^4)}{8\alpha_1\beta_1}\right)\right|^2, \tag{17}$$

three approximate trajectory equations are defined by $H \pm \frac{\ln(4194304\alpha_1^8\beta_1^8t^4)}{8\alpha_1\beta_1}=0$ and $H=0$, where $H=4t\alpha_1^2-4t\beta_1^2+x$.

Proof 2 As we can see in Fig. 3(a), the undetermined form of q_{3-p} can be written as

$$|q_{3-p}|^2 \approx |q_{1-s}(H+c_2)|^2+|q_{1-s}(H)|^2 + |q_{1-s}(H-c_2)|^2, \tag{18}$$

when $|t| \to \infty$; c_2 is "phase shift" to be determined, and $q_{1-s}(H \pm c_2)$ are obtained by changing c_1 in Eq. (16). Substituting $q_{1-s}(H+c_2)$, $q_{1-s}(H)$, $q_{1-s}(H-c_2)$ into Eq. (18) and considering the approximation in the neighborhood of $H=0$, we can obtain $c_2 \approx \frac{\ln(4194304\alpha_1^8\beta_1^8t^4)}{8\alpha_1\beta_1}$.

When $n=4$ in Eq. (12), the four-positon solution q_{4-p} of the GI equation can be obtained via the similar method. We omit the formula of the four-positon solution because of the rather complexity form. The three-dimensional plot and density plot have shown in Fig. 4, and higher-order positon solutions can be constructed by the similar way.

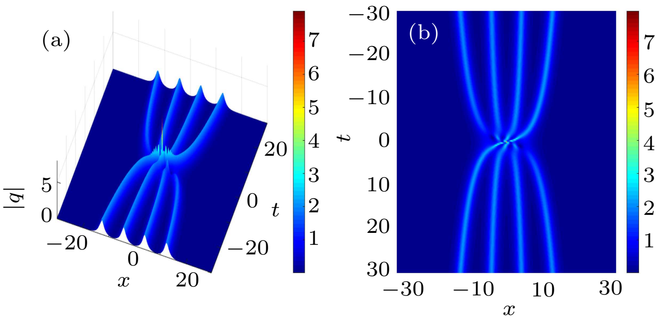

Fig. 4. The evolution of a four-positon $|q_{4-p}|$ with $\alpha_1=1/2$, $\beta_1=1/2$ of the GI equation on (x,t)-plane: (a) the 3D plot, (b) the density plot.

5. Hybrid solutions of solitons and positons

In this section, we discuss the hybrid solutions of the soliton solutions and the positons solutions of the GI equation.

Proposition 4 A hybrid solution of m-order smooth positon and l solitons has the following form based on semi-degenerate DT,

$$q_{m-l}=2\mathrm{i}\frac{N'_{2n}}{W'_{2n}}, \tag{19}$$

where

$$N'_{2n}=\left(\frac{\partial^{h(i)}}{\partial\varepsilon^{h(i)}}\bigg|_{\varepsilon=0}(\Omega_{11})_{ij}(\lambda_1+\varepsilon)\right)_{2n\times 2n},$$

$$W'_{2n}=\left(\frac{\partial^{h(i)}}{\partial\varepsilon^{h(i)}}\bigg|_{\varepsilon=0}(\Omega_{12})_{ij}(\lambda_1+\varepsilon)\right)_{2n\times 2n},$$

$$h(i)=\begin{cases}\left[\frac{i-1}{2}\right], & i\le 2m,\\ 0, & i>2m,\end{cases} \tag{20}$$

where $n=m+l$, $\lambda_1=\lambda_3=\cdots=\lambda_{2\,m-1}$, $[i]$ denotes the floor function, $\lambda_{2m+1},\ldots,\lambda_{2m+2l-1}$ mutual inequality.

The n-positon solution can be obtained by performing higher-order expansion with $\lambda_j \to \lambda_1$ in the n-soliton solution. The two-positon solution is the degenerated case of the two-soliton solution. Inspired by employing the Hirota bilinear method to construct interaction solutions, we consider the degeneration on partial eigenvalues. In proposition 4, if $l=0$, the hybrid solutions will be converted to higher-order smooth positons. Let $m=2$ and $l=1$, we can get hybrid of one-soliton and two-positon, see Fig. 5. As $m=3$ and $l=1$, the hybrid of one-soliton and three-positon can be generated, see Fig. 6. As $m=2$ and $l=2$, the hybrid of two-soliton and two-positon can be generated, see Fig. 7. We can roughly observe that there is no change except the phase shift before and after the collision between solitons and higher-order smooth positons. Some interaction phenomena have similar properties.[36] Upon collision, large pulses are formed, this is very similar to soliton collision in a fibre laser. In Ref. [37], authors demonstrated a conceptually different type of soliton explosions induced by soliton collision. In first experimental demonstration of a new mechanism for rogue wave generation, the nonlinear interactions and collisions of ultrashort optical pulses in a strongly dissipative regime lead to extreme peak-optical-intensity events.[38]

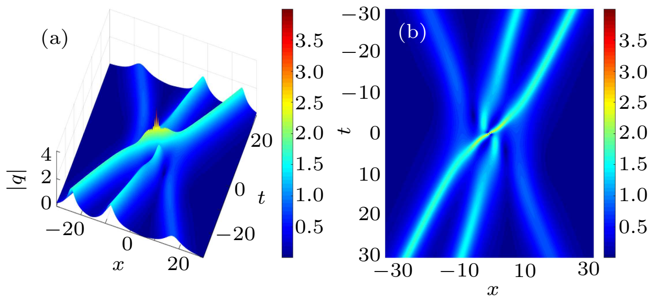

Fig. 5. The evolution of hybrid solution consisting of a soliton and two-positon with $\alpha_1=2/5$, $\beta_1=1/5$, $\alpha_3=1/5$, $\beta_3=2/5$ of the GI equation on (x,t)-plane: (a) the 3D plot, (b) the density plot.

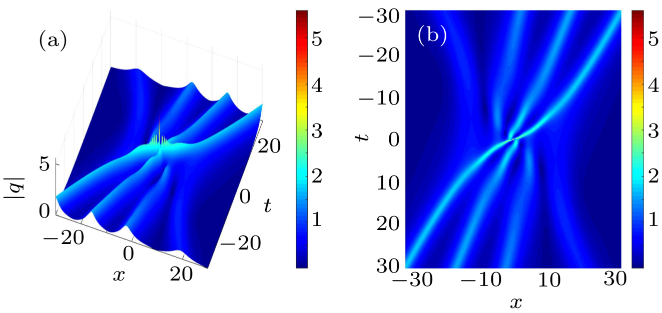

Fig. 6. The evolution of hybrid solution consisting of a soliton and three-positon with $\alpha_1=2/5$, $\beta_1=1/5$, $\alpha_3=1/5$, $\beta_3=2/5$ of the GI equation on (x,t)-plane: (a) the 3D plot, (b) the density plot.

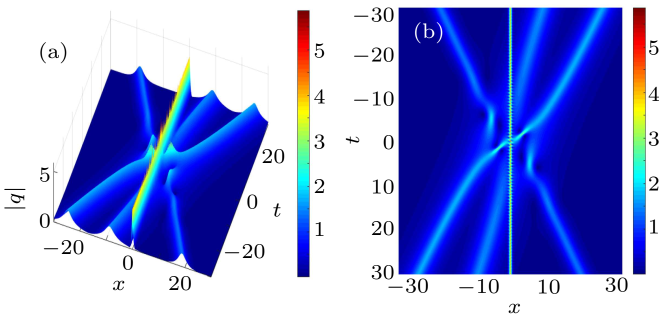

Fig. 7. The evolution of hybrid solution consisting of two solitons and two-positon with $\alpha_1 = 1/2$, $\beta_1 = 1/2$, $\alpha_3 = 1/2$, $\beta_3 = 1/3$, $\alpha_5 = 1/5$, $\beta_5 = 2/5$ of the GI equation on (x,t)-plane: (a) the 3D plot, (b) the density plot.

6. Conclusion

In this paper, we have presented the velocity resonance method to generate soliton molecules of the GI equation, see Fig. 1. The determinant representation of the n-positon solution to the GI equation is presented based on degenerate DT in proposition 1. Two-positon, three-positon and four-positon solutions are given as an example, and the trajectory of positons is a slowly changing curve, see Figs. 2(a), 3(a), and 4. The decomposition of the positons for the GI equation, the time-dependent "phase shift" and approximate trajectories are obtained in propositions 2 and 3, see Figs. 2(b) and 3(b). We also discuss the hybrid solutions of the solutions and positons in proposition 4. The interaction of positons and solitons can be seen from Figs. 5–7, and the complicated dynamics are revealed.

The interaction phenomenon of soliton molecules can be approached from many equations, such as modified nonlinear Schrödinger equation (MNSE),[39] complex modified Korteweg-de Vries equation,[40] B-type Kadomtsev–Petviashvili equation.[41] Liu *et al.* used numerical simulations to reveal the soliton formation through beating dynamics on the MNSE. How to investigate beating behavior by the GI equation, this is a meaningful question. In Ref. [42], the soliton (kink) molecules, half periodic kink molecules and breathing soliton molecules of the Sharma–Tasso–Olver–Burgers equation were derived, and the fission and fusion phenomena have been analyzed among kink molecules and half periodic kink molecules by multiple solitary wave solution. Recently, Liu and Pang[43] unveiled that the multiple harmonic mode-locking pulses originate from a single-pulse splitting phenomenon and breathing behavior occurs in an early stage of the harmonic mode-locking buildup process. We cannot reach such results theoretically by DT at present, the corresponding work of this problem will be studied in future.

References

[1] Johnson R S 1977 *Proc. R. Soc. A* **357** 131
[2] Anderson D and Lisak M 1983 *Phys. Rev. A* **27** 1393
[3] Clarkson P A and Tuszynski J A 1990 *J. Phys. A: Math. Gen.* **23** 4269
[4] Rogister A 1971 *Phys. Fluids* **14** 2733
[5] Kaup D J and Newell A C 1978 *J. Math. Phys.* **19** 798
[6] Chen H H, Lee Y C and Liu C S 1979 *Phys. Scr.* **20** 490
[7] Gerdjikov V S and Ivanov I 1983 *Bulg. J. Phys.* **10** 130
[8] Kadkhoda N and Jafari H 2017 *Optik* **139** 72
[9] Ding C C, Gao Y T and Li L Q 2019 *Chaos Solitons Fractals* **120** 259
[10] Fan E G 2000 *J. Phys. A: Math. Gen.* **33** 6925
[11] Xu S W and He J S 2012 *J. Math. Phys.* **53** 063507
[12] Guo L J, Zhang Y S, Xu S W, Wu Z W and He J S 2014 *Phys. Scr.* **89** 035501
[13] Crasovan L, Kartashov Y, Mihalache D, Torner L, Kivshar Y and Pérez-García V 2003 *Phys. Rev. E* **67** 046610
[14] Weng W, Bouchand R, Lucas E, Obrzud E, Herr T and Kippenberg T J 2020 *Nat. Commun.* **11** 2402
[15] Peng J S and Zeng H P 2018 *Laser Photon. Rev.* **12** 1800009
[16] Herink G, Kurtz F, Jalali B, Solli D R and Ropers C 2017 *Science* **356** 50
[17] Liu X M, Yao X K and Cui Y D 2018 *Phys. Rev. Lett.* **121** 023905
[18] Peng J S, Sorokina M, Sugavanam S, Tarasov N, Churkin D V, Turitsyn S K and Zeng H P 2018 *Commun. Phys.* **1** 20
[19] Peng J S and Zeng H P 2019 *Opt. Lett.* **44** 2899
[20] Peng J S, Boscolo S, Zhao Z H and Zeng H P 2019 *Sci. Adv.* **5** eaax1110
[21] Lou S Y 2020 *J. Phys. Commun.* **4** 041002
[22] Zhang Z, Yang X Y and Li B 2020 *Appl. Math. Lett.* **103** 106168
[23] Wang B, Zhang Z and Li B 2020 *Chin. Phys. Lett.* **37** 030501
[24] Zhang Z, Yang S X and Li B 2019 *Chin. Phys. Lett.* **36** 120501
[25] Matveev V B 1992 *Phys. Lett. A* **166** 205
[26] Maisch H and Stahlhofen A A 1995 *Phys. Scr.* **52** 228
[27] Chow K W, Lai W C, Shek C K and Tso K 1998 *Chaos Solitons Fractals.* **9** 1901
[28] Shao Y J and Zeng Y B 2005 *J. Phys. A: Math. Gen.* **38** 2441
[29] Dubard P, Gaillard P, Klein C and Matveev V B 2010 *Eur. Phys. J. Spec. Top.* **185** 247
[30] Liu Y K and Li B 2017 *Chin. Phys. Lett.* **34** 010202
[31] Qian C, Rao J G, Liu Y B and He J S 2016 *Chin. Phys. Lett.* **33** 110201
[32] Silem A, Zhang C and Zhang D J 2019 *Chin. Phys. B* **28** 020202
[33] Liu W, Zhang Y S and He J S 2018 *Waves Random Complex Media* **28** 203
[34] Song W J, Xu S W, Li M H and He J S 2019 *Nonliner Dyn.* **97** 2135
[35] Qiu D Q and Cheng W G 2019 *Commun. Nonlinear Sci. Numer. Simulat.* **78** 104887
[36] Zhang Z, Yang X Y, Li W T and Li B 2019 *Chin. Phys. B* **28** 110201
[37] Peng J S and Zeng H P 2019 *Commun. Phys.* **2** 34
[38] Lecaplain C, Grelu P, Soto-Crespo J M and Akhmediev N 2012 *Phys. Rev. Lett.* **108** 233901
[39] Liu X M, Popa D and Akhmediev N 2019 *Phys. Rev. Lett.* **123** 093901
[40] Zhang Z, Yang X Y and Li B 2020 *Nonlinear Dyn.* **100** 1551
[41] Yang X Y, Fan R and Li B 2020 *Phys. Scr.* **95** 045213
[42] Yan Z W and Lou S Y 2020 *Appl. Math. Lett.* **104** 106271
[43] Liu X M and Pang M 2019 *Laser Photon. Rev.* **13** 1800333