Damping of displaced chaotic light field in amplitude dissipation channel*

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We explore how a displaced chaotic light (DCL) behaves in an amplitude dissipation channel, and what is its time evolution formula of photon number distribution. With the use of the method of integration within ordered product product of operator (IWOP) and the new binomial theorem involving two-variable Hermite polynomials we obtain the evolution law of DCL in the channel.

Keywords: displaced chaotic light, amplitude dissipation channel, time evolution formula, IWOP

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1. Introduction

In nature the sun light is a kind of photon chaotic fields from the point of view of quantum optics. It is an important subject of quantum optics to discover new quantum states of light field and study its non-classical properties.[1–4] Since the light field is a multi-degree-of-freedom system, it is usually described by density operator.[5–8] The density operator describing a chaotic field is

$$\rho_c = \left(1-\mathrm{e}^{f}\right)\mathrm{e}^{fa^\dagger a}, \tag{1}$$

where f is the chaotic parameter, $a^\dagger$ and a represent photon annihilator and creator, respectively, $\left[a, a^\dagger\right] = 1$. Using $\sum_{n=0}^{\infty}|n\rangle\langle n| = 1$, the completeness relation of number state $|n\rangle$, the ensemble average of $a^\dagger a$ can be calculated by

$$\begin{aligned}
\operatorname{tr}\left(\rho_c a^\dagger a\right) &= \left(1-\mathrm{e}^{f}\right)\frac{\partial}{\partial f}\operatorname{Tr}\left[\mathrm{e}^{fa^\dagger a}\sum\nolimits_{n=0}^{\infty}|n\rangle\langle n|\right] \\
&= \left(1-\mathrm{e}^{f}\right)\frac{\partial}{\partial f}\sum_{n=0}^{\infty}\mathrm{e}^{fn} \\
&= \left(1-\mathrm{e}^{f}\right)\frac{\partial}{\partial f}\frac{1}{1-\mathrm{e}^{f}} \\
&= \left(\mathrm{e}^{-f}-1\right)^{-1},
\end{aligned} \tag{2}$$

which is just the Planck photon distribution formula $1/(\mathrm{e}^{\beta\hbar\omega}-1)$, when taking $f=-\omega\hbar/kT$, $\beta = 1/kT$, k is Boltzmann constant, T denotes temperature.

Now we generalize ρ_c to

$$\rho_d = C\,\mathrm{e}^{\lambda a^\dagger}\,\mathrm{e}^{fa^\dagger a}\,\mathrm{e}^{\lambda^* a}, \tag{3}$$

which describes a displaced chaotic light (DCL), here λ is a displacement parameter, C is the normalization constant to be determined shortly later by $\operatorname{tr}\rho_d = 1$.

The motivation of doing this lies in that a displaced vacuum state $\mathrm{e}^{\lambda a^\dagger}|0\rangle\langle 0|\mathrm{e}^{\lambda^* a}$ is a coherent state, then what is a displaced chaotic light and how it behaves so far as the photon statistics concerned? Generally speaking, a harmonic oscillator system with external source will produce DCL, the corresponding Hamiltonian is $H = fa^\dagger a + \sigma a^\dagger + \sigma^* a$. So due to the effect of external sources, when the initial state of quantum system is vacuum state, the final state will be transformed into coherent state, then if the initial state is chaotic light field, the final state will be transformed into a displaced chaotic light field. In this paper we explore how a displaced chaotic light behaves in an amplitude dissipation channel, and how its photon number decreases; in another word, we shall deduce a photon distribution formula (with time evolution) for DCL. With the use of the method of integration (summation) within ordered product of operator[9–12] and the new binomial theorem involving two-variable Hermite polynomials,[13–15] we obtain the evolution law of DCL in the channel.

2. The normalization of ρ_d

Using the coherent state representation[16–18]

$$\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = 1, \tag{4}$$

where

$$|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2 + \alpha a^\dagger\right]|0\rangle, \quad a|\alpha\rangle = \alpha|\alpha\rangle. \tag{5}$$

We can evaluate

$$1 = \operatorname{Tr}\rho_d = \operatorname{Tr}\left[\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha|\rho_d\right]$$

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$$
\begin{aligned}
&= C\int \frac{\mathrm{d}^2\alpha}{\pi}\langle\alpha|\,\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{fa^\dagger a}\mathrm{e}^{\lambda^* a}|\alpha\rangle \\
&= C\int \frac{\mathrm{d}^2\alpha}{\pi}\mathrm{e}^{\lambda\alpha^*+\lambda^*\alpha}\langle\alpha|:\exp\left[\left(\mathrm{e}^f-1\right)a^\dagger a\right]:|\alpha\rangle \\
&= C\int \frac{\mathrm{d}^2\alpha}{\pi}\mathrm{e}^{\lambda\alpha^*+\lambda^*\alpha}\exp\left[-\left(1-\mathrm{e}^f\right)|\alpha|^2\right] \\
&= \frac{C}{1-\mathrm{e}^f}\exp\left[\frac{|\lambda|^2}{1-\mathrm{e}^f}\right],
\end{aligned}
\tag{6}
$$

so the normalization constant is

$$
C=\left(1-\mathrm{e}^f\right)\exp\left[\frac{-|\lambda|^2}{1-\mathrm{e}^f}\right]. \tag{7}
$$

Thus the normalized ρ_{d} is

$$
\rho_{\mathrm{d}}=\left(1-\mathrm{e}^f\right)\exp\left[\frac{-|\lambda|^2}{1-\mathrm{e}^f}\right]\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{fa^\dagger a}\mathrm{e}^{\lambda^* a}. \tag{8}
$$

Obviously, by noting the normally ordered form[19–21] of e^{fa^+a}

$$
\mathrm{e}^{fa^\dagger a}=\;:\exp\left[\left(\mathrm{e}^f-1\right)a^\dagger a\right]:, \tag{9}
$$

we see that when $T\to 0$,

$$
f=-\omega\hbar/kT\to-\infty,\quad \mathrm{e}^{fa^\dagger a}\to:\mathrm{e}^{-a^\dagger a}:\;=|0\rangle\langle 0|,
$$

when it is replaced by the pure vacuum state, then $\rho_{\mathrm{d}}\to\exp\left[-|\lambda|^2\right]\mathrm{e}^{\lambda a^\dagger}|0\rangle\langle 0|\,\mathrm{e}^{\lambda^* a}$ stands for a pure coherent state. Hence ρ_{d} can be treated as an intermediate quantum state between the chaotic state ($\lambda=0$) and the coherent state. In another word, a displaced chaotic light can be produced when a coherent light undergoes a diffusion process. The photon number in DCL is

$$
\begin{aligned}
\operatorname{Tr}\left(\rho_{\mathrm{d}}a^\dagger a\right) &= \operatorname{Tr}\left(\rho_{\mathrm{d}}aa^\dagger\right)-1 \\
&= \operatorname{Tr}\left[\rho_{\mathrm{d}}a\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha|a^\dagger\right]-1 \\
&= \left(1-\mathrm{e}^f\right)\exp\left(\frac{-|\lambda|^2}{1-\mathrm{e}^f}\right) \\
&\quad\times\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha|^2\langle\alpha|\,\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{fa^\dagger a}\mathrm{e}^{\lambda^* a}|\alpha\rangle-1 \\
&= \left(1-\mathrm{e}^f\right)\exp\left(\frac{-|\lambda|^2}{1-\mathrm{e}^f}\right)\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha|^2\,\mathrm{e}^{\lambda\alpha^*+\lambda^*\alpha} \\
&\quad\times\exp\left[\left(\mathrm{e}^f-1\right)|\alpha|^2\right]-1 \\
&= \frac{1}{\mathrm{e}^{-f}-1}+\frac{|\lambda|^2}{\left(1-\mathrm{e}^f\right)^2},
\end{aligned}
\tag{10}
$$

we see photon number in DCL is more than that in Eq. (2) by an amount $|\lambda|^2/\left(1-\mathrm{e}^f\right)^2$. In the following, we study how a DCL dissipates in an amplitude damping channel.

3. Damping of DCL in an amplitude dissipation channel

The master equation describing an amplitude dissipation channel is

$$
\frac{\mathrm{d}\rho}{\mathrm{d}t}=k\left(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a\right), \tag{11}
$$

where k is the damping rate. By using the entangled state representation,[22–24] one can directly derive the infinitive-sum solution to Eq. (11), *i.e.*,

$$
\begin{aligned}
\rho(t) &= \sum_{n=0}^{\infty}\frac{T'^n}{n!}\mathrm{e}^{-kta^\dagger a}a^n\rho_0 a^{\dagger n}\mathrm{e}^{-kta^\dagger a} \\
&= \sum_{n=0}^{\infty}M_n\rho_0 M_n^\dagger,
\end{aligned}
\tag{12}
$$

where $T'=1-\mathrm{e}^{-2kt}$

$$
M_n=\sqrt{\frac{T'^n}{n!}}\,\mathrm{e}^{-kta^\dagger a}a^n, \tag{13}
$$

M_n satisfies $\sum_{n=0} M_nM_n^\dagger=1$, in fact using

$$
\mathrm{e}^{\lambda a^\dagger a}a\,\mathrm{e}^{-\lambda a^\dagger a}=a\,\mathrm{e}^{-\lambda} \tag{14}
$$

and the summation method within normal ordering : : we really have

$$
\begin{aligned}
\sum_{n=0}^{\infty}M_nM_n^+ &= \sum_{n=0}^{\infty}\frac{T'^n}{n!}a^{\dagger n}\mathrm{e}^{-2kta^\dagger a}a^n \\
&= \sum_{n=0}^{\infty}\frac{T'^n}{n!}\mathrm{e}^{2nkt}:a^{\dagger n}a^n:\mathrm{e}^{-2kta^\dagger a} \\
&= :\mathrm{e}^{T\mathrm{e}^{2kt}a^\dagger a}:\mathrm{e}^{-2kta^\dagger a} \\
&= :\mathrm{e}^{\left(\mathrm{e}^{2kt}-1\right)a^\dagger a}:\mathrm{e}^{-2kta^\dagger a} \\
&= 1.
\end{aligned}
\tag{15}
$$

Thus

$$
\operatorname{Tr}\rho(t)=\operatorname{Tr}\sum_{n=0}^{\infty}M_n\rho_0M_n^\dagger=\operatorname{Tr}\rho_0. \tag{16}
$$

Substituting Eq. (8) as ρ_0 into Eq. (12) yields

$$
\begin{aligned}
\rho_{\mathrm{d}}(t) &= \sum_{n=0}^{\infty}M_n\rho_{\mathrm{d}}M_n^\dagger \\
&= C\sum_{n=0}^{\infty}\frac{T'^n}{n!}\mathrm{e}^{-kta^\dagger a}a^n\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{fa^\dagger a}\mathrm{e}^{\lambda^* a}a^{\dagger n}\mathrm{e}^{-kta^\dagger a},
\end{aligned}
\tag{17}
$$

at this point we are facing the summation over n, but it is hard, since these operators are not commutable. We must put them in a definite order before we can perform the summation. Noting

$$
a^n\mathrm{e}^{\lambda a^\dagger}=\mathrm{e}^{\lambda a^\dagger}(a+\lambda)^n=\mathrm{e}^{\lambda a^\dagger}\sum_{l=0}\binom{n}{l}\lambda^{n-l}a^l, \tag{18}
$$

we see

$$
a^n\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{fa^\dagger a}\mathrm{e}^{\lambda^* a}a^{\dagger n}
$$

$$= \mathrm{e}^{\lambda a^{\dagger}} \sum_{l=0} \sum_{k=0} \binom{n}{l} \binom{n}{k} \lambda^{n-l} a^{l} \mathrm{e}^{f a^{\dagger} a} a^{\dagger k} \lambda^{* n-k} \mathrm{e}^{\lambda^{*} a}. \quad (19)$$

Then using the anti-normally ordered form[25,26]

$$\mathrm{e}^{f a^{\dagger} a} = \mathrm{e}^{-f} \vdots \exp\left[\left(1-\mathrm{e}^{-f}\right) a a^{\dagger}\right] \vdots \quad (20)$$

and the coherent state's completeness relation as well as the method of integration within ordered product (IWOP) of operators we can convert $a^{l} \mathrm{e}^{f a^{\dagger} a} a^{\dagger k}$ into its normally ordered form

$$\begin{aligned}
& a^{l} \mathrm{e}^{f a^{\dagger} a} a^{\dagger k} \\
= & \mathrm{e}^{-f} a^{l} \vdots \exp\left[\left(1-\mathrm{e}^{-f}\right) a a^{\dagger}\right] \vdots a^{\dagger k} \\
= & \mathrm{e}^{-f} \int \frac{\mathrm{d}^{2} z}{\pi} z^{l} \mathrm{e}^{\left(1-\mathrm{e}^{-f}\right)|z|^{2}} |z\rangle \langle z| z^{* k} \\
= & \mathrm{e}^{-f} \int \frac{\mathrm{d}^{2} z}{\pi} z^{l} z^{* k} : \mathrm{e}^{-\mathrm{e}^{-f}|z|^{2}+z a^{\dagger}+z^{*} a-a^{\dagger} a} : \\
= & \left(\mathrm{e}^{f}\right)^{(l+k)/2} \int \frac{\mathrm{d}^{2} z}{\pi} z^{l} z^{* k} : \mathrm{e}^{-|z|^{2}+\mathrm{e}^{f/2} z a^{\dagger}+\mathrm{e}^{f/2} z^{*} a-a^{\dagger} a} : \\
= & \left(\mathrm{e}^{f}\right)^{(l+k)/2} (-\mathrm{i})^{l+k} : \mathrm{e}^{\left(\mathrm{e}^{f}-1\right) a^{\dagger} a} \\
& \times \sum_{m=0} \frac{l! k! (-1)^{m}}{m! (l-m)! (k-m)!} \left(\mathrm{i}\, \mathrm{e}^{f/2} a^{\dagger}\right)^{k-m} \left(\mathrm{i}\, \mathrm{e}^{f/2} a\right)^{l-m} : \\
= & (-\mathrm{i})^{l+k} \left(\mathrm{e}^{f}\right)^{(l+k)/2} : \mathrm{e}^{\left(\mathrm{e}^{f}-1\right) a^{\dagger} a} \\
& \times \mathrm{H}_{k,l}\left(\mathrm{i} a^{\dagger} \mathrm{e}^{f/2}, \mathrm{i} a \mathrm{e}^{f/2}\right) :
\end{aligned} \quad (21)$$

where $\mathrm{H}_{k,l}$ is a two-variable Hermite polynomial, introduced via

$$\begin{aligned}
\mathrm{H}_{m,n}(x,y) & = \frac{\partial^{n+m}}{\partial t^{m} \partial \tau^{n}} \exp(t x+\tau y-t \tau)|_{t=\tau=0} \\
& = \frac{\partial^{m}}{\partial t^{m}} \mathrm{e}^{t x} \frac{\partial^{n}}{\partial \tau^{n}} \exp[\tau(y-t)]|_{t=\tau=0} \\
& = \frac{\partial^{m}}{\partial t^{m}} \left[\mathrm{e}^{t x}(y-t)^{n}\right]|_{t=0} \\
& = \sum_{l=0} \binom{m}{l} \frac{\partial^{l}}{\partial t^{l}} (y-t)^{n} \frac{\partial^{m-l}}{\partial t^{m-l}} \mathrm{e}^{t x}|_{t=0} \\
& = \sum_{l=0}^{\min(m,n)} \frac{m! n! (-1)^{l}}{l! (m-l)! (n-l)!} x^{m-l} y^{n-l}
\end{aligned} \quad (22)$$

or we can say that the two-variable Hermite polynomial is introduced through its generating function formula

$$\sum_{m,n=0} \frac{t^{m} \tau^{n}}{m! n!} \mathrm{H}_{m,n}(x,y) = \exp(t x+\tau y-t \tau). \quad (23)$$

Substituting Eq. (21) into the double-summation term in Eq. (19) we are facing

$$\begin{aligned}
& \sum_{l=0} \sum_{k=0} \binom{n}{l} \binom{n}{k} \lambda^{n-l} a^{l} \mathrm{e}^{f a^{\dagger} a} a^{\dagger k} \lambda^{* n-k} \\
= & \sum_{l=0} \sum_{k=0} \binom{n}{l} \binom{n}{k} \lambda^{n-l} (-i)^{l+k} \left(\mathrm{e}^{f}\right)^{(l+k)/2} : \mathrm{e}^{\left(\mathrm{e}^{f}-1\right) a^{\dagger} a} \mathrm{H}_{k,l}\left(\mathrm{i} a^{\dagger} \mathrm{e}^{f/2}, \mathrm{i} a \mathrm{e}^{f/2}\right) : \lambda^{* n-k} \\
= & |\lambda|^{2n} : \mathrm{e}^{\left(\mathrm{e}^{f}-1\right) a^{\dagger} a} \sum_{l=0}^{n} \sum_{k=0}^{n} \binom{n}{l} \binom{n}{k} \left(\frac{-\mathrm{i}\, \mathrm{e}^{f/2}}{\lambda}\right)^{l} \left(\frac{-\mathrm{i}\, \mathrm{e}^{f/2}}{\lambda^{*}}\right)^{k} \mathrm{H}_{k,l}\left(\mathrm{i} a^{\dagger} \mathrm{e}^{f/2}, \mathrm{i} a \mathrm{e}^{f/2}\right) : .
\end{aligned} \quad (24)$$

Now we can perform the above summation within normal ordering, since all the operators within : : can be permutable. Before performing the summation we must set up a new generalized binomial theorem regarding to two-variable Hermite polynomials.

4. The generalized binomial theorem

We now prove the generalized binomial theorem[27,28]

$$\begin{aligned}
& \sum_{r=0}^{l} \sum_{q=0}^{k} \binom{l}{r} \binom{k}{q} \mathrm{H}_{r,q}(x,y) f^{r} g^{q} \\
= & f^{l} t^{k} \mathrm{H}_{l,k}\left(x+\frac{1}{f}, y+\frac{1}{g}\right).
\end{aligned} \quad (25)$$

To prove it we need an operator identity

$$a^{\dagger q} a^{r} = \vdots \mathrm{H}_{r,q}\left(a, a^{\dagger}\right) \vdots. \quad (26)$$

In fact, using the Baker–Hausdorff formula[29–31] and Eq. (23) we have

$$\begin{aligned}
\mathrm{e}^{t a^{\dagger}} \mathrm{e}^{t' a} & = \mathrm{e}^{t' a} \mathrm{e}^{t a^{\dagger}} \mathrm{e}^{-t t'} = \vdots \mathrm{e}^{t' a+t a^{\dagger}-t t'} \vdots \\
& = \sum_{r=0} \sum_{q=0} \frac{(t)^{r} (t')^{q}}{q! r!} \vdots \mathrm{H}_{r,q}\left(a, a^{\dagger}\right) \vdots,
\end{aligned} \quad (27)$$

comparing with the power series expansion

$$\mathrm{e}^{t a^{\dagger}} \mathrm{e}^{t' a} = \sum_{r=0} \sum_{q=0} \frac{t'^{q} t^{r}}{q! r!} a^{\dagger r} a^{q}, \quad (28)$$

we can see equation (26) really holds.

Then we replace $\mathrm{H}_{r,q}(x,y)$ in Eq. (25) by $\vdots \mathrm{H}_{r,q}\left(a, a^{\dagger}\right) \vdots$ (this is named the operator Hermite polynomial method, OHP method)[32,33] and using Eq. (26) to perform the summation

$$\begin{aligned}
& \sum_{r=0}^{l} \sum_{q=0}^{k} \binom{l}{r} \binom{k}{q} \vdots \mathrm{H}_{r,q}\left(a, a^{\dagger}\right) \vdots f^{r} g^{q} \\
= & \sum_{r=0}^{l} \sum_{q=0}^{k} \binom{l}{r} \binom{k}{q} : a^{\dagger q} a^{r} : f^{r} g^{q}
\end{aligned}$$

$$= :\left(ga^{\dagger}+1\right)^{k}(fa+1)^{l}:, \tag{29}$$

where the ordinary binomial theorem is used. Then we make up the following power series summation and convert the normal ordering into its anti-normal ordering, we can have

$$\begin{aligned}
&\sum_{l,k=0}^{\infty}\frac{s^{l}t^{k}:\left(ga^{\dagger}+1\right)^{k}(fa+1)^{l}:}{l!k!}\\
&= \mathrm{e}^{t\left(ga^{\dagger}+1\right)}\mathrm{e}^{s(fa+1)}\\
&= \mathrm{e}^{sf(a+1/f)}\mathrm{e}^{tg\left(a^{\dagger}+1/g\right)}\mathrm{e}^{-sftg}\\
&= \vdots\,\mathrm{e}^{sf(a+1/f)+tg\left(a^{\dagger}+1/g\right)}\mathrm{e}^{-sftg}\,\vdots\\
&= \sum_{l,k}^{\infty}\frac{(sf)^{l}(gt)^{k}}{l!k!}\,\vdots\,\mathrm{H}_{l,k}\left(a+\frac{1}{f},a^{\dagger}+\frac{1}{g}\right)\vdots,
\end{aligned} \tag{30}$$

where equation (23) is again used in the final step. From this equation we deduce

$$:\left(ga^{\dagger}+1\right)^{k}(fa+1)^{l}:=f^{l}t^{k}\,\vdots\,\mathrm{H}_{l,k}\left(a+\frac{1}{f},a^{\dagger}+\frac{1}{g}\right)\vdots. \tag{31}$$

Combining Eq. (31) and Eq. (29) leads to

$$\begin{aligned}
&\sum_{r=0}^{l}\sum_{q=0}^{k}\binom{l}{r}\binom{k}{q}\,\vdots\,\mathrm{H}_{r,q}\left(a,a^{\dagger}\right)\vdots\,f^{r}g^{q}\\
&= f^{l}t^{k}\,\vdots\,\mathrm{H}_{l,k}\left(a+\frac{1}{f},a^{\dagger}+\frac{1}{g}\right)\vdots.
\end{aligned} \tag{32}$$

Since both sides of Eq. (32) are in anti-normal ordering, we restore to

$$\begin{aligned}
&\sum_{r=0}^{l}\sum_{q=0}^{k}\binom{l}{r}\binom{k}{q}\mathrm{H}_{r,q}(x,y)f^{r}g^{q}\\
&= f^{l}t^{k}\mathrm{H}_{l,k}\left(x+\frac{1}{f},y+\frac{1}{g}\right).
\end{aligned} \tag{33}$$

Thus the theorem is thus proved.

5. Evolution law of DCL and photon number

The summation in Eq. (24) can be proceeded as

$$\begin{aligned}
&\sum_{l=0}\sum_{k=0}\binom{n}{l}\binom{n}{k}\lambda^{n-l}a^{l}\mathrm{e}^{fa^{\dagger}a}a^{\dagger k}\lambda^{*n-k}\\
&= \sum_{l=0}\sum_{k=0}\binom{n}{l}\binom{n}{k}\lambda^{n-l}(-\mathrm{i})^{l+k}\left(\mathrm{e}^{f}\right)^{(l+k)/2}:\mathrm{e}^{\left(\mathrm{e}^{f}-1\right)a^{\dagger}a}\mathrm{H}_{k,l}\left(\mathrm{i}a^{\dagger}\mathrm{e}^{f/2},\mathrm{i}a\mathrm{e}^{f/2}\right):\lambda^{*n-k}\\
&= |\lambda|^{2n}:\mathrm{e}^{\left(\mathrm{e}^{f}-1\right)a^{\dagger}a}\sum_{l=0}^{n}\sum_{k=0}^{n}\binom{n}{l}\binom{n}{k}\left(\frac{-\mathrm{i}\,\mathrm{e}^{f/2}}{\lambda}\right)^{l}\left(\frac{-\mathrm{i}\,\mathrm{e}^{f/2}}{\lambda^{*}}\right)^{k}\mathrm{H}_{k,l}\left(\mathrm{i}a^{\dagger}\mathrm{e}^{f/2},\mathrm{i}a\mathrm{e}^{f/2}\right):\\
&= \left(-\mathrm{e}^{f}\right)^{n}:\mathrm{e}^{\left(\mathrm{e}^{f}-1\right)a^{\dagger}a}\mathrm{H}_{n,n}\left(\mathrm{i}a^{\dagger}\mathrm{e}^{f/2}+\mathrm{i}\lambda^{*}\mathrm{e}^{-f/2},\mathrm{i}a\mathrm{e}^{f/2}+\mathrm{i}\,\mathrm{e}^{-f/2}\lambda\right):.
\end{aligned} \tag{34}$$

Then using the relation between $\mathrm{H}_{n,n}$ with the Laguerre polynomial L_n

$$\mathrm{L}_{n}(xy)=\frac{(-1)^{n}}{n!}\mathrm{H}_{n,n}(x,y), \tag{35}$$

we see

$$\begin{aligned}
a^{n}\mathrm{e}^{\lambda a^{\dagger}}\mathrm{e}^{fa^{\dagger}a}\mathrm{e}^{\lambda^{*}a}a^{\dagger n} &= \mathrm{e}^{\lambda a^{\dagger}}\sum_{l=0}\sum_{k=0}\binom{n}{l}\binom{n}{k}\lambda^{n-l}a^{l}\mathrm{e}^{fa^{\dagger}a}a^{\dagger k}\lambda^{*n-k}\mathrm{e}^{\lambda^{*}a}\\
&= \mathrm{e}^{\lambda a^{\dagger}}\left(-\mathrm{e}^{f}\right)^{n}:\mathrm{e}^{\left(\mathrm{e}^{f}-1\right)a^{\dagger}a}\mathrm{H}_{n,n}\left(\mathrm{i}a^{\dagger}\mathrm{e}^{f/2}+\mathrm{i}\lambda^{*}\mathrm{e}^{-f/2},\mathrm{i}a\mathrm{e}^{f/2}+\mathrm{i}\lambda\,\mathrm{e}^{-f/2}\right):\mathrm{e}^{\lambda^{*}a}\\
&= \mathrm{e}^{\lambda a^{\dagger}}\left(-\mathrm{e}^{f}\right)^{n}:\mathrm{e}^{\left(\mathrm{e}^{f}-1\right)a^{\dagger}a}n!(-1)^{n}\mathrm{L}_{n}\left[-\left(a^{\dagger}\mathrm{e}^{f/2}+\lambda^{*}\mathrm{e}^{-f/2}\right)\left(a\mathrm{e}^{f/2}+\lambda\,\mathrm{e}^{-f/2}\right)\right]:\mathrm{e}^{\lambda^{*}a}.
\end{aligned} \tag{36}$$

Let

$$W=\sum_{n=0}^{\infty}\frac{T'^{n}}{n!}a^{n}\mathrm{e}^{\lambda a^{\dagger}}\mathrm{e}^{fa^{\dagger}a}\mathrm{e}^{\lambda^{*}a}a^{\dagger n}, \tag{37}$$

then

$$\rho_{\mathrm{d}}(t)=\left(1-\mathrm{e}^{f}\right)\exp\left[\frac{-|\lambda|^{2}}{1-\mathrm{e}^{f}}\right]\mathrm{e}^{-\kappa ta^{\dagger}a}W\mathrm{e}^{-\kappa ta^{\dagger}a}. \tag{38}$$

By noticing the generating function of the Laguerre polynomial[34]

$$(1-z)^{-1}\exp\left[\frac{zx}{z-1}\right]=\sum_{l=0}\mathrm{L}_{n}(x)z^{n} \tag{39}$$

and using Eq. (36) we obtain

$$\begin{aligned} W &= \sum_{n=0}^{\infty} \frac{T'^n}{n!} a^n \mathrm{e}^{\lambda a^\dagger} \mathrm{e}^{f a^\dagger a} \mathrm{e}^{\lambda^* a} a^{\dagger n} \\ &= \mathrm{e}^{\lambda a^\dagger} \sum_{n=0}^{\infty} \frac{T'^n}{n!} \left(-\mathrm{e}^f\right)^n : \mathrm{e}^{\left(\mathrm{e}^f-1\right)a^\dagger a} n! (-1)^n \mathrm{L}_n \left[-\left(a^\dagger \mathrm{e}^{f/2} + \lambda^* \mathrm{e}^{-f/2}\right)\left(a \mathrm{e}^{f/2} + \lambda \mathrm{e}^{-f/2}\right)\right] : \mathrm{e}^{\lambda^* a} \\ &= \mathrm{e}^{\lambda a^\dagger} \sum_{n=0}^{\infty} \left(T' \mathrm{e}^f\right)^n : \mathrm{e}^{\left(\mathrm{e}^f-1\right)a^\dagger a} \mathrm{L}_n \left[-\left(a^\dagger \mathrm{e}^{f/2} + \lambda \mathrm{e}^{-f/2}\right)\left(a \mathrm{e}^{f/2} + \lambda^* \mathrm{e}^{-f/2}\right)\right] : \mathrm{e}^{\lambda^* a} \\ &= \frac{1}{1-T'\mathrm{e}^f} \mathrm{e}^{\lambda a^\dagger} : \mathrm{e}^{\left(\mathrm{e}^f-1\right)a^\dagger a} \exp\left[\frac{-T'\mathrm{e}^f\left(a^\dagger \mathrm{e}^{f/2} + \lambda^* \mathrm{e}^{-f/2}\right)\left(a \mathrm{e}^{f/2} + \lambda \mathrm{e}^{-f/2}\right)}{T'\mathrm{e}^f - 1}\right] : \mathrm{e}^{\lambda^* a}, \end{aligned} \tag{40}$$

then noticing

$$\mathrm{e}^{-kta^\dagger a} = a^\dagger \mathrm{e}^{-kta^\dagger a} \mathrm{e}^{-kt} \tag{41}$$

and remembering $T' = 1 - \mathrm{e}^{-2kt}$, we see

$$\begin{aligned} \rho_{\mathrm{d}}(t) &= \left(1-\mathrm{e}^f\right) \exp\left[\frac{-|\lambda|^2}{1-\mathrm{e}^f}\right] \mathrm{e}^{-\kappa t a^\dagger a} W \mathrm{e}^{-\kappa t a^\dagger a} \\ &= \frac{1-\mathrm{e}^f}{1-T'\mathrm{e}^f} \exp\left[\frac{T'-1}{\left(\mathrm{e}^f-1\right)\left(T'\mathrm{e}^f-1\right)} |\lambda|^2\right] \\ &\quad \times \exp\left[\left(\frac{\lambda a^\dagger}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] \\ &\quad \times \mathrm{e}^{-\kappa t a^\dagger a} : \exp\left[\left(\frac{\mathrm{e}^f}{1-T'\mathrm{e}^f} - 1\right) a^\dagger a\right] : \mathrm{e}^{-\kappa t a^\dagger a} \\ &\quad \times \left[\exp\left(\frac{\lambda^* a}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] \\ &= \frac{1-\mathrm{e}^f}{1-T'\mathrm{e}^f} \exp\left[\frac{T'-1}{\left(\mathrm{e}^f-1\right)\left(T'\mathrm{e}^f-1\right)} |\lambda|^2\right] \\ &\quad \times \exp\left[\left(\frac{\lambda a^\dagger}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] \\ &\quad \times \exp\left[a^\dagger a \left(\ln \frac{\mathrm{e}^f}{1-T'\mathrm{e}^f} - 2kt\right)\right] \\ &\quad \times \exp\left[\left(\frac{\lambda^* a}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] \\ &= \frac{1-\mathrm{e}^f}{1-T'\mathrm{e}^f} \exp\left[\frac{T'-1}{\left(\mathrm{e}^f-1\right)\left(T'\mathrm{e}^f-1\right)} |\lambda|^2\right] \\ &\quad \times \exp\left[\left(\frac{\lambda a^\dagger}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] \\ &\quad \times \exp\left[a^\dagger a \ln \frac{\mathrm{e}^f\left(1-T'\right)}{1-T'\mathrm{e}^f}\right] \\ &\quad \times \exp\left[\left(\frac{\lambda^* a}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right]. \end{aligned} \tag{42}$$

We conclude that the final state is still a displaced chaotic state, but the displacing amount and the chaotic parameter are both changed, remarkably, the displacing amount is affected by the chaotic parameter, λ becomes to $\lambda \mathrm{e}^{-kt}/1-T'\mathrm{e}^f$. Clearly, when $t=0$, $T'=0$, equation (42) reduces to Eq. (8). To further confirm this result we calculate

$$\begin{aligned} \operatorname{tr}\rho_{\mathrm{d}}(t) &= \frac{1-\mathrm{e}^f}{1-T'\mathrm{e}^f} \exp\left[\frac{T'-1}{\left(\mathrm{e}^f-1\right)\left(T'\mathrm{e}^f-1\right)} |\lambda|^2\right] \int \frac{\mathrm{d}^2 z}{\pi} \langle z| \exp\left[\left(\frac{\lambda a^\dagger}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] \\ &\quad \times \exp\left[a^\dagger a \ln \frac{\mathrm{e}^f\left(1-T'\right)}{1-T'\mathrm{e}^f}\right] \exp\left[\left(\frac{\lambda^* a}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] |z\rangle \\ &= \frac{1-\mathrm{e}^f}{1-T'\mathrm{e}^f} \exp\left[\frac{T'-1}{\left(\mathrm{e}^f-1\right)\left(T'\mathrm{e}^f-1\right)} |\lambda|^2\right] \int \frac{\mathrm{d}^2 z}{\pi} \exp\left[-\left(1 - \frac{\mathrm{e}^f\left(1-T'\right)}{1-T'\mathrm{e}^f}\right)|z|^2 + \frac{\lambda z^* \mathrm{e}^{-kt}}{1-T'\mathrm{e}^f} + \frac{\lambda^* z \mathrm{e}^{-kt}}{1-T'\mathrm{e}^f}\right] \\ &= \exp\left\{\left[\frac{\mathrm{e}^{-2kt}}{\left(1-\mathrm{e}^f\right)\left(1-T'\mathrm{e}^f\right)} + \frac{T'-1}{\left(1-\mathrm{e}^f\right)\left(1-T'\mathrm{e}^f\right)}\right] |\lambda|^2\right\} \\ &= 1. \end{aligned} \tag{43}$$

Now using the coherent state's completeness relation we can evaluate

$$\begin{aligned} \operatorname{tr}\left[\rho(t) a a^\dagger\right] &= \operatorname{tr}\left[\rho(t) a \int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\langle z| a^\dagger\right] \\ &= \frac{1-\mathrm{e}^f}{1-T'\mathrm{e}^f} \exp\left[\frac{T'-1}{\left(\mathrm{e}^f-1\right)\left(T'\mathrm{e}^f-1\right)} |\lambda|^2\right] \int \frac{\mathrm{d}^2 z}{\pi} |z|^2 \langle z| \exp\left[\left(\frac{\lambda a^+}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] \\ &\quad \times \exp\left[a^\dagger a \ln \frac{\mathrm{e}^f\left(1-T'\right)}{1-T'\mathrm{e}^f}\right] \exp\left[\left(\frac{\lambda^* a}{1-T'\mathrm{e}^f}\right) \mathrm{e}^{-kt}\right] |z\rangle \\ &= \frac{1-\mathrm{e}^f}{1-T'\mathrm{e}^f} \exp\left[\frac{T'-1}{\left(\mathrm{e}^f-1\right)\left(T'\mathrm{e}^f-1\right)} |\lambda|^2\right] \int \frac{\mathrm{d}^2 z}{\pi} |z|^2 \exp\left[-\frac{1-\mathrm{e}^f}{1-T'\mathrm{e}^f} |z|^2 + \frac{\lambda z^*}{1-T'\mathrm{e}^f} \mathrm{e}^{-kt} + \frac{\lambda^* z}{1-T'\mathrm{e}^f} \mathrm{e}^{-kt}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1-\mathrm{e}^{f}}{1-T'\mathrm{e}^{f}} \exp\left[\frac{T'-1}{(\mathrm{e}^{f}-1)(T'\mathrm{e}^{f}-1)}|\lambda|^2\right] (1-T'\mathrm{e}^{f})^2 \mathrm{e}^{2kt} \frac{\partial^2}{\partial\lambda\partial\lambda^*} \\
&\quad \times \int \frac{\mathrm{d}^2 z}{\pi} \exp\left[-\frac{1-\mathrm{e}^{f}}{1-T'\mathrm{e}^{f}}|z|^2 + \frac{\lambda z^*}{1-T'\mathrm{e}^{f}}\mathrm{e}^{-kt} + \frac{\lambda^* z}{1-T'\mathrm{e}^{f}}\mathrm{e}^{-kt}\right] \\
&= \exp\left[\frac{T'-1}{(\mathrm{e}^{f}-1)(T'\mathrm{e}^{f}-1)}|\lambda|^2\right] (1-T'\mathrm{e}^{f})^2 \mathrm{e}^{2kt} \frac{\partial^2}{\partial\lambda\partial\lambda^*} \exp\left[\frac{\mathrm{e}^{-2kt}}{(\mathrm{e}^{f}-1)(T'\mathrm{e}^{f}-1)}|\lambda|^2\right], \qquad (44)
\end{aligned}
$$

then let $\mathrm{e}^{-kt}/\sqrt{T'\mathrm{e}^{f}-1}\lambda = \lambda'$, equation (44) becomes to

$$
\begin{aligned}
&\operatorname{tr}\left[\rho(t) a a^{\dagger}\right] \\
&= (T'\mathrm{e}^{f}-1) \exp\left[\frac{\mathrm{e}^{-2kt}}{(\mathrm{e}^{f}-1)(T'\mathrm{e}^{f}-1)}|\lambda|^2\right] \\
&\quad \times \frac{\partial^2}{\partial\lambda'\partial\lambda'^*} \exp\left[\frac{1}{\mathrm{e}^{f}-1}|\lambda'|^2\right] \\
&= (T'\mathrm{e}^{f}-1) \exp\left[\frac{\mathrm{e}^{-2kt}}{(\mathrm{e}^{f}-1)(T'\mathrm{e}^{f}-1)}|\lambda|^2\right] \frac{\partial}{\partial\lambda'} \\
&\quad \times \left\{\frac{\lambda'}{\mathrm{e}^{f}-1} \exp\left[\frac{1}{\mathrm{e}^{f}-1}|\lambda'|^2\right]\right\} \\
&= (T'\mathrm{e}^{f}-1) \exp\left[\frac{\mathrm{e}^{-2kt}}{(\mathrm{e}^{f}-1)(T'\mathrm{e}^{f}-1)}|\lambda|^2\right] \\
&\quad \times \left\{\frac{1}{\mathrm{e}^{f}-1} + \frac{|\lambda'|^2}{(\mathrm{e}^{f}-1)^2}\right\} \exp\left[\frac{1}{\mathrm{e}^{f}-1}|\lambda'|^2\right] \\
&= (T'\mathrm{e}^{f}-1) \left\{\frac{1}{\mathrm{e}^{f}-1} + \frac{|\lambda'|^2}{(\mathrm{e}^{f}-1)^2}\right\} \\
&= (T'\mathrm{e}^{f}-1) \left\{\frac{1}{\mathrm{e}^{f}-1} + \frac{|\lambda|^2 \mathrm{e}^{-2kt}}{(\mathrm{e}^{f}-1)^2 (T'\mathrm{e}^{f}-1)}\right\} \\
&= \frac{T'\mathrm{e}^{f}-1}{\mathrm{e}^{f}-1} + \frac{|\lambda|^2 \mathrm{e}^{-2kt}}{(\mathrm{e}^{f}-1)^2}. \qquad (45)
\end{aligned}
$$

The photon number distribution formula is

$$
\begin{aligned}
\operatorname{tr}\left[\rho(t) a^{\dagger} a\right] &= \operatorname{tr}\left[\rho(t) a a^{\dagger}\right] - 1 \\
&= \frac{\mathrm{e}^{f}(T'-1)}{\mathrm{e}^{f}-1} + \frac{|\lambda|^2 \mathrm{e}^{-2kt}}{(\mathrm{e}^{f}-1)^2} \\
&= \left[\frac{1}{\mathrm{e}^{-f}-1} + \frac{|\lambda|^2}{(\mathrm{e}^{f}-1)^2}\right] \mathrm{e}^{-2kt}. \qquad (46)
\end{aligned}
$$

Comparing Eq. (46) with Eq. (10) we see clearly the damping factor e^{-2kt}.

6. Conclusions

In summary, for the first time we introduced the density operator of DCL by determining the normalization constant, its photon statistics is governed by $\left[1/(\mathrm{e}^{-f}-1)+|\lambda|^2/(\mathrm{e}^{f}-1)^2\right]$. We also studied the evolution law of passing DCL through an amplitude dissipation channel, we have found that the final state is still a displaced chaotic state, and the displacing amount is affected by the chaotic parameter. We also derived the time evolution formula of the photon number distribution in DCL, which manifestly exhibits the damping factor e^{-2kt}. We have fulfilled our task with the use of the method of integration (summation) within ordered product of operator and the new binomial theorem involving two-variable Hermite polynomials.

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