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Ordered product expansions of operators $(AB)^{\pm m}$ with arbitrary positive integer*

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We arrange quantum mechanical operators $(a^{\dagger}a)^{m}$ in their normally ordered product forms by using Touchard polynomials. Moreover, we derive the anti-normally ordered forms of $(a^{\dagger}a)^{\pm m}$ by using special functions as well as Stirling-like numbers together with the general mutual transformation rule between normal and anti-normal orderings of operators. Further, the $\mathbb{Q}$- and $\mathbb{P}$-ordered forms of $(QP)^{\pm m}$ are also obtained by using an analogy method.

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1. Introduction

Operators, in quantum mechanics and quantum optics, have been considered as an important characteristic by endowing Bose annihilation and creation operators with the basic commutative relation, such as $\left[a,a^{\dagger}\right]=1$. In order to deal with a lot of problems involved in quantum optics, say, calculating some expectation values of physical quality or various matrix elements, the operator ordering is often fallen back on due to its convenience. It is the main reason that the matrix elements of the normally ordered operator function $:f(a,a^{\dagger}):$ in coherent states $|z\rangle=\exp\left(-\frac{1}{2}zz^{*}+za^{\dagger}\right)|0\rangle$ yield[1,2]

$$\langle z|:f(a,a^{\dagger}):|z'\rangle=f(z',z^{*})\exp\left(-\frac{1}{2}zz^{*}-\frac{1}{2}z'z'^{*}+z^{*}z'\right), \quad (1)$$

while the P-representation of density operator can be obtained directly by the anti-normal ordering operators[3,4]

$$\hat{\rho}=\int\frac{\mathrm{d}^{2}z}{\pi}P(z,z^{*})|z\rangle\langle z|=\vdots P(a,a^{\dagger})\vdots, \quad (2)$$

where the symbol $:\,:$ denotes the normal ordering (all creation operators stand on the left of all annihilation operators), and the symbol $\vdots\ \vdots$ refers to the anti-normal ordering (all annihilation operators stand on the left of all creation operators). Apart from these operator ordering forms, some others are proposed, such as the $\mathbb{Q}$-ordering and $\mathbb{P}$-ordering of operators.[5–10] The matrix elements of $\mathbb{Q}$-ordered or $\mathbb{P}$-ordered operator in phase space directly yield

$$\langle q|\mathbb{Q}g(Q,P)\mathbb{Q}|p\rangle=g(q,p)\langle q|p\rangle,$$
$$\langle p|\mathbb{P}h(Q,P)\mathbb{P}|q\rangle=h(q,p)\langle p|q\rangle, \quad (3)$$

where the symbol $\mathbb{Q}\cdots\mathbb{Q}$ represents the $\mathbb{Q}$-ordering (all coordinate operators stand on the left of all momentum operators), and the symbol $\mathbb{P}\cdots\mathbb{P}$ denote the $\mathbb{P}$-ordering (all momentum operators stand on the left of all coordinate operators). Therefore, it is an important task to obtain the various ordered forms of operators as directly as possible.

In quantum optics, these operators such as $(QP)^{m}$ and $(QP)^{-m}$ keep coming up and It is troublesome in handling these operators due to a lack of proper mathematical tools. To our knowledge, there has been no report on a good solution to this problem. In this paper, we shall recast the quantum mechanical operators $(a^{\dagger}a)^{m}$, with m being an arbitrary positive integer, into their normally ordered expressions by using Touchard polynomials. Also, we shall derive the anti-normally ordered expressions of $(a^{\dagger}a)^{\pm m}$ by using special functions as well as Stirling-like numbers together with the general mutual transformation rule between normal and anti-normal orderings of operators.[4–11] Moreover, we shall obtain the $\mathbb{Q}$- and $\mathbb{P}$-ordered forms of $(QP)^{\pm m}$ by using an analogy method. Finally, some applications in the deduced operator ordering identities are discussed.

2. Normally and anti-normally ordered expansion of $(a^{\dagger}a)^{m}$ and $(aa^{\dagger})^{m}$

In this section, we deduce the normally and anti-normally ordered expansions of operators $(a^{\dagger}a)^{m}$ and $(aa^{\dagger})^{m}$, with m being an arbitrary positive integer. In this work we will make full use of a parameter differential method, special functions, and general mutual transformation rule between normal and anti-normal orderings of operators.[4,11]

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2.1. Normally and anti-normally ordered expansion of $(a^{\dagger}a)^m$

By using the completeness relation of Fock space $\sum_{n=0}^{\infty}|n\rangle\langle n|=1$ and $a^{\dagger}a|n\rangle=n|n\rangle$ and IWOP technique, we obtain

$$
\begin{aligned}
(a^{\dagger}a)^m &= (a^{\dagger}a)^m\sum_{n=0}^{\infty}|n\rangle\langle n| = \sum_{n=0}^{\infty}n^m|n\rangle\langle n| \\
&= :\mathrm{e}^{-a^{\dagger}a}\sum_{n=0}^{\infty}\frac{n^m}{n!}(a^{\dagger}a)^n:, \qquad (4)
\end{aligned}
$$

which is the basic normally ordered form of $(a^{\dagger}a)^m$. In the above calculations, the normally ordered vacuum projection operator $|0\rangle\langle 0|=:\mathrm{e}^{-a^{\dagger}a}:$ has been used. In fact, by the parameter differential method

$$
n^m=\frac{\partial^m}{\partial t^m}\mathrm{e}^{tn}\Big|_{t=0}
$$

we may rewrite Eq. (4) as

$$
\begin{aligned}
(a^{\dagger}a)^m &= \frac{\partial^m}{\partial t^m}:\mathrm{e}^{-a^{\dagger}a}\sum_{n=0}^{\infty}\frac{(a^{\dagger}a)^n\mathrm{e}^{tn}}{n!}:\Big|_{t=0} \\
&= \frac{\partial^m}{\partial t^m}:\mathrm{e}^{-a^{\dagger}a}\sum_{n=0}^{\infty}\frac{(a^{\dagger}a\mathrm{e}^{t})^n}{n!}:\Big|_{t=0} \\
&= \frac{\partial^m}{\partial t^m}:\mathrm{e}^{a^{\dagger}a(\mathrm{e}^{t}-1)}:\Big|_{t=0}. \qquad (5)
\end{aligned}
$$

where t is a real parameter. Equation (5) is just the Touchard polynomials defined by

$$
T_m(\xi)=\frac{\partial^m}{\partial t^m}\mathrm{e}^{\xi(\mathrm{e}^{t}-1)}\Big|_{t=0}. \qquad (6)
$$

That is to say, $T_m(\xi)$ is generated from the mother function $\exp[\xi(\mathrm{e}^{t}-1)]$, with

$$
\mathrm{e}^{\xi(\mathrm{e}^{t}-1)}=\sum_{m=0}^{\infty}\frac{T_m(\xi)}{m!}t^m, \qquad (7)
$$

which is a particular case of the (complete) Bell polynomials.[12–14] Further, the differential form of $T_m(\xi)$ can also be obtained as follows (see Appendix A):

$$
T_m(\xi)=\mathrm{e}^{-\xi}\left(\xi\frac{\partial}{\partial\xi}\right)^m\mathrm{e}^{\xi}. \qquad (8)
$$

The more properties of $T_m(\xi)$ may be seen in Appendix A. Hence we have

$$
(a^{\dagger}a)^m=:T_m(a^{\dagger}a):, \qquad (9)
$$

which is just the normally ordered expansion of $(a^{\dagger}a)^m$. As examples, the first several cases are listed below:

$$
\begin{aligned}
(a^{\dagger}a)^1 &= :T_1(a^{\dagger}a):=:a^{\dagger}a:=a^{\dagger}a, \\
(a^{\dagger}a)^2 &= :T_2(a^{\dagger}a):=:(a^{\dagger}a)^2+(a^{\dagger}a): \\
&= a^{\dagger 2}a^2+a^{\dagger}a, \\
(a^{\dagger}a)^3 &= :T_3(a^{\dagger}a):=:(a^{\dagger}a)^3+3(a^{\dagger}a)^2+(a^{\dagger}a): \\
&= a^{\dagger 3}a^3+3a^{\dagger 2}a^2+a^{\dagger}a, \\
(a^{\dagger}a)^4 &= :T_4(a^{\dagger}a):=:(a^{\dagger}a)^4+6(a^{\dagger}a)^3+7(a^{\dagger}a)^2+(a^{\dagger}a): \\
&= a^{\dagger 4}a^4+6a^{\dagger 3}a^3+7a^{\dagger 2}a^2+a^{\dagger}a.
\end{aligned}
$$

To derive the anti-normally ordered form of $(a^{\dagger}a)^m$, recall the general mutual transformation rules between normal and anti-normal orderings[4,11]

$$
\begin{aligned}
:F(a,a^{\dagger}): &= \vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^{\dagger}}\right)F(a,a^{\dagger})\vdots, \\
\vdots F(a,a^{\dagger})\vdots &= :\exp\left(\frac{\partial^2}{\partial a\partial a^{\dagger}}\right)F(a,a^{\dagger}):, \qquad (10)
\end{aligned}
$$

then by using Eqs. (5) and (10) we will be able to obtain

$$
\begin{aligned}
(a^{\dagger}a)^m &= \vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^{\dagger}}\right)\frac{\partial^m}{\partial t^m}\mathrm{e}^{aa^{\dagger}(\mathrm{e}^{t}-1)}\vdots\Big|_{t=0} \\
&= \frac{\partial^m}{\partial t^m}\vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^{\dagger}}\right)\mathrm{e}^{aa^{\dagger}(\mathrm{e}^{t}-1)}\vdots\Big|_{t=0} \\
&= \frac{\partial^m}{\partial t^m}\vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^{\dagger}}\right) \\
&\quad\times\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{-zz^*+za+z^*a^{\dagger}(\mathrm{e}^{t}-1)}\vdots\Big|_{t=0} \\
&= \frac{\partial^m}{\partial t^m}\vdots\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{-\mathrm{e}^{t}zz^*+za+z^*a^{\dagger}(\mathrm{e}^{t}-1)}\vdots\Big|_{t=0} \\
&= \frac{\partial^m}{\partial t^m}\vdots\mathrm{e}^{-t+aa^{\dagger}(1-\mathrm{e}^{-t})}\vdots\Big|_{t=0}, \qquad (11)
\end{aligned}
$$

which is the basic anti-normally ordered form of $(a^{\dagger}a)^m$. In the above calculations the integral formula

$$
\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{-\zeta zz^*+\eta z+z^*\xi}=\frac{1}{\zeta}\mathrm{e}^{\eta\xi/\zeta}
$$

with $\mathrm{Re}(\zeta)>0$ being used. In order to simplify formula (11), one can write

$$
X_m(\xi)=\frac{\partial^m}{\partial t^m}\mathrm{e}^{-t+\xi(1-\mathrm{e}^{-t})}\Big|_{t=0}. \qquad (12)
$$

That is to say, $X_m(\xi)$ is generated from the mother function $\exp[-t+\xi(1-\mathrm{e}^{-t})]$, with

$$
\mathrm{e}^{-t+\xi(1-\mathrm{e}^{-t})}=\sum_{m=0}^{\infty}\frac{X_m(\xi)}{m!}t^m. \qquad (13)
$$

The more properties of $X_m(\xi)$ may be seen in Appendix B. Thus we have

$$
(a^{\dagger}a)^m=\vdots X_m(a^{\dagger}a)\vdots, \qquad (14)
$$

which is just the anti-normally ordered expansion of $(a^{\dagger}a)^m$, whose first several cases are listed as follows:

$$
\begin{aligned}
(a^{\dagger}a)^1 &= \vdots X_1(a^{\dagger}a)\vdots=\vdots a^{\dagger}a-1\vdots=aa^{\dagger}-1, \\
(a^{\dagger}a)^2 &= \vdots X_3(a^{\dagger}a)\vdots=\vdots(a^{\dagger}a)^2-3(a^{\dagger}a)+1\vdots \\
&= \hat{a}^2\hat{a}^{\dagger 2}-3aa^{\dagger}+1, \\
(a^{\dagger}a)^3 &= \vdots X_3(a^{\dagger}a)\vdots=\vdots(a^{\dagger}a)^3-6(a^{\dagger}a)^2+7(a^{\dagger}a)-1\vdots \\
&= a^3a^{\dagger 3}-6a^2a^{\dagger 2}+7aa^{\dagger}-1, \\
(a^{\dagger}a)^4 &= \vdots X_4(a^{\dagger}a)\vdots
\end{aligned}
$$

$$
\begin{aligned}
&= \vdots (a^\dagger a)^4 - 10(a^\dagger a)^3 + 25(a^\dagger a)^2 - 15(a^\dagger a) + 1 \vdots \\
&= a^4 a^{\dagger 4} - 10 a^3 a^{\dagger 3} + 25 a^2 a^{\dagger 2} - 15 a a^\dagger + 1.
\end{aligned}
$$

Although $X_m(\xi)$ may be related to Bell polynomials in some way, it is still a good polynomial due to its brevity and physical use, just as Hermite polynomials and Laguerre polynomials.

2.2. Normally and anti-normally ordered expansions of $(aa^\dagger)^m$

In this subsection we deduce the normally and anti-normally ordered expansions of operators $(aa^\dagger)^m$, with m being an arbitrary positive integer. Considering $[a^\dagger, a] = -1 = [-a, a^\dagger]$ and comparing formula (9), we can deduce that

$$
(aa^\dagger)^m = (-)^m [(-a)a^\dagger]^m = (-)^m \vdots T_m(-aa^\dagger) \vdots . \tag{15}
$$

This is just the anti-normally ordered expansion of operator $(aa^\dagger)^m$.

Moreover, by analogy with formula (14), we can obtain

$$
(aa^\dagger)^m = (-)^m [(-a)a^\dagger]^m = (-)^m : X_m(-aa^\dagger) : , \tag{16}
$$

which is just the normally ordered expansion of operators $(aa^\dagger)^m$.

2.3. Normally and antinormally ordered expansions of $\mathrm{e}^{\lambda a^\dagger a}$ and $\mathrm{e}^{\lambda a a^\dagger}$

Now we deduce the normally and anti-normally ordered expansions of operators $\mathrm{e}^{\lambda a^\dagger a}$ and $\mathrm{e}^{\lambda a a^\dagger}$. By using the power series expansion of $\mathrm{e}^{\lambda a^\dagger a}$ and Eqs. (9) and (7) we have

$$
\begin{aligned}
\mathrm{e}^{\lambda a^\dagger a} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (a^\dagger a)^n \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} : T_m(a^\dagger a) : = : \mathrm{e}^{a^\dagger a (\mathrm{e}^\lambda - 1)} : .
\end{aligned} \tag{17}
$$

This is just the normally ordered expansion of exponential operator $\mathrm{e}^{\lambda a^\dagger a}$. Let $\mathrm{e}^\lambda - 1 = \mu$ then the following equation is directly obtained:

$$
: \mathrm{e}^{\mu a^\dagger a} : = \mathrm{e}^{a^\dagger a \ln(1+\mu)}, \tag{18}
$$

which is the formula where the normal ordering symbol $::$ from $: \mathrm{e}^{a^\dagger a(\mathrm{e}^\lambda - 1)} :$ has been taken off.

On the other hand, using the power series expansion of $\mathrm{e}^{\lambda a^\dagger a}$ and Eqs. (14) and (13) turn into

$$
\mathrm{e}^{\lambda a^\dagger a} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \vdots X_m(a^\dagger a) \vdots = \vdots \mathrm{e}^{-\lambda + a^\dagger a(1 - \mathrm{e}^{-\lambda})} \vdots . \tag{19}
$$

This is just the anti-normally ordered expansion of exponential operator $\mathrm{e}^{\lambda a^\dagger a}$. Letting $1 - \mathrm{e}^{-\lambda} = \mu$ the following is directly obtained:

$$
\vdots \mathrm{e}^{\mu a^\dagger a} \vdots = \frac{1}{1-\mu} \mathrm{e}^{-a^\dagger a \ln(1-\mu)}, \tag{20}
$$

which is the formula where the anti-normal ordering symbol $\vdots\ \vdots$ has been removed from $\vdots \mathrm{e}^{a^\dagger a(\mathrm{e}^\lambda - 1)} \vdots$.

Similarly , by using Eqs. (15), (7), (16), and (13) we can obtain

$$
\mathrm{e}^{\lambda a a^\dagger} = \vdots \mathrm{e}^{a^\dagger a (1 - \mathrm{e}^{-\lambda})} \vdots , \qquad \mathrm{e}^{\lambda a a^\dagger} = \mathrm{e}^\lambda : \mathrm{e}^{a^\dagger a(\mathrm{e}^\lambda - 1)} : , \tag{21}
$$

which are the anti-normal and normal orderings of $\mathrm{e}^{\lambda a a^\dagger}$, respectively.

3. $\mathbb{Q}$- and $\mathbb{P}$-ordered expansions of $(QP)^m$ and $(PQ)^m$

In the following section, we deduce the $\mathbb{Q}$- and $\mathbb{P}$-ordered expansions of operators $(QP)^m$ and $(PQ)^m$, with m being an arbitrary positive integer.

In view of $[Q, \frac{\mathrm{i}}{\hbar} P] = -1 = [a^\dagger, a]$, by analogy with formula (9) we can deduce that

$$
\begin{aligned}
(QP)^m &= (-\mathrm{i}\hbar)^m \left[Q \left(\frac{\mathrm{i}}{\hbar} P \right) \right]^m \\
&= (-\mathrm{i}\hbar)^m \mathbb{Q} T_m \left(\frac{\mathrm{i}}{\hbar} QP \right) \mathbb{Q},
\end{aligned} \tag{22}
$$

which is just the $\mathbb{Q}$-ordered expansion of $(QP)^m$. The correctness of Eq. (22) can be proved by mathematical induction method. Further, by analogy with formula (14) we can also derive

$$
\begin{aligned}
(QP)^m &= (-\mathrm{i}\hbar)^m \left[Q \left(\frac{\mathrm{i}}{\hbar} P \right) \right]^m \\
&= (-\mathrm{i}\hbar)^m \mathbb{P} X_m \left(\frac{\mathrm{i}}{\hbar} QP \right) \mathbb{P},
\end{aligned} \tag{23}
$$

which is just the $\mathbb{P}$-ordered expansion of $(QP)^m$. Similarly, by analogy we can also derive

$$
(PQ)^m = (\mathrm{i}\hbar)^m \left[\left(\frac{1}{\mathrm{i}\hbar} P \right) Q \right]^m = (\mathrm{i}\hbar)^m \mathbb{P} T_m \left(\frac{1}{\mathrm{i}\hbar} QP \right) \mathbb{P}, \tag{24}
$$

$$
(PQ)^m = (\mathrm{i}\hbar)^m \left[\left(\frac{1}{\mathrm{i}\hbar} P \right) Q \right]^m = (\mathrm{i}\hbar)^m \mathbb{Q} X_m \left(\frac{1}{\mathrm{i}\hbar} QP \right) \mathbb{Q}, \tag{25}
$$

which are just the $\mathbb{P}$- and $\mathbb{Q}$-ordered expansions of $(PQ)^m$, respectively.

By using Eqs. (22)–(25), (7) and (13) or by analogy with formulas (17), (19), and (21), we can derive

$$
\begin{aligned}
\mathrm{e}^{\lambda QP} &= \mathbb{Q} \exp \left[\frac{\mathrm{i}}{\hbar} QP (\mathrm{e}^{-\mathrm{i}\hbar\lambda} - 1) \right] \mathbb{Q} \\
&= \mathbb{P} \exp \left[\mathrm{i}\hbar\lambda + \frac{\mathrm{i}}{\hbar} QP (1 - \mathrm{e}^{\mathrm{i}\hbar\lambda}) \right] \mathbb{P},
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\mathrm{e}^{\lambda PQ} &= \mathbb{P} \exp \left[-\frac{\mathrm{i}}{\hbar} QP (\mathrm{e}^{\mathrm{i}\hbar\lambda} - 1) \right] \mathbb{P} \\
&= \mathbb{Q} \exp \left[-\mathrm{i}\hbar\lambda + \frac{\mathrm{i}}{\hbar} QP (\mathrm{e}^{-\mathrm{i}\hbar\lambda} - 1) \right] \mathbb{Q}.
\end{aligned} \tag{27}
$$

These are the $\mathbb{Q}$- and $\mathbb{P}$-ordered expansions of $\mathrm{e}^{\lambda QP}$ and $\mathrm{e}^{\lambda PQ}$, respectively.

4. Ordered expansions of operators $(AB)^{-m}$

Now we deduce the ordered expansions of operators such as $(AB)^{-m}$, with m being an arbitrary positive integer. We know that $(a^\dagger a)\left|n\right\rangle = n\left|n\right\rangle$, $n = 0,1,2,3,\ldots$. To be precise, the eigenvalue of the number operator $(a^\dagger a)$ is discrete and contains zero. So the operator $(a^\dagger a)$ has no inverse, that is to say, $(a^\dagger a)^{-1}$ does not exist.

4.1. Normally and antinormally ordered expansions of $(aa^\dagger)^{-m}$

From $(a^\dagger a)\left|n\right\rangle = n\left|n\right\rangle$ we obtain $(aa^\dagger)^m\left|n\right\rangle = (a^\dagger a+1)^m\left|n\right\rangle = (n+1)^m\left|n\right\rangle$, n= 0, 1, 2, 3, ... Thus we see that the operator $(aa^\dagger)^m$ has inverse, denoted by $(aa^\dagger)^{-m} \equiv 1/(aa^\dagger)^m$. By using the completeness relation of the Fock space, we can obtain

$$\begin{aligned}\frac{1}{(aa^\dagger)^m} &= \sum_{n=0}^{\infty}\frac{1}{(aa^\dagger)^m}\left|n\right\rangle\left\langle n\right| = \sum_{n=0}^{\infty}\frac{1}{(n+1)^m}\left|n\right\rangle\left\langle n\right| \\ &= :\mathrm{e}^{-aa^\dagger}\sum_{n=0}^{\infty}\frac{(aa^\dagger)^n}{n!(n+1)^m}:,\end{aligned}$$

which is a basic normally ordered form of $(aa^\dagger)^{-m}$. In order to simplify this formula, by using the power series expansion $\mathrm{e}^{-x} = \sum_{l=0}^{\infty}(-)^l x^l/l!$, we can obtain

$$\begin{aligned}\frac{1}{(aa^\dagger)^m} &= :\sum_{l=0}^{\infty}\frac{(-)^l}{l!}(aa^\dagger)^l\sum_{n=0}^{\infty}\frac{(aa^\dagger)^n}{n!(n+1)^m}: \\ &= :\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-)^l(aa^\dagger)^{n+l}}{l!n!(n+1)^m}: \\ &= :\sum_{k=0}^{\infty}(aa^\dagger)^k\sum_{l=0}^{k}\frac{(-)^l}{l!(k-l)!(k-l+1)^m}:. \quad (28)\end{aligned}$$

In the last step of the above calculations, we have used the rearranging double summation formula

$$\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}A_m B_l = \sum_{k=0}^{\infty}\sum_{l=0}^{k}A_{k-l}B_l. \quad (29)$$

Now we introduce a kind of special numbers here, defined by

$$S_k(m) = \sum_{l=0}^{k}\frac{(-)^l}{l!(k-l)!(k-l+1)^m}, \quad (30)$$

then equation (28) can be rewritten as

$$\frac{1}{(aa^\dagger)^m} = \sum_{k=0}^{\infty}S_k(m)a^{\dagger k}a^k, \quad (31)$$

which is just the compact normally ordered form of $(aa^\dagger)^{-m}$. From the definition of $S_k(m)$ we can easily derive that $S_k(0) = \delta_{k,0}$, $S_0(m) = 1$ and

$$\begin{aligned}S_k(1) &= \sum_{l=0}^{k}\frac{(-)^l}{l!(k+1-l)!} = \frac{1}{(k+1)!}\sum_{l=0}^{k}\frac{(k+1)!(-)^l}{l!(k+1-l)!} \\ &= \frac{1}{(k+1)!}\left[-(-)^{k+1}+\sum_{l=0}^{k+1}\frac{(k+1)!(-)^l}{l!(k+1-l)!}\right] = \frac{(-)^k}{(k+1)!}.\end{aligned}$$

Then we have

$$\frac{1}{aa^\dagger} = \sum_{k=0}^{\infty}\frac{(-)^k}{(k+1)!}a^{\dagger k}a^k, \quad (32)$$

which is just the normally ordered expansion of $(aa^\dagger)^{-1}$. We can prove that

$$(aa^\dagger)\sum_{k=0}^{\infty}\frac{(-)^k}{(k+1)!}a^{\dagger k}a^k = \sum_{k=0}^{\infty}\frac{(-)^k}{(k+1)!}a^{\dagger k}a^k(aa^\dagger) = 1.$$

That is to say, equation (32) is indeed the inverse operator of $(aa^\dagger)$. The more properties of the defined special number $S_k(m)$ may be seen in Appendix C.

Moreover, by using Eq. (31) and the general mutual transformation rule between normal and anti-normal orderings of operators Eq.(10), we can obtain

$$\begin{aligned}\frac{1}{(aa^\dagger)^m} &= \vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^\dagger}\right)\sum_{k=0}^{\infty}S_k(m)a^{\dagger k}a^k\vdots \\ &= \vdots\sum_{k=0}^{\infty}S_k(m)\exp\left(-\frac{\partial^2}{\partial a\partial a^\dagger}\right)a^{\dagger k}a^k\vdots \\ &= \vdots\sum_{k=0}^{\infty}S_k(m)\mathrm{H}_{k,k}(a,a^\dagger)\vdots \\ &= \vdots\sum_{k=0}^{\infty}(-)^k S_k(m)\mathrm{L}_k(aa^\dagger)\vdots, \quad (33)\end{aligned}$$

which is just the anti-normally ordered form of $(aa^\dagger)^{-m}$. Here, $\mathrm{H}_{m,n}(x,y)$ is a two-variable Hermite polynomial function,[4,15,16] defined as

$$\begin{aligned}\mathrm{H}_{m,n}(x,y) &= \mathrm{e}^{xy}\left(-\frac{\partial}{\partial y}\right)^m\left(-\frac{\partial}{\partial x}\right)^n\mathrm{e}^{-xy} \\ &= \exp\left(-\frac{\partial^2}{\partial x\partial y}\right)x^m y^n.\end{aligned}$$

$\mathrm{L}_m(\xi)$ in Eq. (33) is the Laguerre polynomial, defined by

$$\mathrm{L}_m(\xi) = \mathrm{e}^{\xi}\frac{\partial^m}{\partial\xi^m}\left(\xi^m\mathrm{e}^{-\xi}\right) = \sum_{l=0}^{m}\frac{(m!)^2}{(l!)^2(m-l)!}(-\xi)^l,$$

and the $\mathrm{H}_{m,m}(x,y)$ and $\mathrm{L}_m(\xi)$ are related by

$$\begin{aligned}\mathrm{H}_{m,m}(x,y) &= \mathrm{e}^{xy}\left(-\frac{\partial}{\partial y}\right)^m\left(-\frac{\partial}{\partial x}\right)^m\mathrm{e}^{-xy} \\ &= (-)^m\mathrm{e}^{xy}\frac{\partial^m}{\partial y^m}y^m\mathrm{e}^{-xy} = (-)^m\mathrm{L}_m(\xi)|_{\xi=xy}.\end{aligned}$$

In particular, according to Eq. (32) we have

$$\frac{1}{\hat{a}\hat{a}^\dagger} = \vdots\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\mathrm{L}_k(aa^\dagger)\vdots. \quad (34)$$

4.2. $\mathbb{Q}$- and $\mathbb{P}$-ordered expansions of $(QP)^{-m}$ and $(PQ)^{-m}$

Due to $QP = (QP+PQ)/2 + \mathrm{i}\hbar/2$ and $(QP+PQ)$ being a Hermitian operator, which implies the eigenvalue of (QP) does not contain zero, we know that the operator (QP) has its inverse operator. To be precise, $(QP)^{-1} \equiv 1/(QP)$ does exit.

In view of $[Q, \frac{1}{\mathrm{i}\hbar}P] = 1 = [a,a^\dagger]$, by analogy with the formula (28) we can deduce that

$$\begin{aligned}\frac{1}{(QP)^m} &= \frac{1}{(\mathrm{i}\hbar)^m}\frac{1}{\left[Q\left(\frac{1}{\mathrm{i}\hbar}P\right)\right]^m} \\ &= \frac{1}{(\mathrm{i}\hbar)^m}\mathbb{P}\sum_{k=0}^{\infty}(\mathrm{i}\hbar)^{-k}S_k(m)Q^kP^k\mathbb{P} \\ &= \frac{1}{(\mathrm{i}\hbar)^m}\sum_{k=0}^{\infty}(\mathrm{i}\hbar)^{-k}S_k(m)P^kQ^k, \qquad (35)\end{aligned}$$

which is just the $\mathbb{P}$-ordered expansion of $(QP)^{-m}$, expressed by the special number $S_k(m)$.

Similarly, by analogy with formula (33) we obtain

$$\begin{aligned}\frac{1}{(QP)^m} &= \frac{1}{(\mathrm{i}\hbar)^m}\frac{1}{\left[Q\left(\frac{1}{\mathrm{i}\hbar}P\right)\right]^m} \\ &= \frac{1}{(\mathrm{i}\hbar)^m}\mathbb{Q}\sum_{k=0}^{\infty}(-)^kS_k(m)\mathrm{L}_k\left(\frac{1}{\mathrm{i}\hbar}QP\right)\mathbb{Q}, \qquad (36)\end{aligned}$$

which is just the $\mathbb{Q}$-ordered expansion of $(QP)^{-m}$, expressed by the special number $S_k(m)$ and Laguerre polynomial.

In a manner completely similar to the way of deriving Eqs. (35) and (36), we may derive that

$$\begin{aligned}\frac{1}{(PQ)^m} &= \left(\frac{\mathrm{i}}{\hbar}\right)^m\frac{1}{\left[\left(\frac{\mathrm{i}}{\hbar}P\right)Q\right]^m} \\ &= \left(\frac{\mathrm{i}}{\hbar}\right)^m\mathbb{Q}\sum_{k=0}^{\infty}S_k(m)\left(\frac{\mathrm{i}}{\hbar}\right)^kQ^kP^k\mathbb{Q}, \\ \frac{1}{(PQ)^m} &= \left(\frac{\mathrm{i}}{\hbar}\right)^m\frac{1}{\left[\left(\frac{\mathrm{i}}{\hbar}P\right)Q\right]^m} \\ &= \left(\frac{\mathrm{i}}{\hbar}\right)^m\mathbb{P}\sum_{k=0}^{\infty}(-)^kS_k(m)\mathrm{L}_k\left(\frac{\mathrm{i}}{\hbar}QP\right)\mathbb{P}.\end{aligned}$$

These are the $\mathbb{Q}$- and $\mathbb{P}$-ordered product expressions of the operator $(PQ)^{-m}$.

5. Applications of some identities above

In the present section we discuss some applications of the above new identities. Since the operator $(QP)^m$ is a complex one, with m being a big positive integer, it is tricky to calculate its matrix elements in phase space without Eqs. (22) and (23). Here we can see that it is easy to derive the matrix elements by using the new identity (22). According to Eq. (22) we can immediately obtain

$$\begin{aligned}\langle q|\,(QP)^m\,|p\rangle &= \frac{(-\mathrm{i}\hbar)^m}{\sqrt{2\pi\hbar}}X_m\left(\frac{\mathrm{i}}{\hbar}qp\right)\exp\left(\frac{\mathrm{i}}{\hbar}qp\right) \\ &= \frac{(-\mathrm{i}\hbar)^m}{\sqrt{2\pi\hbar}}\exp\left(\frac{\mathrm{i}}{\hbar}qp\right) \\ &\quad\times\sum_{n=0}^{m}\left(\frac{\mathrm{i}}{\hbar}qp\right)^n\sum_{l=0}^{n}(-)^l\frac{(n-l)^m}{l!\,(n-l)!}, \qquad (37)\end{aligned}$$

which is just the matrix elements of $(QP)^m$ in $q-p$ phase space. In the last step of the above calculations the equation (A4) in Appendix A has been used.

As another example, by using Eq. (34) we easily obtain the $\mathbb{P}$-representations of $(aa^\dagger)^{-1}$ as follows:

$$\begin{aligned}P(z,z^*) &= \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\mathrm{L}_k(zz^*) \\ &= \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\sum_{l=0}^{k}\frac{k!}{(k-l)!}\binom{k}{l}(-zz^*)^{k-l}.\end{aligned}$$

As the last example, we discuss the application of the general mutual transformation rules of normal and antinormal orderings Eq. (10). Using $[a^n,a^\dagger]=na^{n-1}$ and $[a,a^{\dagger n}]=na^{\dagger n-1}$, we have

$$\begin{aligned}&a^{\dagger n+1}a^m = a^{\dagger n}a^ma^\dagger - ma^{\dagger n}a^{m-1}, \\ &a^{\dagger n}a^{m+1} = aa^{\dagger n}a^m - na^{\dagger n-1}a^m. \qquad (38)\end{aligned}$$

On the other hand, by using Eq. (10) we can obtain

$$\begin{aligned}a^{\dagger n+1}a^m &= \vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^\dagger}\right)a^{\dagger n+1}a^m\vdots = \vdots\,\mathrm{H}_{n+1,m}(a^\dagger a)\vdots, \\ a^{\dagger n}a^m &= \vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^\dagger}\right)a^{\dagger n}a^m\vdots = \vdots\,\mathrm{H}_{n,m}(a^\dagger a)\vdots, \\ a^{\dagger n}a^{m-1} &= \vdots\,\mathrm{e}^{-\frac{\partial^2}{\partial a\partial a^\dagger}}a^{\dagger n}a^{m-1}\vdots = \vdots\,\mathrm{H}_{n,m-1}(a^\dagger a)\vdots. \qquad (39)\end{aligned}$$

$$\begin{aligned}a^{\dagger n}a^{m+1} &= \vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^\dagger}\right)a^{\dagger n}a^{m+1}\vdots = \vdots\,\mathrm{H}_{n,m+1}(a^\dagger a)\vdots, \\ a^{\dagger n}a^m &= \vdots\exp\left(-\frac{\partial^2}{\partial a\partial a^\dagger}\right)a^{\dagger n}a^m\vdots = \vdots\,\mathrm{H}_{n,m}(a^\dagger a)\vdots, \\ a^{\dagger n-1}a^m &= \vdots\,\mathrm{e}^{-\frac{\partial^2}{\partial a\partial a^\dagger}}a^{\dagger n-1}a^m\vdots = \vdots\,\mathrm{H}_{n-1,m}(a^\dagger a)\vdots. \qquad (40)\end{aligned}$$

Substituting Eqs. (39) and (40) into Eq. (38) leads to that

$$\begin{aligned}&\vdots\,\mathrm{H}_{n+1,m}(a^\dagger,a)\vdots = \vdots\,\mathrm{H}_{n,m}(a^\dagger,a)\vdots\,a^\dagger - m\vdots\,\mathrm{H}_{n,m-1}(a^\dagger,a)\vdots, \\ &\vdots\,\mathrm{H}_{n,m+1}(a^\dagger,a)\vdots = a\vdots\,\mathrm{H}_{n,m}(a^\dagger,a)\vdots - n\vdots\,\mathrm{H}_{n-1,m}(a^\dagger,a)\vdots. \qquad (41)\end{aligned}$$

which indicate the recurrence relations of $\mathrm{H}_{n,m}(\eta\xi)$,

$$\begin{aligned}&\mathrm{H}_{n+1,m}(\eta,\xi) = \mathrm{H}_{n,m}(\eta,\xi)\eta - m\vdots\,\mathrm{H}_{n,m-1}(\eta,\xi), \\ &\mathrm{H}_{n,m+1}(\eta,\xi) = \xi\mathrm{H}_{n,m}(\eta,\xi) - n\mathrm{H}_{n-1,m}(\eta,\xi). \qquad (42)\end{aligned}$$

So, it is seen that the general mutual transformation rules of normal and antinormal orderings Eq. (10) also provide an approach to deriving the well-known recurrence relations of two-variable Hermite polynomial.

6. Conclusions

In this work, we recast the quantum mechanical operators $(a^\dagger a)^{\pm m}$ and $(aa^\dagger)^{\pm m}$, with m being an arbitrary positive integer, into their normally ordered expansions by using Touchard polynomials and using special functions as well as a kind of special numbers (called Stirling-like numbers). Also, we derive their antinormally ordered expressions via the general mutual transformation rules between normal and antinomal orderings of operators. Moreover, the $\mathbb{Q}$- and $\mathbb{P}$-ordered forms of $(QP)^{\pm m}$ are also obtained by using an analogy method. Finally, some applications of these new identities are discussed.

Appendix A

The differential form and recursion relation of Touchard polynomials $T_m(\xi)$.

From the definition of Touchard polynomials $T_m(\xi)$ we have

$$T_m(\xi) = \frac{\partial^m}{\partial t^m} \mathrm{e}^{\xi(\mathrm{e}^t-1)}\bigg|_{t=0} = \mathrm{e}^{-\xi}\frac{\partial^m}{\partial t^m}\mathrm{e}^{\xi\,\mathrm{e}^t}\bigg|_{t=0}. \tag{A1}$$

Making transformation $\mathrm{e}^t = \tau$ leads directly to that

$$\begin{aligned} T_m(\xi) &= \mathrm{e}^{-\xi}\left(\frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau}\right)^m \mathrm{e}^{\xi\tau}\bigg|_{\tau=1} = \mathrm{e}^{-\xi}\left(\tau\frac{\partial}{\partial\tau}\right)^m \mathrm{e}^{\xi\tau}\bigg|_{\tau=1} \\ &= \mathrm{e}^{-\xi}\left(\xi\frac{\partial}{\partial\xi}\right)^m \mathrm{e}^{\xi\tau}\bigg|_{\tau=1} = \mathrm{e}^{-\xi}\left(\xi\frac{\partial}{\partial\xi}\right)^m \mathrm{e}^{\xi}, \end{aligned} \tag{A2}$$

which is just the differential form of Touchard polynomial $T_m(\xi)$. From Eq. (A2) we can easily derive

$m=0,\quad T_0(\xi)=1,$

$m=1,\quad T_1(\xi)=\xi,$

$m=2,\quad T_2(\xi)=\xi^2+\xi,$

$m=3,\quad T_3(\xi)=\xi^3+3\xi^2+\xi,$

$m=4,\quad T_4(\xi)=\xi^4+6\xi^3+7\xi^2+\xi,$

$m=5,\quad T_5(\xi)=\xi^5+10\xi^4+25\xi^3+15\xi^2+\xi,\ \cdots$

From Eq. (A2) we can obtain

$$\begin{aligned} T_{m+1}(\xi) &= \mathrm{e}^{-\xi}\left(\xi\frac{\partial}{\partial\xi}\right)^{m+1}\mathrm{e}^{\xi} \\ &= \mathrm{e}^{-\xi}\left(\xi\frac{\partial}{\partial\xi}\right)\mathrm{e}^{\xi}\,\mathrm{e}^{-\xi}\left(\xi\frac{\partial}{\partial\xi}\right)^m\mathrm{e}^{\xi} \\ &= \mathrm{e}^{-\xi}\left(\xi\frac{\partial}{\partial\xi}\right)\mathrm{e}^{\xi}T_m(\xi) \\ &= \mathrm{e}^{-\xi}\xi\left(\mathrm{e}^{\xi}T_m(\xi)+\mathrm{e}^{\xi}\frac{\partial T_m(\xi)}{\partial\xi}\right) \\ &= \xi T_m(\xi)+\xi\frac{\partial T_m(\xi)}{\partial\xi}. \end{aligned} \tag{A3}$$

This is just the recursion relation of Touchard polynomials. In addition, we can deduce the power series expansion of $T_n(\xi)$ as follows:

$$T_m(\xi) = \sum_{n=0}^{m}\xi^n\sum_{l=0}^{n}(-)^l\frac{(n-l)^m}{l!\,(n-l)!} = \sum_{n=0}^{m}S(m,n)\xi^n, \tag{A4}$$

where $S(mn)$ is the Stirling numbers of the second kind.[12]

The complete Bell polynomial $B_n(\xi)$ is defined as[12]

$$\exp\left(\sum_{m\geq 1}y_m\frac{t^m}{m!}\right) = \sum_{n\geq 0}B_n(y_1,y_2,\cdots,y_n)\frac{t^n}{n!}, \tag{A5}$$

where for convenience, B_0 is set to be 1, *i.e.*, $B_0=1$. When $y_m=\xi$ for $m=1,2,3,\ldots$, equation (A5) reduces to

$$\mathrm{e}^{\xi(\mathrm{e}^t-1)} = \sum_{n\geq 0}B_n(\xi,\xi,\ldots,\xi)\frac{t^n}{n!}. \tag{A6}$$

By comparing Eq. (A6) with Eq. (7) one can know that Touchard polynomial is a particular case of the (complete) Bell polynomial, *i.e.*, $T_n(\xi)=B_n(\xi,\xi,\ldots,\xi)$.

Appendix B

The differential form and recursion relation of new polynomial $X_m(\xi)$.

From the definition of the new polynomial $X_m(\xi)$ we can derive

$$\begin{aligned} X_m(\xi) &= \frac{\partial^m}{\partial t^m}\mathrm{e}^{-t+\xi(1-\mathrm{e}^{-t})}\bigg|_{t=0} \\ &= \mathrm{e}^{\xi}\frac{\partial^m}{\partial t^m}\mathrm{e}^{-t}\mathrm{e}^{-\xi\,\mathrm{e}^{-t}}\bigg|_{t=0}. \end{aligned} \tag{B1}$$

Let $\mathrm{e}^{-t}=\tau$, then we will have

$$\begin{aligned} X_m(\xi) &= \mathrm{e}^{\xi}\left(-\tau\frac{\partial}{\partial\tau}\right)^m\tau\mathrm{e}^{-\xi\tau}\bigg|_{\tau=1} = \mathrm{e}^{\xi}\underbrace{\left(-\tau\frac{\partial}{\partial\tau}\right)\left(-\tau\frac{\partial}{\partial t}\right)\cdots\left(-\tau\frac{\partial}{\partial\tau}\right)\left(-\tau\frac{\partial}{\partial\tau}\right)}_{m\text{ times}}\tau\mathrm{e}^{-\xi\tau}\bigg|_{\tau=1} \\ &= \mathrm{e}^{\xi}\tau\underbrace{\left(-\frac{\partial}{\partial\tau}\tau\right)\left(-\frac{\partial}{\partial t}\tau\right)\cdots\left(-\frac{\partial}{\partial\tau}\tau\right)\left(-\frac{\partial}{\partial\tau}\tau\right)}_{m\text{ times}}\mathrm{e}^{-\xi\tau}\bigg|_{\tau=1} \\ &= \mathrm{e}^{\xi}\tau\underbrace{\left(-\frac{\partial}{\partial\tau}\xi\right)\left(-\frac{\partial}{\partial t}\xi\right)\cdots\left(-\frac{\partial}{\partial\tau}\xi\right)\left(-\frac{\partial}{\partial\tau}\xi\right)}_{m\text{ times}}\mathrm{e}^{-\xi\tau}\bigg|_{\tau=1} = \mathrm{e}^{\xi}\left(-\frac{\partial}{\partial\xi}\xi\right)^m\mathrm{e}^{-\xi}. \end{aligned} \tag{B2}$$

This is just the differential form of new polynomial $X_m(\xi)$. From Eq. (B2) we can easily derive

$m=0,\quad X_0(\xi)=1,$

$m=1,\quad X_1(\xi)=\xi-1,$

$m=2,\quad X_2(\xi)=\xi^2-3\xi+1,$

$m=3,\quad X_3(\xi)=\xi^3-6\xi^2+7\xi-1,$

$m=4,\quad X_4(\xi)=\xi^4-10\xi^3+25\xi^2-15\xi+1,$

$m=5,\quad X_5(\xi)=\xi^5-15\xi^4+65\xi^3-90\xi^2+31\xi-1,$

$\cdots$

Further, we can also obtain the recursion relation of

$X_m(\xi)$, say,

$$X_{m+1}(\xi)=(\xi-1)X_m(\xi)-\xi\frac{\partial X_m(\xi)}{\partial\xi}. \tag{B3}$$

The power series expansion of $X_m(\xi)$ reads

$$X_m(\xi)=\sum_{n=0}^{m}\xi^n\sum_{l=0}^{n}(-)^{m+l}\frac{(l+1)^m}{l!(n-l)!}. \tag{B4}$$

Appendix C

The special number is defines as

$$S_k(m)=\sum_{l=0}^{k}\frac{(-)^l}{l!(k-l)!(k+1-l)^m}.$$

This new defined special number $S_k(m)$ is a little like the Stirling number of the second kind

$$S(m,n)=\frac{1}{n!}\sum_{j=0}^{n}(-)^j\frac{n!}{j!(n-j)!}(n-j)^m\equiv\left\{\begin{matrix}m\\n\end{matrix}\right\},$$

but extremely different from it. The purpose for defining the new special number $S_k(m)$ here is to express the expansion of the operator such as $(aa^\dagger)^{-m}$ and $(QP)^{-m}$. We may call $S_k(m)$ the Stirling-like number. The first part of the numbers of $S_k(m)$ are listed in Table C1 below.

Table C1. The first part of the numbers for $S_k(m)$.

$S_k(m)$		k							
		0	1	2	3	4	5	6	7
	0	1	0	0	0	0	0	0	0
	1	1	$\frac{-1}{2!}$	$\frac{1}{3!}$	$\frac{-1}{4!}$	$\frac{1}{5!}$	$\frac{-1}{6!}$	$\frac{1}{7!}$	$\frac{-1}{8!}$
	2	1	$\frac{-3}{4}$	$\frac{11}{36}$	$\frac{-25}{288}$	$\frac{137}{7200}$	$\frac{-49}{14400}$	$\frac{121}{235200}$	$\frac{-761}{11289600}$
	3	1	$\frac{-7}{8}$	$\frac{85}{216}$	$\frac{-415}{3456}$	$\frac{12019}{432000}$	$\frac{-13489}{2592000}$	…	…
m	4	1	$\frac{-15}{16}$	$\frac{575}{1296}$	$\frac{-5845}{41472}$	$\frac{874853}{25920000}$	$\frac{-336581}{51840000}$	…	…
	5	1	$\frac{-31}{32}$	$\frac{3661}{7776}$	$\frac{-76111}{497664}$	$\frac{58067611}{1555200000}$	$\frac{-68165041}{9331200000}$	…	…
	6	1	$\frac{-63}{64}$	$\frac{22631}{46656}$	$\frac{-952525}{5971968}$	…	…	…	…
	7	1	$\frac{-127}{128}$	$\frac{137845}{279936}$	$\frac{-11679655}{71663616}$	…	…	…	…
	8	1	$\frac{-255}{256}$	$\frac{833375}{1679616}$	$\frac{-141710965}{859963392}$	…	…	…	…

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