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Bifurcation analysis and exact traveling wave solutions for (2+1)-dimensional generalized modified dispersive water wave equation*

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We investigate (2+1)-dimensional generalized modified dispersive water wave (GMDWW) equation by utilizing the bifurcation theory of dynamical systems. We give the phase portraits and bifurcation analysis of the plane system corresponding to the GMDWW equation. By using the special orbits in the phase portraits, we analyze the existence of the traveling wave solutions. When some parameter takes special values, we obtain abundant exact kink wave solutions, singular wave solutions, periodic wave solutions, periodic singular wave solutions, and solitary wave solutions for the GMDWW equation.

Keywords: bifurcation theory, generalized modified dispersive water wave equation, traveling wave solution

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1. Introduction

The (2+1)-dimensional modified dispersive water wave equation[1–8]

$$\begin{cases} u_{yt}+u_{xxy}-2w_{xx}-2(uu_y)_x=0, \\ w_t-w_{xx}-2(uw)_x=0 \end{cases} \tag{1}$$

is used to describe the nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth. Equation (1) has been widely used in fluid dynamics, nonlinear optics, and plasma physics. Zheng[3] used the variable separation approach to obtain a number of structures of the localized solutions of Eq. (1). Li and Zhang[4] obtained abundant non-traveling wave solutions of Eq. (1) by utilizing the generalized projective Riccati equation method. Ma[5] used the projective Riccati equation expansion method to obtained three variable separation solutions of Eq. (1). Huang[6] obtained periodic folded wave patterns of Eq. (1) by using the WTC truncation method. Wen and Xu[7] applied the Bäcklund transformation and the Hirota bilinear method to obtain multiple soliton solutions of Eq. (1). Ren *et al.*[8] used the standard Hirota bilinear method to get a number of lump solutions of Eq. (1).

In this work, we employ the bifurcation theory of dynamical systems[9–18] to study the following (2+1)-dimensional generalized modified dispersive water wave (GMDWW) equation:

$$\begin{cases} u_{yt}+u_{xxy}-aw_{xx}-a(uu_y)_x=0, \\ w_t-w_{xx}-a(uw)_x=0, \end{cases} \tag{2}$$

where a is a positive constant. We obtain the phase portraits of plane system corresponds to the GMDWW equation. We analyze the existence of the traveling wave solutions by using the special orbits in the phase portraits. When some parameter takes special values, we obtain abundant exact kink wave solutions, singular wave solutions, periodic wave solutions, and periodic singular wave solutions, and exact solitary wave solutions for the GMDWW equation.

The remainder of this work is organized as follows. In Section 2, we study the phase portraits and bifurcation analysis for Eq. (2). In Section 3, we obtain abundant exact traveling wave solutions of Eq. (2). The profiles of some exact traveling wave solutions are given in Section 4. A brief conclusion is given in Section 5.

2. Phase portraits and bifurcation analysis

By using the following transformation:

$$u=\phi(\zeta), \quad w=\psi(\zeta), \quad \zeta=x+y-ct, \tag{3}$$

where c is a positive constant wave speed, equation (2) can be reduced to the following equation:

$$\begin{cases} -c\phi''+\phi'''-a\psi''-a(\phi\phi')'=0, \\ -c\psi'-\psi''-a(\phi\psi)'=0. \end{cases} \tag{4}$$

Integrating the above first equation twice with regard to ζ and letting the integral constants be zero, we obtain

$$-c\phi+\phi'-a\psi-\frac{a}{2}\phi^2=0. \tag{5}$$

Integrating the second equation of Eq. (4) once with regard to ζ, we obtain

$$-c\psi-\psi'-a\phi\psi=g_1, \tag{6}$$

where g_1 is the integral constant.

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Using Eqs. (5) and (6), we obtain

$$\phi'' = \frac{a^2}{2}\phi^3 + \frac{3ac}{2}\phi^2 + c^2\phi + g. \tag{7}$$

where $g = -ag_1$.

From Eq. (7), we establish a planar system

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \varphi, \\ \dfrac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = \dfrac{a^2}{2}\phi^3 + \dfrac{3ac}{2}\phi^2 + c^2\phi + g. \end{cases} \tag{8}$$

Evidently, the system (8) is a Hamiltonian system with the following Hamiltonian function:

$$H(\phi,\varphi) = \frac{1}{2}\varphi^2 - \frac{a^2}{8}\phi^4 - \frac{ac}{2}\phi^3 - \frac{c^2}{2}\phi^2 - g\phi = h, \tag{9}$$

where h is the Hamiltonian.

Suppose that

$$f(\phi) = \frac{a^2}{2}\phi^3 + \frac{3ac}{2}\phi^2 + c^2\phi, \tag{10}$$

we know that the equation $f(\phi) = 0$ has following three real roots:

$$\phi_0 = 0, \quad \phi_1 = -\frac{c}{a}, \quad \phi_2 = -\frac{2c}{a}, \tag{11}$$

and function (10) has two extreme points

$$\phi_\pm^* = \frac{-3c \pm \sqrt{3}c}{3a}, \tag{12}$$

and two extreme values

$$g_0 = f(\phi_-^*) = \frac{c^3}{3\sqrt{3}a}, \quad -g_0 = f(\phi_+^*) = -\frac{c^3}{3\sqrt{3}a}. \tag{13}$$

By using the qualitative theory of dynamical systems,[19,20] we draw the phase portraits of system (8) in Fig. 1.

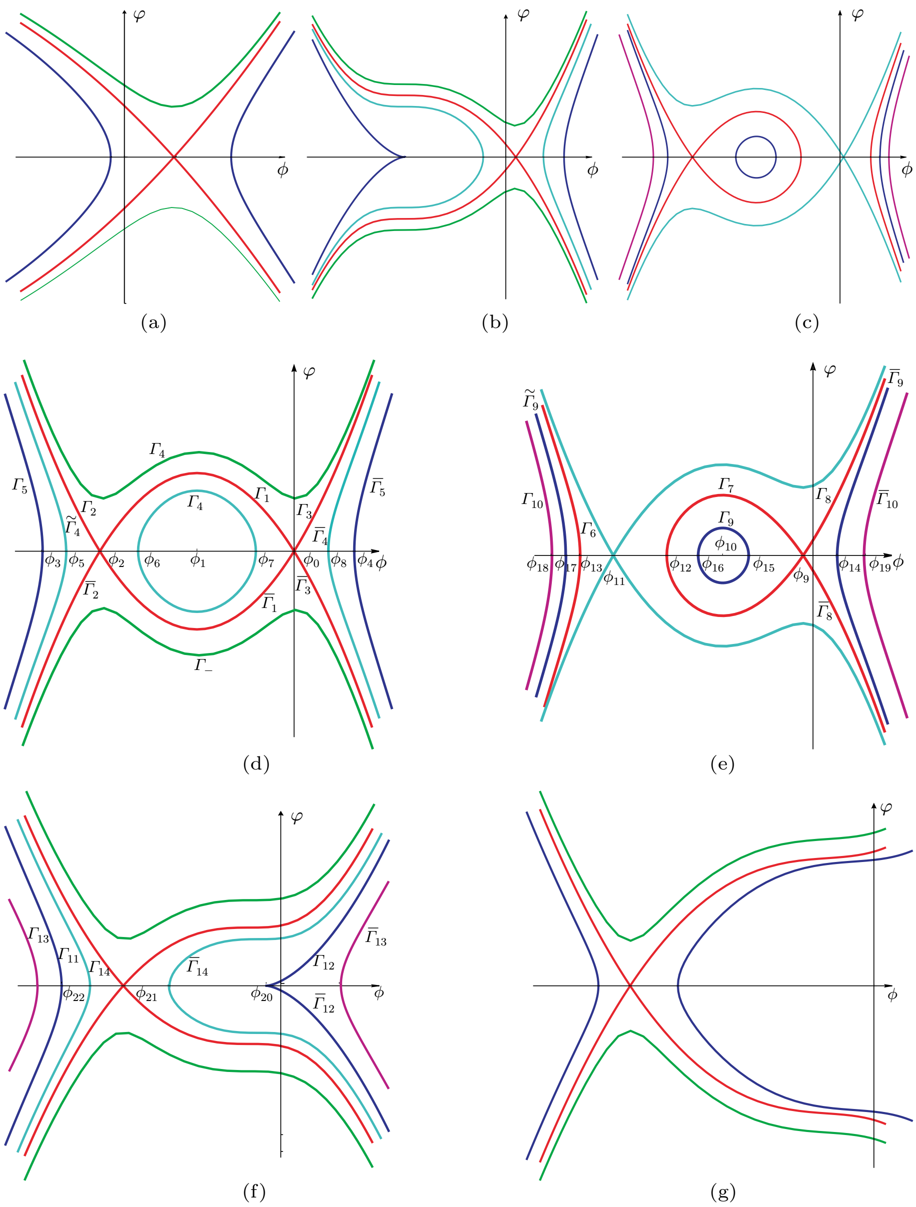

Fig. 1. The phase portraits of system (8): (a) $g < -g_0$, (b) $g = -g_0$, (c) $-g_0 < g < 0$, (d) $g = 0$, (e) $0 < g < g_0$, (f) $g = g_0$, (g) $g > g_0$.

If let

$$h_i = H(\phi_i, 0), \quad i = 0, 1, 9, 10, 20, 21, \tag{14}$$

where $\phi_9 \in (\frac{-3c+\sqrt{3}c}{3a}, 0)$ and

$$\phi_{10} = \frac{-3ac - a^2\phi_9 + \sqrt{a^2c^2 - 6a^3c\phi_9 - 3a^4\phi_9^2}}{2a^2}$$

are the roots of $f(\phi) + g = 0$, $0 < g < g_0$, $\phi_{20} = \frac{-3c+\sqrt{3}c}{3a}$ and $\phi_{21} = \frac{-3c-\sqrt{3}c}{3a}$ are the roots of $f(\phi) + g_0 = 0$, then we can derive the relations between the orbits of system (8), traveling wave solutions of Eq. (2), and the Hamiltonian h_i as the the following propositions 1–4.

Proposition 1 When $g = 0$:

(i) Suppose that $h = h_0$, system (8) has two heteroclonic orbits Γ_1 and $\overline{\Gamma}_1$ corresponding to two kink wave solutions of Eq. (2) and four special orbits $\Gamma_2, \overline{\Gamma}_2$, Γ_3, and $\overline{\Gamma}_3$ corresponding to two singular wave solutions of Eq. (2).

(ii) Suppose that $h_1 < h < h_0$, system (8) has three periodic orbits $\widetilde{\Gamma}_4, \Gamma_4$, and $\overline{\Gamma}_4$ corresponding to three periodic wave solutions of Eq. (2).

(iii) Suppose that $h \le h_1$, system (8) has two periodic orbits Γ_5 and $\overline{\Gamma}_5$ corresponding to four periodic wave solutions of Eq. (2).

(iv) Suppose that $h > 0$, system (8) has two special orbits Γ_+ and Γ_-.

Proposition 2 When $0 < g < g_0$:

(i) Suppose that $h = h_9$, system (8) has a homoclinic orbit Γ_6 corresponding to a solitary wave solution of Eq. (2) and three special orbits Γ_7, Γ_8, and $\overline{\Gamma}_8$ corresponding to two singular wave solutions of Eq. (2).

(ii) Suppose that $h_{10} < h < h_9$, system (8) has three periodic orbits $\widetilde{\Gamma}_9, \Gamma_9$, and $\overline{\Gamma}_9$ corresponding to three periodic wave solutions of Eq. (2).

(iii) Suppose that $h \le h_{10}$, system (8) has two periodic orbits Γ_{10} and $\overline{\Gamma}_{10}$ corresponding to two periodic wave solutions of Eq. (2).

(iv) Suppose that $h > h_9$, system (8) does not hanve any closed orbit.

Proposition 3 When $g = g_0$:

(i) Suppose that $h = h_{20}$, system (8) has three special orbits Γ_{11}, Γ_{12}, and $\overline{\Gamma}_{12}$ corresponding to three singular wave solutions of Eq. (2).

(ii) Suppose that $h < h_{20}$, system (8) has two special orbits Γ_{13} and $\overline{\Gamma}_{13}$.

(iii) Suppose that $h_{20} < h < h_{21}$, system (8) has two special orbits Γ_{14} and $\overline{\Gamma}_{14}$.

(iv) Suppose that $h \ge h_{21}$, system (8) does not have any closed orbit.

Proposition 4 When $g > g_0$ and h is an arbitrary constant, system (8) does not have any closed orbit.

3. Traveling wave solutions

For the convenience of exposition, we will omit the expressions of w with $w(x,y,t) = \psi(\zeta) = \frac{1}{a}\phi'(\zeta) - \frac{1}{2}\phi^2(\zeta) - \frac{c}{a}\phi(\zeta)$ in this work.

Proposition 5 For the given positive constant c and transformation $\zeta = x + y - ct$, we have the following results.

(i) When $g = 0$, equation (2) has two kink wave solutions

$$u(x,y,t) = \frac{2c\tilde{\phi}}{(2c + a\tilde{\phi})\exp(c\zeta) - a\tilde{\phi}}, \tag{15}$$

$$u(x,y,t) = \frac{2c\tilde{\phi}\exp(c\zeta)}{2c + a\tilde{\phi}(1 - \exp(c\zeta))}, \tag{16}$$

where $\tilde{\phi} \in (-\frac{2c}{a}, -\frac{c}{a})$, two singular wave solutions

$$u(x,y,t) = \frac{2c\exp(c\zeta)}{a(1 - \exp(c\zeta))}, \tag{17}$$

$$u(x,y,t) = -\frac{2c}{a(1 - \exp(c\zeta))}, \tag{18}$$

four periodic singular periodic wave solutions

$$u_{\pm}(x,y,t) = \frac{c}{a}\Big(-1 \pm \sqrt{2}\sec\frac{\sqrt{2}c\zeta}{2}\Big), \tag{19}$$

$$u_{\pm}(x,y,t) = \frac{c}{a}\Big(-1 \pm \sqrt{2}\csc\frac{\sqrt{2}c\zeta}{2}\Big), \tag{20}$$

and three periodic wave solutions

$$u(x,y,t) = \frac{\phi_5(\phi_8 - \phi_6) + \phi_6(\phi_5 - \phi_8)\mathrm{sn}^2(\frac{a}{2g_1}\zeta, k_1)}{\phi_8 - \phi_6 + (\phi_5 - \phi_8)\mathrm{sn}^2(\frac{a}{2g_1}\zeta, k_1)}, \tag{21}$$

$$u(x,y,t) = \frac{\phi_6(\phi_5 - \phi_7) + \phi_5(\phi_7 - \phi_6)\mathrm{sn}^2(\frac{a}{2g_1}\zeta, k_1)}{\phi_5 - \phi_7 + (\phi_7 - \phi_6)\mathrm{sn}^2(\frac{a}{2g_1}\zeta, k_1)}, \tag{22}$$

$$u(x,y,t) = \frac{\phi_8(\phi_7 - \phi_5) + \phi_7(\phi_5 - \phi_8)\mathrm{sn}^2(\frac{a}{2g_1}\zeta, k_1)}{\phi_7 - \phi_5 + (\phi_5 - \phi_8)\mathrm{sn}^2(\frac{a}{2g_1}\zeta, k_1)}, \tag{23}$$

where $h \in (-\frac{c^4}{8a^2}, 0)$ is the Hamiltonian,

$$\phi_5 = \frac{-c - \sqrt{c^2 + 2a\sqrt{-2h}}}{a}, \quad \phi_6 = \frac{-c - \sqrt{c^2 - 2a\sqrt{-2h}}}{a},$$

$$\phi_7 = \frac{-c + \sqrt{c^2 - 2a\sqrt{-2h}}}{a}, \quad \phi_8 = \frac{-c + \sqrt{c^2 + 2a\sqrt{-2h}}}{a},$$

$$g_1 = \frac{\sqrt{2}a}{\sqrt{c^2 + \sqrt{c^4 + 8a^2h}}}, \quad k_1 = \sqrt{\frac{2\sqrt{c^4 + 8a^2h}}{c^2 + \sqrt{c^4 + 8a^2h}}}.$$

(ii) When $0 < g < g_0$, equation (2) has one solitary wave solution

$$u(x,y,t) = \phi_9 + \frac{2\sqrt{2}\delta\beta\exp(\frac{\sqrt{2\delta}\zeta}{2})}{a^2\phi_9(a\phi_9 + 2c)(1 + e^{\sqrt{2\delta}\zeta}) - 2\sqrt{2}a\beta(a\phi_9 + c)\exp(\frac{\sqrt{2\delta}\zeta}{2})}, \tag{24}$$

one singular wave solution

$$u(x,y,t) = \phi_9 - \frac{2\delta(2c + 2a\phi_9 + \sqrt{2\delta})\exp(\frac{\sqrt{2\delta}\zeta}{2})}{a^2\phi_9(2c + a\phi_9) + a(\gamma - a\phi_9(2c + a\phi_9))\exp(\frac{\sqrt{2\delta}\zeta}{2}) - a\gamma\exp(\sqrt{2\delta}\zeta)}, \tag{25}$$

where $\phi_9 \in (\frac{-3c+\sqrt{3}c}{3a}, 0)$ is the initial value, $\beta = \sqrt{-a\phi_9(2c+a\phi_9)}$, $\delta = 2c^2+6ac\phi_9+3a^2\phi_9^2$, $\gamma = 4c^2+10ac\phi_9+5a^2\phi_9^2+(2c+2a\phi_9)\sqrt{2\delta}$, and three periodic wave solutions

$$u(x,y,t) = \frac{\phi_{16}(\phi_{17}-\phi_{15})+\phi_{17}(\phi_{15}-\phi_{16})\mathrm{sn}^2(\frac{a}{2g_2}\zeta,k_2)}{\phi_{17}-\phi_{15}+(\phi_{15}-\phi_{16})\mathrm{sn}^2(\frac{a}{2g_2}\zeta,k_2)}, \quad (26)$$

$$u(x,y,t) = \frac{\phi_{17}(\phi_{16}-\phi_{14})+\phi_{16}(\phi_{14}-\phi_{17})\mathrm{sn}^2(\frac{a}{2g_2}\zeta,k_2)}{\phi_{16}-\phi_{14}+(\phi_{14}-\phi_{17})\mathrm{sn}^2(\frac{a}{2g_2}\zeta,k_2)}, \quad (27)$$

$$u(x,y,t) = \frac{\phi_{14}(\phi_{15}-\phi_{17})+\phi_{15}(\phi_{17}-\phi_{14})\mathrm{sn}^2(\frac{a}{2g_2}\zeta,k_2)}{\phi_{15}-\phi_{17}+(\phi_{17}-\phi_{14})\mathrm{sn}^2(\frac{a}{2g_2}\zeta,k_2)}, \quad (28)$$

where

$$g_2 = \frac{2}{\sqrt{(\phi_{14}-\phi_{16})(\phi_{15}-\phi_{17})}},$$
$$k_2 = \sqrt{\frac{(\phi_{15}-\phi_{16})(\phi_{14}-\phi_{17})}{(\phi_{14}-\phi_{16})(\phi_{15}-\phi_{17})}},$$
$$\phi_{14} = \frac{-2c+a\sqrt{\mu}+\sqrt{a^2\tau-\eta}}{2a},$$
$$\phi_{15} = \frac{-2c+a\sqrt{\mu}-\sqrt{a^2\tau-\eta}}{2a},$$
$$\phi_{16} = \frac{-2c-a\sqrt{\mu}+\sqrt{a^2\tau+\eta}}{2a},$$
$$\phi_{17} = \frac{-2c-a\sqrt{\mu}-\sqrt{a^2\tau+\eta}}{2a},$$
$$\theta = \sqrt[3]{\varepsilon+\sqrt{\varepsilon^2-2\lambda^3}}, \quad \mu = \frac{1}{3a^2}\Big(4c^2+\frac{\lambda}{\theta}+\frac{\theta}{\sqrt[3]{2}}\Big),$$
$$\tau = \frac{1}{3a^2}\Big(8c^2-\frac{\lambda}{\theta}-\frac{\theta}{\sqrt[3]{2}}\Big),$$
$$\varepsilon = 128c^6-1152acg+1728a^2g^2+1152a^2c^2h,$$
$$\lambda = 16\sqrt[3]{2}(c^4-6acg+6a^2h), \quad h\in(h_{10},h_9).$$

(iii) When $g=g_0$, equation (2) has three singular wave solutions

$$u(x,y,t) = \frac{9(1+\sqrt{3})c-(3-\sqrt{3})c^3\zeta^2}{3a(c^2\zeta^2-3)}, \quad (29)$$

$$u(x,y,t) = \frac{12\sqrt{3}-6(1-\sqrt{3})c\zeta-(3-\sqrt{3})c^2\zeta^2}{3a\zeta(c\zeta-2\sqrt{3})}, \quad (30)$$

$$u(x,y,t) = \frac{12\sqrt{3}+6(1-\sqrt{3})c\zeta-(3-\sqrt{3})c^2\zeta^2}{3a\zeta(c\zeta+2\sqrt{3})}. \quad (31)$$

Proof (i) There are two heteroclinic orbits Γ_1 and Γ_1^*, four special orbits Γ_2, $\overline{\Gamma}_2$, Γ_3, and $\overline{\Gamma}_3$ in Fig. 1(d). The orbits are defined by $H(\phi,\varphi)=H(\phi_0,0)$, which can be reduced to

$$\varphi = \pm\frac{a}{2}\phi(\phi-\phi_2), \quad (32)$$

where $\phi_2=-2c/a$.

Substituting Eq. (32) into $\mathrm{d}\phi/\mathrm{d}\zeta=\varphi$ and integrating along the above orbits, we can obtain the following integrals:

$$\pm\int_{\widetilde{\phi}}^{\phi}\frac{1}{r(r-\phi_2)}\mathrm{d}r = \frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \quad (33)$$

$$\pm\int_{\phi}^{+\infty}\frac{1}{r(r-\phi_2)}\mathrm{d}r = \frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \quad (34)$$

where $\widetilde{\phi}\in(-\frac{2c}{a},-\frac{c}{a})$ is the initial value.

Completing the integrals (33), (34) and utilizing the transformation (3), we obtain two kink solutions (15), (16) and two singular wave solutions (17), (18).

There are two special orbits Γ_5 and $\overline{\Gamma}_5$ in Fig. 1(d). The orbits are defined by $H(\phi,\varphi)=H(\phi_1,0)$, which can be reduced to

$$\varphi = \pm\frac{a}{2}(\phi+\phi_1)\sqrt{(\phi-\phi_3)(\phi-\phi_4)}, \quad (35)$$

where $\phi_1=-\frac{c}{a}$, $\phi_3=\frac{-c-\sqrt{2}c}{a}$, and $\phi_4=\frac{-c+\sqrt{2}c}{a}$.

Substituting Eq. (35) into $\mathrm{d}\phi/\mathrm{d}\zeta=\varphi$ and integrating along the above orbits, we can obtain the following integrals:

$$\pm\int_{\phi_3}^{\phi}\frac{1}{(r-\phi_1)\sqrt{(r-\phi_3)(r-\phi_4)}}\mathrm{d}r = \frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \quad (36)$$

$$\pm\int_{\phi}^{\infty}\frac{1}{(r-\phi_1)\sqrt{(r-\phi_3)(r-\phi_4)}}\mathrm{d}r = \frac{a}{2}\int_0^{\zeta}\mathrm{d}r. \quad (37)$$

Completing the integrals (36), (37) and utilizing the transformation (3), we obtain the periodic signal wave solutions (19) and (20).

There are one periodic orbit and two special orbits $\widetilde{\Gamma}_4$, Γ_4, and $\overline{\Gamma}_4$ in Fig. 1(d). The orbits are given by $H(\phi,\varphi)=h, h\in(h_1,h_0)$, which can be converted to

$$z = \pm\frac{a}{2}\sqrt{(\phi-\phi_5)(\phi-\phi_6)(\phi-\phi_7)(\phi-\phi_8)}, \quad (38)$$

where

$$\phi_5 = \frac{-c-\sqrt{c^2+2a\sqrt{-2h}}}{a},$$
$$\phi_6 = \frac{-c-\sqrt{c^2-2a\sqrt{-2h}}}{a},$$
$$\phi_7 = \frac{-c+\sqrt{c^2-2a\sqrt{-2h}}}{a},$$
$$\phi_8 = \frac{-c+\sqrt{c^2+2a\sqrt{-2h}}}{a}.$$

Substituting Eq. (38) into $\mathrm{d}\phi/\mathrm{d}\zeta=\varphi$ and integrating along the above orbits, we can obtain the following integrals:

$$\pm\int_{\phi_6}^{\phi}\frac{1}{\sqrt{(r-\phi_5)(r-\phi_6)(\phi_7-r)(\phi_8-r)}}\mathrm{d}r = \frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \quad (39)$$

$$\pm\int_{\phi_5}^{\phi}\frac{1}{\sqrt{(\phi_5-r)(\phi_6-r)(\phi_7-r)(\phi_8-r)}}\mathrm{d}r = \frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \quad (40)$$

$$\pm\int_{\phi_8}^{\phi}\frac{1}{\sqrt{(r-\phi_5)(r-\phi_6)(r-\phi_7)(r-\phi_8)}}\mathrm{d}r = \frac{a}{2}\int_0^{\zeta}\mathrm{d}r. \quad (41)$$

Completing the integrals (39)–(41) and using the transformation (3), we obtain the periodic wave solutions (21)–(23).

(ii) In Fig. 1(e), there are one homoclinic orbit Γ_7 and three special orbits Γ_6, Γ_8, and $\overline{\Gamma}_8$. The orbits are defined by

$H(\phi,\varphi)=H(\phi_9,0)$, which can be converted to

$$\varphi=\pm\frac{a}{2}(\phi-\phi_9)\sqrt{(\phi-\phi_{12})(\phi-\phi_{13})}, \tag{42}$$

where $\phi_9\in(\frac{-3c+\sqrt{3}c}{3a},0)$ is the initial value,

$$\phi_{12}=\frac{-2c-a\phi_9+\sqrt{-4ac\phi_9-2a^2\phi_9^2}}{a},$$
$$\phi_{13}=\frac{-2c-a\phi_9-\sqrt{-4ac\phi_9-2a^2\phi_9^2}}{a}.$$

Substituting Eq. (42) into $\mathrm{d}\phi/\mathrm{d}\zeta=\varphi$, and integrating along the above orbits, we can obtain the following integrals:

$$\pm\int_{\phi_{12}}^{\phi}\frac{1}{(r-\phi_9)\sqrt{(r-\phi_{12})(r-\phi_{13})}}\mathrm{d}r=\frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \tag{43}$$

$$\pm\int_{\phi}^{+\infty}\frac{1}{(r-\phi_9)\sqrt{(r-\phi_{12})(r-\phi_{13})}}\mathrm{d}r=\frac{a}{2}\int_0^{\zeta}\mathrm{d}r. \tag{44}$$

Completing the integrals (43), (44) and using the transformation (3), we obtain the solitary wave solution (24) and the singular wave solution (25).

Similarly, there are a periodic orbit Γ_9, two special orbits $\widetilde{\Gamma}_9$ and $\overline{\Gamma}_9$ in Fig. 1(e). The orbits are defined by $H(\phi,\varphi)=h, h\in(h_{10},h_9)$, which can be converted to

$$\varphi=\pm\frac{a}{2}\sqrt{(\phi-\phi_{14})(\phi-\phi_{15})(\phi-\phi_{16})(\phi-\phi_{17})}, \tag{45}$$

where

$$\phi_{14}=\frac{-2c+a\sqrt{\mu}+\sqrt{a^2\tau-\eta}}{2a},$$
$$\phi_{15}=\frac{-2c+a\sqrt{\mu}-\sqrt{a^2\tau-\eta}}{2a},$$
$$\phi_{16}=\frac{-2c-a\sqrt{\mu}+\sqrt{a^2\tau+\eta}}{2a},$$
$$\phi_{17}=\frac{-2c-a\sqrt{\mu}-\sqrt{a^2\tau+\eta}}{2a},$$
$$\theta=\sqrt[3]{\varepsilon+\sqrt{\varepsilon^2-2\lambda^3}},\ \mu=\frac{1}{3a^2}\left(4c^2+\frac{\lambda}{\theta}+\frac{\theta}{\sqrt[3]{2}}\right),$$
$$\tau=\frac{1}{3a^2}\left(8c^2-\frac{\lambda}{\theta}-\frac{\theta}{\sqrt[3]{2}}\right),$$
$$\varepsilon=128c^6-1152acg+1728a^2g^2+1152a^2c^2h,$$
$$\lambda=16\sqrt[3]{2}(c^4-6acg+6a^2h),\quad h\in(h_{10},h_9).$$

Substituting Eq. (45) into $\mathrm{d}\phi/\mathrm{d}\zeta=\varphi$ and integrating along the special orbits, we can obtain the following integrals:

$$\pm\int_{\phi_{14}}^{\phi}\frac{1}{\sqrt{(r-\phi_{14})(r-\phi_{15})(r-\phi_{16})(r-\phi_{17})}}\mathrm{d}r=\frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \tag{46}$$

$$\pm\int_{\phi_{16}}^{\phi}\frac{1}{\sqrt{(\phi_{14}-r)(\phi_{15}-r)(r-\phi_{16})(r-\phi_{17})}}\mathrm{d}s=\frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \tag{47}$$

$$\pm\int_{\phi_{17}}^{\phi}\frac{1}{\sqrt{(\phi_{14}-r)(\phi_{15}-r)(\phi_{16}-r)(\phi_{17}-r)}}\mathrm{d}r=\frac{a}{2}\int_0^{\zeta}\mathrm{d}r. \tag{48}$$

Completing the integrals (46)–(48) and utilizing the transformation (3), we obtain the periodic wave solutions (26)–(28).

(iii) In Fig. 1(e), three are three special orbits Γ_{11}, Γ_{12}, and $\overline{\Gamma}_{12}$. The orbits are defined by $H(\phi,\varphi)=H(\phi_{20},0)$, which can be converted to

$$\varphi=\pm\frac{a}{2}(\phi-\phi_{20})\sqrt{(\phi-\phi_{20})(\phi-\phi_{22})}, \tag{49}$$

where $\phi_{20}=\frac{-3c+\sqrt{3}c}{3a}$ and $\phi_{22}=\frac{-c+\sqrt{3}c}{a}$.

Substituting Eq. (49) into $\mathrm{d}\phi/\mathrm{d}\zeta=\varphi$ and integrating along the above orbits, we can obtain the following integrals:

$$\pm\int_{\phi_{22}}^{\phi}\frac{1}{(r-\phi_{20})\sqrt{(r-\phi_{20})(r-\phi_{22})}}\mathrm{d}r=\frac{a}{2}\int_0^{\zeta}\mathrm{d}r, \tag{50}$$

$$\pm\int_{\phi}^{+\infty}\frac{1}{(r-\phi_{20})\sqrt{(r-\phi_{20})(r-\phi_{22})}}\mathrm{d}r=\frac{a}{2}\int_0^{\zeta}\mathrm{d}r. \tag{51}$$

Completing the integrals (50), (51) and utilizing the transformation (3), we obtain the periodic wave solutions (29)–(31).

4. Profiles of some traveling wave solutions

In this section, we draw the profiles of some traveling wave solutions when some parameters take special values.

(i) When $a=2$, $c=2$, and $\widetilde{\phi}=-1.8$, we draw the profiles of the kink wave solution (15), singular wave solution (17), and periodic singular wave solution (19) in Fig. 2.

(ii) When $a=2$, $c=2$, $\phi_9=-0.1$, and $h=-0.2$, we draw the profiles of the periodic wave solution (22), solitary wave solution (24), and singular wave solution (29) in Fig. 3.

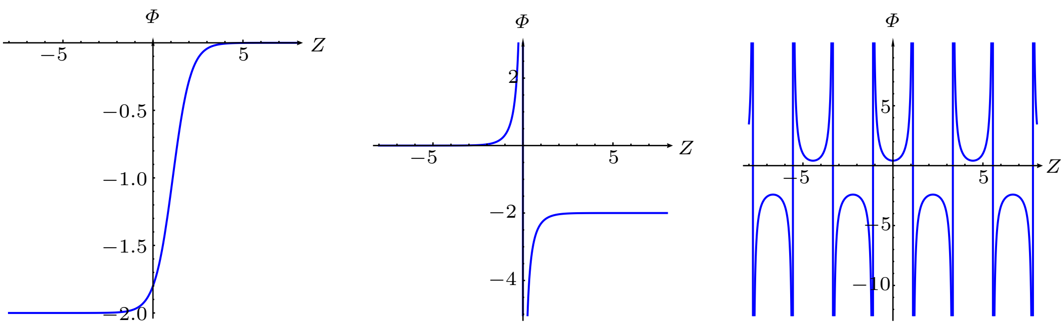

Fig. 2. The profiles of traveling wave solutions of Eq. (2). (a) Kink wave solution (15), (b) singular wave solution (17), (c) periodic singular wave solution (19).

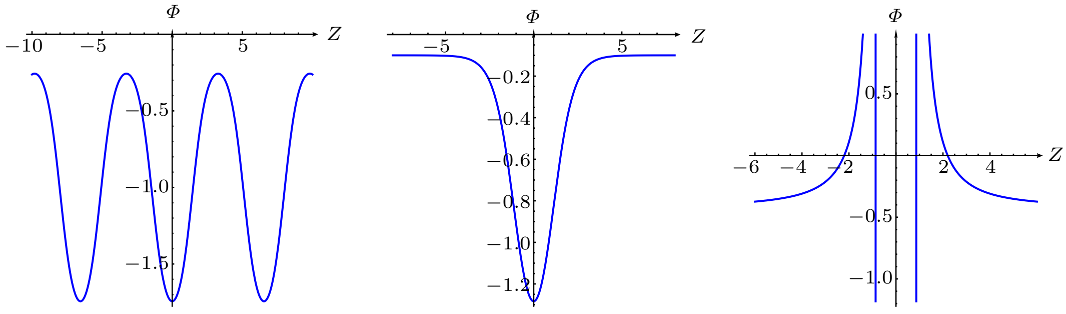

Fig. 3. The profiles of traveling wave solutions of Eq. (2). (a) Periodic wave solution (22), (b) solitary wave solution (24), (c) singular wave solution (29).

5. Conclusion

In this work, we utilize the bifurcation theory of planar dynamical systems to obtain the phase portraits and bifurcations analysis of the traveling wave system corresponding to Eq. (2). Furthermore, we have obtained abundant exact kink wave solutions, singular wave solutions, periodic wave solutions, periodic singular wave solutions, and solitary wave solutions. We are convinced that, in the future, the method can also be applied to other partial differential equations and may find more new results.

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