On superintegrable systems with a position-dependent mass in polar-like coordinates*

Hai Zhang(章海)†

School of Mathematics and Physics, Anqing Normal University, Anqing 246133, China

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For a superintegrable system defined in plane polar-like coordinates introduced by Szumiński *et al.* and studied by Fordy, we show that the system with a position-dependent mass is separable in three distinct coordinate systems. The corresponding separation equations and additional integrals of motion are derived explicitly. The closure algebra of integrals is deduced. We also make a generalization of this system by employing the classical Jacobi method. Lastly a sufficient condition which ensures flatness of the underlying space is derived via explicit calculation.

Keywords: superintegrable system, separation of variables, position-dependent mass, polar-like coordinates, Jacobi method

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1. Introduction

Recently Hamiltonian systems on the plane defined in polar-like coordinates have been studied.[1–4] A prototypical example of such a system is given in the polar-like coordinates by

$$H=\frac{1}{2}r^{m-k}\left(p_r^2+\frac{p_\phi^2}{r^2}\right)+r^m U(\phi), \tag{1}$$

where $k\neq 0$ and m are integers, $U(\phi)$ is a complex meromorphic function. In Ref. [1] Szumiński *et al.* have determined six cases to ensure the Liouville integrability of system (1). In Ref. [4] Fordy has derived an integrable system with the Hamilton function of the form

$$H=\frac{1}{2}r^n\left(p_r^2+\frac{p_\phi^2}{r^2}\right)+c_1 r^{1-n/2}\cos\frac{(n-2)\phi}{2} \tag{2}$$

with two constants $n\neq 2$ and c_1, by assuming that the Hamilton function and two additional integrals of motion have fixed quadratic parts as

$$H=T+h,\quad F_1=K_2K_3+f_1,\quad F_2=K_1K_2+f_2,$$

where K_i $(i=1,2,3)$ correspond to three basic Killing vectors of underlying Riemannian space, $h=h(r,\phi)$, $f_1=f_1(r,\phi)$ and $f_2=f_2(r,\phi)$ are functions of configuration space variables. The Fordy system (2) has generalized the integrable cases 3 and 4 given in Ref. [1], which correspond to the specific values $n=0$ and $n=4$, respectively.

The transformation between the original variables (r,ϕ) and flat Cartesian coordinates (Q_1,Q_2) is given by[4]

$$Q_1=br^{1-\frac{n}{2}}\cos\frac{(n-2)\phi}{2},$$

$$Q_2=br^{1-\frac{n}{2}}\sin\frac{(n-2)\phi}{2}, \tag{3}$$

where the constant $b=2/(2-n)$, $n\neq 2$, the polar-like variables $r>0$, $\phi\in\mathbb{R}$. Note that for ϕ differing by a multiple of $4\pi/(2-n)$ the coordinates (Q_1,Q_2) coincide. The parabolic coordinates (u,v) on plane are defined by[5]

$$u=\frac{1}{2}\left(Q_1+\sqrt{Q_1^2+Q_2^2}\right),$$

$$v=\frac{1}{2}\left(-Q_1+\sqrt{Q_1^2+Q_2^2}\right), \tag{4}$$

with inverse

$$Q_1=u-v,\quad Q_2=2\sqrt{uv}.$$

By combining point transformations (3) and (4), the Hamiltonian of Fordy system (2) transforms to[4]

$$H=\frac{1}{2}\frac{up_u^2+vp_v^2}{u+v}+\frac{1}{2}k(u-v), \tag{5}$$

where the constant k is related with c_1 in Eq. (2) through $k=(2-n)c_1$.

In the theory of integrable systems there are various approaches to generalize a given integrable system. One of them is to consider the original system with a position-dependent mass (PDM). Hamiltonian systems with a PDM have been proposed and studied extensively,[6–11] including classical integrable systems and quantum integrable ones. Many common models, e.g., harmonic oscillator and Kepler system, together with a PDM have been intensively studied, see e.g. Refs. [6,11].

In this paper we consider the Fordy system with a PDM. We show that it is separable in three coordinate systems and

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†Corresponding author. E-mail: haizhang@mail.ustc.edu.cn

we deduce its integrals of motion in explicit forms. The nontrivial polynomial relation among integrals is worked out. We also propose a generalization of the Fordy system with a PDM by the Jacobi method.

This paper is organized as follows. In Section 2 we review first integrals and separability of the original Fordy system. In Section 3 the Fordy system with a PDM is proposed. We show the multi-separability of the system by explicitly writing its separation equations. The integrals of motion are computed and their algebraic relation established. In Section 4 we apply the Jacobi method and make a generalization of the Fordy system with a PDM. A condition for the flatness of underlying Riemannian space is also deduced. Finally, Section 5 is devoted to the conclusion and some discussions.

2. Integrability of the Fordy system

According to Ref. [4] the Fordy system (5) is superintegrable, endowed with two quadratic integrals of motion,

$$F_1 = \frac{uv(p_u^2 - p_v^2)}{u+v} + kuv, \tag{6}$$

$$F_2 = \frac{\sqrt{uv}(p_u + p_v)(up_u - vp_v)}{(u+v)^2} + k\sqrt{uv}, \tag{7}$$

and a linear integral

$$K_2 = \frac{\sqrt{uv}(p_u + p_v)}{u+v}. \tag{8}$$

The four integrals of motion satisfy the algebraic relation[4]

$$F_2^2 = 2HK_2^2 - K_2^4 + kF_1. \tag{9}$$

In terms of the flat coordinates (Q_1, Q_2) in Eq. (3) the quadratic Hamiltonian H and linear integral K_2 satisfy the equations

$$2H - K_2^2 = P_1^2 + kQ_1, \quad K_2 = P_2. \tag{10}$$

It is obvious that the pair of integrals (H, K_2) is separated in the flat coordinates (Q_1, Q_2).

We can define other coordinates $(\hat{Q}_1, \hat{Q}_2)$, which is a rotation by an angle of $\pi/4$ of flat coordinates (Q_1, Q_2),

$$Q_1 = \frac{1}{\sqrt{2}}(\hat{Q}_1 - \hat{Q}_2), \quad Q_2 = \frac{1}{\sqrt{2}}(\hat{Q}_1 + \hat{Q}_2). \tag{11}$$

In terms of these new coordinates the integrals of motion H and F_2 have the relation

$$H + F_2 = \hat{P}_1^2 + \frac{k}{\sqrt{2}}\hat{Q}_1, \quad H - F_2 = \hat{P}_2^2 - \frac{k}{\sqrt{2}}\hat{Q}_2. \tag{12}$$

It shows that the integrals H and F_2 are Stäckel separable in the new $(\hat{Q}_1, \hat{Q}_2)$ coordinates.

In the parabolic coordinates (u, v) defined by Eq. (4) the pair of quadratic first integrals H and F_1 satisfy the equalities

$$2uH + F_1 = up_u^2 + ku^2, \quad 2vH - F_1 = vp_v^2 - kv^2, \tag{13}$$

which are also separation equations of Stäckel type.

The following proposition summarizes the above analysis.

Proposition 1 For the Fordy system (2) the pairs of quadratic (or linear) integrals (H, K_2), (H, F_2), (H, F_1) are separable in flat coordinates (Q_1, Q_2), rotated coordinates $(\hat{Q}_1, \hat{Q}_2)$ and parabolic coordinates (u, v), respectively. The corresponding separation equations are given by Eqs. (10), (12) and (13), respectively.

3. The Fordy system endowed with a position-dependent mass

The Hamiltonian system with a position-dependent mass has been widely studied, which serves as a vital technique to generalize a given system. In this section we consider the Fordy system with a particular PDM as $\mu = \varepsilon u - \varepsilon v + 1$, where ε is a suitable real constant.

The Hamiltonian of the new system is a scalar multiple of the original one (5) with the scalar factor $\lambda = 1/\mu$,

$$\tilde{H} = \lambda H = \frac{up_u^2 + vp_v^2}{2(1+\varepsilon u - \varepsilon v)(u+v)} + \frac{k(u-v)}{2(1+\varepsilon u - \varepsilon v)}. \tag{14}$$

In terms of the original polar-like coordinates (r, ϕ) the Hamiltonian reads

$$\tilde{H} = \frac{(2-n)r^n\left(p_r^2 + \frac{p_\phi^2}{r^2}\right)}{(4-2n) + 4\varepsilon r^{1-\frac{n}{2}}\cos\frac{(n-2)\phi}{2}} + \frac{kr^{1-\frac{n}{2}}\cos\frac{(n-2)\phi}{2}}{(2-n) + 2\varepsilon r^{1-\frac{n}{2}}\cos\frac{(n-2)\phi}{2}}. \tag{15}$$

As the parameter ε approaches zero the mass tends to $\mu = 1$ and the original system (2) is recovered. Therefore the system with a PDM, i.e., Eq. (15), can be regarded as a generalization or deformation of the Fordy system (2) with deformation parameter ε.

3.1. Separability in flat coordinates (Q_1, Q_2) and rotated coordinates $(\hat{Q}_1, \hat{Q}_2)$

When written in the flat coordinates (Q_1, Q_2) defined by Eq. (3), the Hamiltonian has the form of

$$\tilde{H} = \frac{P_1^2 + P_2^2}{2(1+\varepsilon Q_1)} + \frac{kQ_1}{2(1+\varepsilon Q_1)}. \tag{16}$$

The integrals $\tilde{H}$ and K_2 satisfy the equalities

$$2(1+\varepsilon Q_1)\tilde{H} - K_2^2 = P_1^2 + kQ_1, \quad K_2 = P_2. \tag{17}$$

We obviously have the following proposition.

Proposition 2 The Fordy system with the PDM, Eq. (15), is separable in flat coordinates (Q_1, Q_2) defined by Eq. (3).

The separation equations are given by Eq. (17), and the additional integral of motion is also K_2.

In terms of rotated flat coordinates $(\hat{Q}_1, \hat{Q}_2)$, the PDM Hamiltonian $\tilde{H}$ has the form as

$$\tilde{H} = \frac{\hat{P}_1^2 + \hat{P}_2^2}{2 + \sqrt{2}\varepsilon(\hat{Q}_1 - \hat{Q}_2)} + \frac{k}{2}\frac{\hat{Q}_1 - \hat{Q}_2}{\sqrt{2} + \varepsilon(\hat{Q}_1 - \hat{Q}_2)}. \tag{18}$$

The Hamilton–Jacobi equation $\tilde{H} = \alpha$ turns out to be

$$\frac{\hat{P}_1^2 + \hat{P}_2^2}{1 + \varepsilon Q_1} + \frac{k}{\sqrt{2}}\frac{(\hat{Q}_1 - \hat{Q}_2)}{1 + \varepsilon Q_1} = 2\alpha,$$

which is equivalent to

$$\begin{aligned} &\hat{P}_1^2 + \frac{k}{\sqrt{2}}\hat{Q}_1 - \sqrt{2}\alpha\varepsilon\hat{Q}_1 - \alpha \\ &= -\hat{P}_2^2 + \frac{k}{\sqrt{2}}\hat{Q}_2 - \sqrt{2}\alpha\varepsilon\hat{Q}_2 + \alpha, \end{aligned}$$

both sides of which depend upon a canonical pair $(\hat{Q}_i, \hat{P}_i)$ only. It follows that

$$\tilde{F}_2 = \hat{P}_1^2 + \frac{k}{\sqrt{2}}\hat{Q}_1 - \sqrt{2}\alpha\varepsilon\hat{Q}_1 - \alpha \tag{19}$$

is an additional first integral of the system with a PDM, i.e., Eq. (15).

The pair of first integrals $(\tilde{H}, \tilde{F}_2)$ satisfies the following separation equations:

$$\begin{aligned} (1 + \sqrt{2}\varepsilon\hat{Q}_1)\tilde{H} + \tilde{F}_2 &= \hat{P}_1^2 + \frac{k}{\sqrt{2}}\hat{Q}_1, \\ (1 - \sqrt{2}\varepsilon\hat{Q}_2)\tilde{H} - \tilde{F}_2 &= \hat{P}_2^2 - \frac{k}{\sqrt{2}}\hat{Q}_2. \end{aligned} \tag{20}$$

Substituting Eq. (18) into Eq. (19) leads to the explicit form of the first integral $\tilde{F}_2$ as

$$\begin{aligned} \tilde{F}_2 &= \frac{(1 - \sqrt{2}\varepsilon\hat{Q}_2)\hat{P}_1^2 - (1 + \sqrt{2}\varepsilon\hat{Q}_1)\hat{P}_2^2}{2(1 + \varepsilon Q_1)} \\ &\quad + \frac{k(\hat{Q}_1 + \hat{Q}_2)}{2\sqrt{2}(1 + \varepsilon Q_1)} \end{aligned} \tag{21}$$

in the rotated flat coordinates $(\hat{Q}_1, \hat{Q}_2)$, or

$$\begin{aligned} \tilde{F}_2 &= \frac{\hat{P}_1^2 - \hat{P}_2^2 - \sqrt{2}\varepsilon(\hat{Q}_2\hat{P}_1^2 + \hat{Q}_1\hat{P}_2^2)}{2(1 + \varepsilon Q_1)} + \frac{kQ_2}{2(1 + \varepsilon Q_1)} \\ &= \frac{-\varepsilon Q_2(P_1^2 + P_2^2) + 2P_1P_2(1 + \varepsilon Q_1)}{2(1 + \varepsilon Q_1)} + \frac{kQ_2}{2(1 + \varepsilon Q_1)} \end{aligned} \tag{22}$$

in the flat coordinates (Q_1, Q_2), where we have, in the calculation, used the equality

$$\hat{Q}_2\hat{P}_1^2 + \hat{Q}_1\hat{P}_2^2 = \frac{1}{\sqrt{2}}\left(Q_2(P_1^2 + P_2^2) - 2Q_1P_1P_2\right).$$

Proposition 3 The Fordy system with a PDM, Eq. (15), is Stäckel separable in rotated flat coordinates $(\hat{Q}_1, \hat{Q}_2)$. The separation equations are given by Eq. (20). The quadratic integral of motion $\tilde{F}_2$ has the form of Eq. (21) or (22).

3.2. Separability in parabolic coordinates (u, v)

When written in the parabolic coordinates (u, v), the Hamiltonian $\tilde{H}$ has the form

$$\tilde{H} = \frac{up_u^2 + vp_v^2}{2(1 + \varepsilon u - \varepsilon v)(u + v)} + \frac{k(u - v)}{2(1 + \varepsilon u - \varepsilon v)}.$$

The Hamilton–Jacobi equation $\tilde{H} = \alpha$ turns out to be

$$2(u + \varepsilon u^2)\tilde{H} - up_u^2 - ku^2 + 2(v - \varepsilon v^2)\tilde{H} - vp_v^2 + kv^2 = 0,$$

where the parts of (u, p_u) and (v, p_v) are separated. It follows that

$$2(u + \varepsilon u^2)\tilde{H} - up_u^2 - ku^2 = -\tilde{F}_1$$

is an additional first integral. The pair of first integrals $(\tilde{H}, \tilde{F}_1)$ satisfies the separation equations

$$\begin{aligned} 2(u + \varepsilon u^2)\tilde{H} + \tilde{F}_1 &= up_u^2 + ku^2, \\ 2(v - \varepsilon v^2)\tilde{H} - \tilde{F}_1 &= vp_v^2 - kv^2. \end{aligned} \tag{23}$$

The explicit form of first integral $\tilde{F}_1$ is given by

$$\tilde{F}_1 = \frac{uv(1 - \varepsilon v)p_u^2 - uv(1 + \varepsilon u)p_v^2}{(u + v)(1 + \varepsilon u - \varepsilon v)} + \frac{kuv}{1 + \varepsilon u - \varepsilon v} \tag{24}$$

in parabolic coordinates (u, v), or

$$\begin{aligned} \tilde{F}_1 &= \frac{Q_2^2}{4R(1 + \varepsilon Q_1)}\left((p_u^2 - p_v^2) - \varepsilon(vp_u^2 + up_v^2)\right) + \frac{k}{1 + \varepsilon Q_1}\frac{Q_2^2}{4} \\ &= Q_2P_1P_2 - Q_1P_2^2 - \frac{\varepsilon Q_2^2}{4(1 + \varepsilon Q_1)}(P_1^2 + P_2^2) \\ &\quad + \frac{k}{4(1 + \varepsilon Q_1)}Q_2^2 \end{aligned} \tag{25}$$

in the flat coordinates (Q_1, Q_2), where the quantities

$$\begin{aligned} p_u^2 - p_v^2 &= -4P_2^2\frac{Q_1R}{Q_2^2} + 4P_1P_2\frac{R}{Q_2}, \quad R = \sqrt{Q_1^2 + Q_2^2}, \\ vp_u^2 + up_v^2 &= RP_1^2 + P_2^2\frac{R}{Q_2^2}(R^2 + 3Q_1^2) - 4P_1P_2\frac{Q_1R}{Q_2} \end{aligned}$$

have been employed during the calculation.

Proposition 4 The Fordy system with a PDM, Eq. (15), is Stäckel separable in parabolic coordinates (u, v). The separation equations are given by Eq. (23). The quadratic integral of motion $\tilde{F}_1$ has the explicit form of Eq. (24) or (25).

3.3. Algebra of integrals of motion

The system with a PDM, Eq. (14), has four first integrals, $\tilde{H}, K_2, \tilde{F}_1, \tilde{F}_2$. In this subsection we will determine the algebraic relation and Poisson algebra among them. A straightforward computation gives the nonvanishing Poisson brackets between them as

$$\begin{aligned} &\{K_2, \tilde{F}_1\} = -\tilde{F}_2, \quad \{K_2, \tilde{F}_2\} = \varepsilon\tilde{H} - \frac{k}{2}, \\ &\{\tilde{F}_1, \tilde{F}_2\} = 2K_2(\tilde{H} - K_2^2). \end{aligned}$$

It can be seen that the Poisson algebra is a polynomial algebra as right-hand sides of the above equations are polynomials in the integrals.

Since the number of integrals exceeds the number of degrees of freedom of the system there exists at most three functionally independent integrals of motion and the four integrals of motion must be functionally dependant.[5] However, it is a nontrivial task to find the suitable algebraic equation for the integrals especially when the integrals have complicated forms.

An instructive principle is that, in order to establish such an algebraic relation, it is necessary to make the potential parts of the integrals canceled out. We denote by $\tilde{h}$, $\tilde{f}_1$, $\tilde{f}_2$ the potential parts of integrals $\tilde{H}$, $\tilde{F}_1$, $\tilde{F}_2$, respectively. It is crucial to rewrite $\tilde{h}$ as

$$\tilde{h} = \frac{k}{2\varepsilon} - \frac{k}{2\varepsilon(1+\varepsilon Q_1)}.$$

From this we observe that these potentials can be canceled out by the relation

$$2\left(\varepsilon\tilde{h} - \frac{k}{2}\right)\tilde{f}_1 + \tilde{f}_2^2 = 0. \tag{26}$$

This inspires us to calculate the expression

$$2\left(\varepsilon\tilde{H} - \frac{k}{2}\right)\tilde{F}_1 + \tilde{F}_2^2,$$

the result of which is found to be a combination of $\tilde{H}$ and K_2 only. Lastly we obtain the algebraic equation among integrals as follows:

$$\tilde{F}_2^2 = 2\tilde{H}K_2^2 + (k - 2\varepsilon\tilde{H})\tilde{F}_1 - K_2^4. \tag{27}$$

Compared with the algebraic relation (9) for integrals of motion of the original Fordy system (2), Eq. (27) can be treated as a deformation of the former by the parameter ε. Both Eqs. (9) and (27) are polynomial relations of order four.

4. Generalization by the Jacobi method

The classical Jacobi method can be used to construct new integrable systems from known ones. It originated in Ref. [12] where the author proposed elliptic coordinates and employed them to integrate some vital mechanical systems. This method reverses the conventional path from a given system to its separation variables and formulate new systems from known separation variables and arbitrary separation relations. For more discussions and applications of it, one can see, e.g., Refs. [13–16].

4.1. Generalization of the PDM Fordy system

Note that the separation Eq. (23) of the Fordy system with a PDM can be viewed as a deformation of Eq. (13) for the Fordy system. Alternatively to the PDM approach, we can also obtain the new system by applying the classical Jacobi method to the separation equations of the Fordy system, i.e., by adding quadratic terms εu^2 and $-\varepsilon v^2$.

According to this idea, we can further generalize the Fordy system with a PDM, i.e., Eq. (15). By applying the Jacobi method to separation equations (23) we can get a family of generalized integrable systems with the following separation equations:

$$\begin{aligned} 2(u+\varepsilon u^2)\tilde{\mathcal{H}} + \tilde{\mathcal{F}}_1 &= (u+f(u))p_u^2 + k_1u^2 + k_2u^3, \\ 2(v-\varepsilon v^2)\tilde{\mathcal{H}} - \tilde{\mathcal{F}}_1 &= (v+h(v))p_v^2 + k_3v^2 + k_4v^3, \end{aligned} \tag{28}$$

where f and h are functions of independent variables u and v, respectively; ε and k_i $(i = 1,\ldots,4)$ are arbitrary constants. It follows from Eq. (28) that the generalized system has an explicit Hamiltonian as

$$\begin{aligned} \tilde{\mathcal{H}} = {} & \frac{(u+f(u))p_u^2 + (v+h(v))p_v^2}{2(u+v+\varepsilon u^2 - \varepsilon v^2)} \\ & + \frac{k_1u^2 + k_2u^3 + k_3v^2 + k_4v^3}{2(u+v+\varepsilon u^2 - \varepsilon v^2)}. \end{aligned} \tag{29}$$

Contrast to the system (14) which is multi-separable, the generalized system (29) for general parameters k_i and functions f,h is separable in parabolic coordinates (u,v) only. It will be an interesting problem to determine the values of k_i and forms of f and h for which the system is also separable in other coordinate systems.

From the generalized separation equations (28) we obtain

$$\begin{aligned} W_1(u,\tilde{\alpha},\tilde{\alpha}_1) &= \int^u \sqrt{\frac{2(u+\varepsilon u^2)\tilde{\alpha} + \tilde{\alpha}_1 - k_1u^2 - k_2u^3}{u+f(u)}}\,\mathrm{d}u, \\ W_2(v,\tilde{\alpha},\tilde{\alpha}_1) &= \int^v \sqrt{\frac{2(v-\varepsilon v^2)\tilde{\alpha} - \tilde{\alpha}_1 - k_3v^2 - k_4v^3}{v+h(v)}}\,\mathrm{d}v, \end{aligned}$$

which leads to a separable form of Hamilton's characteristic function,

$$W = W_1(u,\tilde{\alpha},\tilde{\alpha}_1) + W_2(v,\tilde{\alpha},\tilde{\alpha}_1). \tag{30}$$

The equations

$$\tilde{\beta} = -t + \frac{\partial W}{\partial\tilde{\alpha}}, \quad \tilde{\beta}_1 = \frac{\partial W}{\partial\tilde{\alpha}_1}, \tag{31}$$

can be inverted to obtain expressions of u and v as functions of time t and the constants $\tilde{\alpha},\tilde{\alpha}_1,\tilde{\beta},\tilde{\beta}_1$, which means that the system (29) is integrated in principle.

4.2. A particular condition for flat underlying space

In general the generalized system (29) is defined on cotangent bundle of non-flat Riemannian manifold. In the following, we analyze the conditions which imply the flatness of the underlying space.

The underlying Riemannian space has the metric

$$\mathrm{d}s^2 = (u+v+\varepsilon u^2 - \varepsilon v^2)\left(\frac{\mathrm{d}u^2}{u+f(u)} + \frac{\mathrm{d}v^2}{v+h(v)}\right). \tag{32}$$

By denoting $U = u + f(u)$, $V = v + h(v)$,

$$J = u + v + \varepsilon u^2 - \varepsilon v^2,$$

it has the form of

$$ds^2 = \frac{J\,du^2}{U(u)} + \frac{J\,dv^2}{V(v)} = E\,du^2 + G\,dv^2, \tag{33}$$

where $E = J/U$, $G = J/V$. In an orthogonal coordinate system (u,v) the Gaussian curvature of metric (33) is given by the following formula[17]

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right). \tag{34}$$

A detailed calculation gives

$$\sqrt{EG} = \frac{J^2}{\sqrt{UV}}, \quad G_u = \frac{2\varepsilon u+1}{V}, \quad E_v = \frac{-2\varepsilon v+1}{U},$$

and finally the curvature of metric (32) has an explicit form as

$$\begin{aligned} K = \frac{1}{2J^3}\Bigg[& -\left(\varepsilon u+\frac{1}{2}\right)Jf'(u) + \left(\varepsilon v-\frac{1}{2}\right)Jh'(v) \\ & + \left(1+2(u^2+v^2)\varepsilon^2+2(u-v)\varepsilon\right) \\ & \times (f(u)+h(v)) + \varepsilon^2(u+v)^3\Bigg]. \end{aligned} \tag{35}$$

It is difficult to find the most general forms of f, g and ε which imply the flatness of the Riemannian space, $K = 0$. We can consider a special case $\varepsilon = 0$. In such a case the flatness condition $K = 0$ is equivalent to the equation

$$-\frac{1}{2}(u+v)(f'(u)+h'(v)) + (f(u)+h(v)) = 0,$$

which has a solution of the form

$$f(u) = \ell_1 u^2 + \ell_2 u, \quad h(v) = -\ell_1 v^2 + \ell_2 v, \tag{36}$$

where ℓ_1 and ℓ_2 are two arbitrary constants. We thus have the following proposition.

Proposition 5 For the condition of $\varepsilon = 0$ and $f(u)$, $g(v)$ given by Eq. (36) and arbitrary parameters k_i $(i = 1,\ldots,4)$, the Hamiltonian system (29) is defined on the cotangent bundle of a flat Riemannian space.

5. Conclusions

In summary, we have made a deformation for the superintegrable system (2) proposed by Fordy, by a position-dependent mass. The new system with a PDM is shown separable in three distinct coordinate systems with separation equations and integrals of motion explicitly demonstrated. By applying the Jacobi method to the Fordy system with a PDM a family of integrable systems containing arbitrary constants and functions are generated.

It will be an interesting direction of study to explore the connection between the new systems proposed in this paper and those known in the literature. In particular, a broad class of the finite-gap systems with position-depend mass has been considered by Bravo and Plyushchay in Ref. [8]. They obtained elliptic finite-gap systems of Lamé and Darboux–Treibich–Verdier types by reduction to Seiffert's spherical spiral and Bernoulli lemniscate with Calogero-like or harmonic oscillator potentials. Our proposed systems have intimate relations with them. While they studied quantum mechanical systems with PDM, we consider classical ones. It will be interesting to develop the quantum counterparts of our proposed systems and explore their applications in areas of theoretical physics such as supersymmetric quantum mechanics or quantum field theory.[18,19]

Another line of research is to investigate the newly proposed systems by employing some common approaches in theory of integrable systems, such as the Lax representation, classical r-matrix formulation, (bi-)Hamiltonian structure, and action-angle variables. Their symmetry properties[20,21] can also be investigated.

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