

量子扩散通道中 Wigner 算符的演化规律*

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众所周知, 量子态的演化可用与其相应的 Wigner 函数演化来代替. 因为量子态的 Wigner 函数和量子态的密度矩阵一样, 都包含了概率分布和相位等信息, 因此对量子态的 Wigner 函数进行研究, 可以更加快速有效地获取量子态在演化过程的重要信息. 本文从经典扩散方程出发, 利用密度算符的 P 表示, 导出了量子态密度算符的扩散方程. 进一步通过引入量子算符的 Weyl 编序记号, 给出了其对应的 Weyl 量子化方案. 另外, 借助于密度算符的另一相空间表示——Wigner 函数, 建立了 Wigner 算符在扩散通道中演化方程, 并给出了其 Wigner 算符解的形式. 本文推导出了 Wigner 算符在量子扩散通道中的演化规律, 即演化过程中任意时刻 Wigner 算符的形式. 在此结论的基础上, 讨论了相干态经过量子扩散通道的演化情况.

关键词: Wigner 函数, Weyl 编序, 量子扩散通道, 演化规律

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1 引言

近来, 量子调控已经成为研究微观世界的一个重要手段, 而用单光子实现量子操控尤为可行, 例如向光腔中逐个注入光子制备非高斯态, 理论上这属于量子扩散机制^[1,2]. 鉴于量子态的 Wigner 函数包含了量子态的概率分布和相位等信息, 量子态的演化可代之以研究相应的 Wigner 函数的演化^[3-5]. 本文旨在研究量子相空间的 Wigner 算符在量子扩散通道的时间演化规律, 它简洁而物理清晰, 展现了从点源函数 $\frac{1}{2} \text{:} \delta(z^* - a^\dagger) \delta(z - a) \text{:}$ 向 t 时刻高斯型函数 $\frac{1}{2kt} \text{:} \exp \left[\frac{-1}{kt} (a^\dagger - z^*)(a - z) \right] \text{:}$ 的演变, k 是扩散系数, 这里 $\text{:} \text{:}$ 代表 Weyl 编序; a^\dagger, a

是玻色产生和湮灭算符. 用有序算符内的积分方法也可进一步将 Wigner 算符的 Weyl 编序式转化为其他排序形式, 如正规乘积序等, 为计算量子态的 Wigner 函数演化规律带来便利. 本文安排如下, 先从经典扩散方程推导出量子扩散方程, 并以相干光场为例, 讨论其量子扩散. 鉴于初始相干光场的反正规乘积形式是 Delta 函数, 其演化就体现在从 $\delta(z - a) \delta(z^* - a^\dagger)$ 演化为

$$\frac{1}{kt} \text{:} \exp \left[\frac{-1}{kt} (z^* - a^\dagger)(z - a) \right] \text{:},$$

这里 $\text{:} \text{:}$ 代表反正规排序^[6,7]. 然后再讨论 Wigner 算符的量子扩散, 鉴于 Wigner 算符的 Weyl 排序形式是 Delta 函数,

$$\frac{1}{2} \text{:} \delta(\alpha^* - a^\dagger) \delta(\alpha - a) \text{:},$$

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符号 \vdots 代表 Weyl 排序, 为范洪义首创, 所以可以立即从扩散方程导出 Wigner 算符在扩散通道中的演化规律.

2 从经典扩散导出量子扩散方程

经典扩散方程是

$$\frac{\partial P(z, t)}{\partial t} = -k \frac{\partial^2 P(z, t)}{\partial z \partial z^*}, \quad (1)$$

其中 k 是扩散率, $P(z, t)$ 是系统的某种密度分布函数. 下面推导相应的量子扩散方程. 我们将密度算符 ρ 用相干态表象中的 P -表示^[8,9]:

$$\rho_t = \int \frac{d^2 z}{\pi} P(z, t) |z\rangle \langle z|, \quad (2)$$

其中

$$|z\rangle = \exp\left(-\frac{|z|^2}{2} + za^\dagger\right) |0\rangle \quad (3)$$

是相干态, 则密度算符的时间演化满足方程为

$$\frac{d\rho_t}{dt} = \int \frac{d^2 z}{\pi} \frac{\partial P(z, t)}{\partial t} |z\rangle \langle z|. \quad (4)$$

将经典扩散方程 (1) 式代入 (4) 式即有

$$\frac{d\rho_t}{dt} = -k \int \frac{d^2 z}{\pi} \frac{\partial^2 P(z, t)}{\partial z \partial z^*} |z\rangle \langle z|, \quad (5)$$

利用分部积分法, 并注意在无穷远处 $P(z, t)$ 消失, 则有

$$\begin{aligned} & \int \frac{d^2 z}{\pi} \frac{\partial^2 P(z, t)}{\partial z \partial z^*} |z\rangle \langle z| \\ &= \int \frac{d^2 z}{\pi} P(z, t) \frac{\partial^2}{\partial z \partial z^*} |z\rangle \langle z|, \end{aligned}$$

所以

$$\frac{d\rho_t}{dt} = -k \int \frac{d^2 z}{\pi} P(z, t) \frac{\partial^2}{\partial z \partial z^*} |z\rangle \langle z|. \quad (6)$$

现在利用相干态投影算符的正规乘积表示

$$|z\rangle \langle z| = \vdots -|z|^2 + za^\dagger + z^* a - a^\dagger a \vdots \quad (7)$$

来分析 $\frac{\partial^2}{\partial z \partial z^*} |z\rangle \langle z|$, 注意到

$$\begin{aligned} a^\dagger |z\rangle \langle z| &= a^\dagger \vdots -|z|^2 + za^\dagger + z^* a - a^\dagger a \vdots \\ &= \left(z^* + \frac{\partial}{\partial z}\right) |z\rangle \langle z|, \end{aligned} \quad (8)$$

$$|z\rangle \langle z| a = \left(z^* + \frac{\partial}{\partial z}\right) |z\rangle \langle z|, \quad (9)$$

则有

$$\begin{aligned} & -\frac{\partial^2}{\partial z \partial z^*} |z\rangle \langle z| \\ &= z \left(z^* + \frac{\partial}{\partial z}\right) |z\rangle \langle z| - \left(z^* + \frac{\partial}{\partial z}\right) \left(z + \frac{\partial}{\partial z^*}\right) |z\rangle \langle z| \\ & \quad - |z|^2 |z\rangle \langle z| + \left(z + \frac{\partial}{\partial z^*}\right) (z^* |z\rangle \langle z|) \\ &= za^\dagger |z\rangle \langle z| - \left(z^* + \frac{\partial}{\partial z}\right) |z\rangle \langle z| a - |z|^2 |z\rangle \langle z| \\ & \quad + \left(z + \frac{\partial}{\partial z^*}\right) |z\rangle \langle z| a^\dagger \\ &= a^\dagger a |z\rangle \langle z| - a^\dagger |z\rangle \langle z| a - a |z\rangle \langle z| a^\dagger + |z\rangle \langle z| a a^\dagger. \end{aligned} \quad (10)$$

将 (10) 式代入 (6) 式得到

$$\begin{aligned} \frac{d\rho_t}{dt} &= k \int \frac{d^2 z}{\pi} P(z, t) (a^\dagger a |z\rangle \langle z| - a^\dagger |z\rangle \langle z| a \\ & \quad - a |z\rangle \langle z| a^\dagger + |z\rangle \langle z| a a^\dagger) \\ &= k \left[a^\dagger a \int \frac{d^2 z}{\pi} P(z, t) |z\rangle \langle z| - a^\dagger \int \frac{d^2 z}{\pi} P(z, t) |z\rangle \langle z| a \right. \\ & \quad \left. - a \int \frac{d^2 z}{\pi} P(z, t) |z\rangle \langle z| a^\dagger + \int \frac{d^2 z}{\pi} P(z, t) |z\rangle \langle z| a a^\dagger \right]. \end{aligned} \quad (11)$$

这说明量子扩散方程为

$$\frac{d\rho_t}{dt} = k (a^\dagger a \rho - a^\dagger \rho a - a \rho a^\dagger + \rho a a^\dagger), \quad (12)$$

这是从经典扩散方程过渡到量子扩散方程的捷径.

3 相干光场的扩散

当初态是纯相干光场时,

$$\rho_0 = |z\rangle \langle z|, \quad (13)$$

它的正规排序是

$$\rho_0 = \vdots \exp[-(z^* - a^\dagger)(z - a)] \vdots, \quad (14)$$

利用范洪义等^[10]给出的把正规乘积排序变为反正规乘积排序的公式

$$\begin{aligned} |z\rangle \langle z| &= \vdots \exp[-(z^* - a^\dagger)(z - a)] \vdots \\ &= \pi \vdots \delta(z - a) \delta(z^* - a^\dagger) \vdots, \end{aligned}$$

所以初态是纯相干光场时的反正规乘积排序是

$$\rho_0 = \pi \vdots \delta(z - a) \delta(z^* - a^\dagger) \vdots, \quad (15)$$

故而它的 P -表示为

$$P_0 = \pi \delta(z^* - \alpha^*) \delta(z - \alpha). \quad (16)$$

由此可以直接验证扩散方程 $\frac{\partial P(z, t)}{\partial t} = -k \frac{\partial^2 P(z, t)}{\partial z \partial z^*}$ 的解是

$$P_t = \frac{1}{kt} \exp \left[\frac{-1}{kt} (z^* - \alpha^*) (z - \alpha) \right], \quad (17)$$

此解满足初始条件, 即:

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{kt} \exp \left[\frac{-1}{kt} (z^* - \alpha^*) (z - \alpha) \right] \\ &= \pi \delta(z^* - \alpha^*) \delta(z - \alpha), \end{aligned} \quad (18)$$

这是 $\delta(x) = \lim_{t \rightarrow 0} \frac{1}{\sqrt{\pi t}} e^{-x^2/t}$ 的推广.

P_t 是密度算符 ρ_t 在相干态表象中的表示, 所以便得到 ρ_t 的反正规乘积形式为

$$P_t = \frac{1}{kt} \text{:} \exp \left[\frac{-1}{kt} (z^* - a^\dagger) (z - a) \right] \text{:}, \quad (19)$$

这就是相干态在扩散通道中的演化公式. 再用相干态表象^[11]和有序算符内的积分技术^[12-14]可以将它化为正规乘积,

$$\begin{aligned} P_t &= \frac{1}{kt} \text{:} \exp \left[\frac{-1}{kt} (z^* - a^\dagger) (z - a) \right] \text{:} \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| \\ &= \frac{1}{kt} \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| \exp \left[\frac{-1}{kt} (z^* - \alpha^*) (z - \alpha) \right] \\ &= \frac{1}{kt} \int \frac{d^2\alpha}{\pi} \text{:} \exp \left[\frac{-1}{kt} (z^* - \alpha^*) (z - \alpha) \right. \\ &\quad \left. - |\alpha|^2 + \alpha a + \alpha^* a - a^\dagger a \right] \text{:} \\ &= \frac{1}{1+kt} e^{\frac{z}{1+kt} a^\dagger} \text{:} e^{\left(\frac{kt}{1+kt} - 1\right) a^\dagger a} \text{:} e^{\frac{z^*}{1+kt} a} e^{-\frac{|z|^2}{1+kt}} \\ &= \frac{1}{1+kt} e^{\frac{z}{1+kt} a^\dagger} e^{a^\dagger a \ln \frac{kt}{1+kt}} e^{\frac{z^*}{1+kt} a} e^{-\frac{|z|^2}{1+kt}}. \end{aligned} \quad (20)$$

通过 (20) 式可发现它不再是纯态. 同时可验证 $\text{tr} \rho_t = 1$, 故而 ρ_t 是一个新光场密度算符, 代表一个广义的混沌光场^[15,16]. 计算 t 时刻的光子数:

$$\text{tr} (a^\dagger a \rho_t) = |z|^2 + kt. \quad (21)$$

比较初始时刻的光子数 $\text{tr} (a^\dagger a \rho_0) = \langle z | a^\dagger a | z \rangle = |z|^2$, 可见光子数增加了 kt , 这是扩散的结果.

4 Wigner 算符的 Weyl 排序形式

现在讨论 Wigner 函数在扩散通道中的演化. 鉴于 t 时刻的密度算符 $\rho(t)$ 的 Wigner 函数是^[17,18]

$$\begin{aligned} W(p, q, t) &= 2\pi \text{tr} [\rho(t) \Delta(p, q, 0)] \\ &= 2\pi \text{tr} [\rho(0) \Delta(p, q, t)], \end{aligned} \quad (22)$$

这里 $\Delta(p, q)$ 是 Wigner 算符, 所以也可转而讨论 Wigner 算符在扩散通道中的时间演化, 即从

$\Delta(p, q, 0)$ 演化为 $\Delta(p, q, t)$. 历史上, Wigner 算符最早是在坐标表象中定义的^[19],

$$\Delta(q, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dv e^{ipv} \left| q + \frac{v}{2} \right\rangle \left\langle q - \frac{v}{2} \right|, \quad (23)$$

利用 $|q+v/2\rangle = e^{-iPv/2}|q\rangle$ 和 $|q\rangle \langle q| = \delta(q-Q)$, Q 是坐标算符, P 是动量算符, (23) 式可化为

$$\begin{aligned} \Delta(q, p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dv e^{ipv} e^{-iPv/2} \delta(q-Q) e^{-iPv/2} \\ &= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} dudv e^{ipv} e^{-iPv/2} e^{iu(q-Q)} e^{-iPv/2} \\ &= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} dudv e^{iv(p-P) + iu(q-Q)}. \end{aligned} \quad (24)$$

为了对 (24) 式进行积分, 我们引入 Weyl 编序[⋮]来标志已经 Weyl 排序好了的算符^[20-22]. 例如 Weyl 编序可以将经典量 $q^m p^r$ 通过积分变换 (积分核就是 Wigner 算符 $\Delta(p, q)$) 量子化为

$$\begin{aligned} q^m p^r &\rightarrow \iint_{-\infty}^{\infty} dpdq \Delta(p, q) q^m p^r \\ &= \left(\frac{1}{2}\right)^m \sum_{l=0}^{\infty} \binom{m}{l} Q^{m-l} P^r Q^l, \end{aligned} \quad (25)$$

它是 Weyl 排序好的, 以[⋮]标识,

$$\begin{aligned} &\left(\frac{1}{2}\right)^m \sum_{l=0}^{\infty} \binom{m}{l} Q^{m-l} P^r Q^l \\ &= \text{:} \left(\frac{1}{2}\right)^m \sum_{l=0}^{\infty} \binom{m}{l} Q^{m-l} P^r Q^l \text{:}, \end{aligned} \quad (26)$$

再注意到在记号[⋮]内部玻色算符是可交换的, 所以有

$$\begin{aligned} &\text{:} \left(\frac{1}{2}\right)^m \sum_{l=0}^{\infty} \binom{m}{l} Q^{m-l} P^r Q^l \text{:} = \text{:} Q^m P^r \text{:} \\ &= \iint_{-\infty}^{\infty} dpdq q^m p^r \text{:} \delta(q-Q) \delta(p-P) \text{:}. \end{aligned} \quad (27)$$

与一般算符 $H(P, Q)$ 及其经典对应 $h(p, q)$ 的 Weyl 对应式为^[23]

$$H(P, Q) = \iint_{-\infty}^{\infty} dqdp \Delta(p, q) h(p, q), \quad (28)$$

通过比较可见 Wigner 算符的 Weyl 排序形式是

$$\Delta(p, q) = \text{:} \delta(q-Q) \delta(p-P) \text{:}, \quad (29)$$

从而

$$H(P, Q) = \text{Tr} \{ h(P, Q) \}. \quad (30)$$

可见 $H(P, Q)$ 的 Weyl 排序形式, 可以直接将 $h(p, q)$ 中的 $p \rightarrow P, q \rightarrow Q$, 并放入 $\text{Tr} \{ \cdot \}$ 内得到. 例如, $\text{Tr} \{ e^{iPv+iQu} \}$ 的经典对应是 $e^{iPv+iQu}$:

$$\begin{aligned} & \iint_{-\infty}^{\infty} dpdq \Delta(p, q) e^{iPv+iQu} \\ &= \iint_{-\infty}^{\infty} dpdq \text{Tr} \{ \delta(q-Q) \delta(p-P) \} e^{iPv+iQu} \\ &= \text{Tr} \{ e^{iPv+iQu} \}, \end{aligned} \quad (31)$$

进一步令

$$\alpha = (q + ip)/\sqrt{2}, \quad a = (Q + iP)/\sqrt{2}, \quad (32)$$

可得 Wigner 算符的 Weyl 排序形式是

$$\Delta(p, q) \rightarrow \Delta(\alpha, \alpha^*) = \frac{1}{2} \text{Tr} \{ \delta(\alpha^* - a^\dagger) \delta(\alpha - a) \}. \quad (33)$$

5 扩散通道中 Wigner 算符的演化方程

由于 Wigner 算符满足:

$$\iint_{-\infty}^{\infty} dpdq \Delta(p, q) = 1 \quad (34)$$

或

$$2 \int d^2\alpha \Delta(\alpha, \alpha^*) = 1, \quad (35)$$

Wigner 算符本身看作是一个混合态的密度算符, 根据 (12) 式, 它所满足的扩散方程是

$$\begin{aligned} \frac{d\Delta(\alpha, \alpha^*)}{dt} &= k [a^\dagger a \Delta(\alpha, \alpha^*) - a^\dagger \Delta(\alpha, \alpha^*) a \\ &\quad - a \Delta(\alpha, \alpha^*) a^\dagger + \Delta(\alpha, \alpha^*) a a^\dagger], \end{aligned} \quad (36)$$

此方程也可从 Wigner 算符的正规乘积形式方程 (37) 直接导出 (具体详见附录 A).

$$\begin{aligned} \Delta(\alpha, \alpha^*) &= \frac{1}{2} \text{Tr} \{ \delta(\alpha^* - a^\dagger) \delta(\alpha - a) \} \\ &= \int \frac{d^2\beta}{2\pi^2} \text{Tr} \{ \exp \{ i [\beta^* (\alpha - a) + \beta (\alpha^* - a^\dagger)] \} \} \\ &= \int \frac{d^2\beta}{2\pi^2} \exp \{ i [\beta^* (\alpha - a) + \beta (\alpha^* - a^\dagger)] \} \\ &= \int \frac{d^2\beta}{2\pi^2} \exp [i (\beta^* a - \beta^* \alpha + \beta a^\dagger - \beta \alpha^*)] e^{-|\beta|^2/2} \\ &= \frac{1}{\pi} \text{Tr} \{ e^{-2(\alpha^* - a^\dagger)(\alpha - a)} \}. \end{aligned} \quad (37)$$

对照本文第 2 节内容可见此扩散方程的经典对应是

$$\frac{\partial W}{\partial t} = -k \frac{\partial^2}{\partial \alpha \partial \alpha^*} W, \quad (38)$$

此方程即为 Wigner 函数 W 应该满足的扩散方程.

6 Wigner 算符的演化方程之解

初始的 Wigner 算符 $\Delta(\alpha, \alpha^*, 0)$ 在 Weyl 排序下是 [24-26]

$$\Delta(\alpha, \alpha^*, 0) = \frac{1}{2} \text{Tr} \{ \delta(\alpha^* - a^\dagger) \delta(\alpha - a) \}, \quad (39)$$

那么类比于本文第 2 节的结果可知:

$$\Delta(\alpha, \alpha^*, t) = \frac{1}{2kt} \text{Tr} \left\{ \exp \left[\frac{-1}{kt} (a^\dagger - \alpha^*) (a - \alpha) \right] \right\}, \quad (40)$$

其满足的初始条件为

$$\begin{aligned} & \lim_{t \rightarrow 0} \Delta(\alpha, \alpha^*, t) \\ &= \lim_{t \rightarrow 0} \frac{1}{2kt} \text{Tr} \left\{ \exp \left[\frac{-1}{kt} (a^\dagger - \alpha^*) (a - \alpha) \right] \right\} \\ &= \frac{1}{2} \text{Tr} \{ \delta(\alpha^* - a^\dagger) \delta(\alpha - a) \}. \end{aligned} \quad (41)$$

(41) 式就是量子扩散通道中 Wigner 算符的演化律公式, 可以看出, 它简洁明了, 同时展现了从点源函数 $\frac{1}{2} \text{Tr} \{ \delta(z^* - a^\dagger) \delta(z - a) \}$ 向高斯型函数 $\frac{1}{2kt} \text{Tr} \left\{ \exp \left[\frac{-1}{kt} (a^\dagger - z^*) (a - z) \right] \right\}$ 的演变, 所以此数学表达式的物理意义十分明晰.

从 (40) 式可知 $\Delta(\alpha, \alpha^*, t)$ 的经典对应是

$$\Delta(\alpha, \alpha^*, t) \rightarrow \frac{1}{2kt} \exp \left[\frac{-1}{kt} (\alpha^* - z^*) (\alpha - z) \right], \quad (42)$$

根据 (28) 式可知

$$\begin{aligned} \Delta(\alpha, \alpha^*, t) &= \int d^2z \frac{1}{2kt} \exp \left[\frac{-1}{kt} (\alpha^* - z^*) (\alpha - z) \right] \\ &\quad \times \Delta(\alpha, \alpha^*, 0). \end{aligned} \quad (43)$$

而对 (24) 式做积分可得:

$$\Delta(\alpha, \alpha^*, 0) = \frac{1}{\pi} \text{Tr} \{ e^{-2(\alpha^* - a^\dagger)(\alpha - a)} \}, \quad (44)$$

所以 $\Delta(\alpha, \alpha^*, t)$ 的正规乘积是

$$\begin{aligned}
 & \Delta(\alpha, \alpha^*, t) \\
 &= \int d^2z \frac{1}{kt} \exp \left[\frac{-1}{kt} (\alpha^* - z^*) (\alpha - z) \right] \\
 & \quad \times \frac{1}{\pi} : e^{-2(\alpha^* - a^\dagger)(\alpha - a)} : \\
 &= \frac{1}{\pi(2kt+1)} : \exp \left[\frac{-2}{2kt+1} (a^\dagger - \alpha^*) (a - \alpha) \right] :.
 \end{aligned} \tag{45}$$

这样就从 (40) 式的 Weyl 编序形式导出了其正规乘积形式. 另外, (45) 式还可进一步得到验证 (具体详见附录 B), 即将正规乘积形式转化为 Weyl 编序形式. 举例, 当初态是纯相干态, 那么经过扩散通道后, 从 (40) 式可知它的 Wigner 函数为

$$\begin{aligned}
 W_F &= \langle z | \Delta(\alpha, \alpha^*, t) | z \rangle \\
 &= \frac{1}{\pi(2kt+1)} \exp \left[\frac{-2}{2kt+1} (\alpha^* - z^*) (\alpha - z) \right], \\
 & \quad z = q + ip.
 \end{aligned} \tag{46}$$

图 1 所示为 Wigner 算符的演化规律, 图 1(a) 描绘了相干态初始的 Wigner 函数, 尖峰象征 Delta 函数; 图 1(b) 描绘了 $kt = 0.8$ 时高斯型 Wigner 函数. 对比两图中 Wigner 函数的峰值及形状, 可以看出相干态在耗散通道的演化情况.

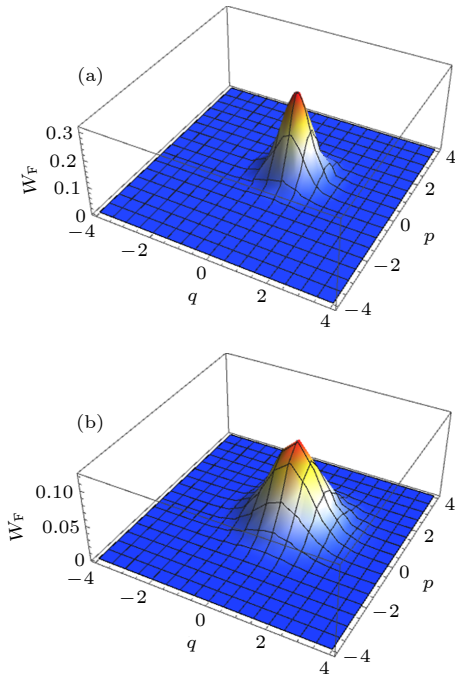


图 1 (a) $kt = 0$, (b) $kt = 0.8$ 时, 相干态下 Wigner 算符的演化规律 ($\alpha = 1/2(1 + i)$)

Fig. 1. Evolution law of Wigner operator for the coherent state with $\alpha = 1/2(1 + i)$ for (a) $kt = 0$; (b) $kt = 0.8$.

7 结论

本文引入算符的 Weyl 编序记号, 导出量子扩散通道中 Wigner 算符的演化律公式, 它简洁而物理清晰, 展现了从点源函数

$$\frac{1}{2} : \delta(z^* - a^\dagger) \delta(z - a) :$$

向高斯型函数

$$\frac{1}{2kt} : \exp \left[\frac{-1}{kt} (a^\dagger - z^*) (a - z) \right] :$$

的演变, k 是扩散系数. 由此也可转化为 Wigner 算符的其他排序形式, 如正规乘积序. 值得指出, 对于相干态的演化用了反正规乘积来讨论, 而对 Wigner 算符的演化用 Weyl 排序来讨论, 这两者的演化在数学形式上是一样的.

附录 A 相空间中 Wigner 算符所满足的扩散方程

本节验证 Wigner 算符所满足的扩散方程. 由 Wigner 算符的正规乘积表达式 (44) 式可算出:

$$\begin{aligned}
 \frac{\partial}{\partial \alpha} \Delta(\alpha, \alpha^*) &= \frac{1}{\pi} \frac{\partial}{\partial \alpha} : e^{-2(\alpha^* - a^\dagger)(\alpha - a)} : \\
 &= 2(a^\dagger - \alpha^*) \Delta(\alpha, \alpha^*),
 \end{aligned} \tag{A1}$$

$$\begin{aligned}
 \frac{\partial}{\partial \alpha^*} \Delta(\alpha, \alpha^*) &= \frac{1}{\pi} \frac{\partial}{\partial \alpha^*} : e^{-2(\alpha^* - a^\dagger)(\alpha - a)} : \\
 &= \Delta(\alpha, \alpha^*) 2(a - \alpha),
 \end{aligned} \tag{A2}$$

因此有

$$a^\dagger \Delta(\alpha, \alpha^*) = \left(\frac{\partial}{2\partial \alpha} + \alpha^* \right) \Delta(\alpha, \alpha^*), \tag{A3}$$

$$\Delta(\alpha, \alpha^*) a = \left(\frac{\partial}{2\partial \alpha^*} + \alpha \right) \Delta(\alpha, \alpha^*). \tag{A4}$$

另一方面, 从 Wigner 算符的反正规乘积表达式

$$\Delta(\alpha, \alpha^*) = -\frac{1}{\pi} : e^{2(\alpha^* - a^\dagger)(\alpha - a)} :, \tag{A5}$$

可以推导出:

$$\begin{aligned}
 \frac{\partial}{\partial \alpha^*} \Delta(\alpha, \alpha^*) &= \Delta(\alpha, \alpha^*) 2(a - \alpha) \\
 &= 2(\alpha - a) \Delta(\alpha, \alpha^*),
 \end{aligned} \tag{A6}$$

$$\begin{aligned}
 \frac{\partial}{\partial \alpha} \Delta(\alpha, \alpha^*) &= 2(a^\dagger - \alpha^*) \Delta(\alpha, \alpha^*) \\
 &= 2\Delta(\alpha, \alpha^*) (\alpha^* - a^\dagger),
 \end{aligned} \tag{A7}$$

所以

$$\left(\alpha - \frac{\partial}{2\partial\alpha^*}\right) \Delta(\alpha, \alpha^*) = a\Delta(\alpha, \alpha^*), \quad (\text{A8})$$

$$\left(\alpha^* - \frac{\partial}{2\partial\alpha}\right) \Delta(\alpha, \alpha^*) = \Delta(\alpha, \alpha^*) a^\dagger, \quad (\text{A9})$$

$$\begin{aligned} & \Delta(\alpha, \alpha^*) a^\dagger a \\ &= \left(\alpha^* - \frac{\partial}{2\partial\alpha}\right) \Delta(\alpha, \alpha^*) a \\ &= \left(\alpha^* - \frac{\partial}{2\partial\alpha}\right) \left(\frac{\partial}{2\partial\alpha^*} + \alpha\right) \Delta(\alpha, \alpha^*), \quad (\text{A10}) \end{aligned}$$

由以上这些关系式就能导出 (36) 式.

附录 B 化算符为 Weyl 排序的公式

相干态 $|z\rangle \langle z|$ 的经典对应是

$$\begin{aligned} & 2\pi \text{tr} [|z\rangle \langle z| \Delta(q, p)] \\ &= 2\pi \frac{1}{\pi} \langle z| : e^{-2(a-\alpha)(a^\dagger-\alpha^*)} : |z\rangle \\ &= 2e^{-2(z^*-\alpha^*)(z-\alpha)}, \quad (\text{B1}) \end{aligned}$$

其 Weyl 对应式则为

$$|z\rangle \langle z| = 4 \int d^2\alpha e^{-2(z^*-\alpha^*)(z-\alpha)} \Delta(\alpha, \alpha^*), \quad (\text{B2})$$

由于是 Delta 函数型, 所以 $|z\rangle \langle z|$ 的 Weyl 排序形式为

$$|z\rangle \langle z| = 2 : e^{-2(z^*-a^\dagger)(z-a)} :, \quad (\text{B3})$$

代入 (2) 式得到

$$\rho = 2 \int \frac{d^2z}{\pi} P(z) : e^{-2(z^*-a^\dagger)(z-a)} :, \quad (\text{B4})$$

鉴于

$$P(z) = e^{|z|^2} \int \frac{d^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle e^{|\beta|^2} e^{\beta^*z-\beta z^*}, \quad (\text{B5})$$

这里 $|\beta\rangle$ 为相干态, $|\beta\rangle = \exp[-|\beta|^2/2 + \beta a^\dagger] |0\rangle$, 所以

$$\begin{aligned} & \rho = 2 \int \frac{d^2z}{\pi} e^{|z|^2} \int \frac{d^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle e^{|\beta|^2} e^{\beta^*z-\beta z^*} \\ & \quad \times : e^{-2(z^*-a^\dagger)(z-a)} : \\ &= 2 \int \frac{d^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle : e^{2(\beta^*a-\beta a^\dagger+a^\dagger a)} :, \quad (\text{B6}) \end{aligned}$$

这就是将算符转化为 Weyl 编序的形式. 当取 ρ 为 (45) 式时,

$$\Delta(\alpha, \alpha^*, t) = \frac{1}{\pi(2kt+1)} : \exp\left[\frac{-2}{2kt+1} (a^\dagger-\alpha^*)(a-\alpha)\right] :, \quad (\text{B7})$$

此为正规乘积形式, 则代入 (B6) 式便可得到 Weyl 编序形式.

$$\begin{aligned} & \Delta(\alpha, \alpha^*, t) \\ &= 2 : \int \frac{d^2\beta}{\pi(2kt+1)} \exp\left[-2|\beta|^2 - \frac{2}{2kt+1} (-\beta^*-\alpha^*)(\beta-\alpha) + 2a^\dagger a + 2a\beta^* - 2\beta a^\dagger\right] : \\ &= 2 \int \frac{d^2\beta}{\pi(2kt+1)} : \exp\left[-|\beta|^2 \frac{4kt}{2kt+1} + 2\beta\left(\frac{\alpha^*}{2kt+1} - a^\dagger\right) - 2\beta^*\left(\frac{\alpha}{2kt+1} - a\right) - \frac{2|\alpha|^2}{2kt+1} + 2a^\dagger a\right] : \\ &= \frac{1}{2kt} : \exp\left[-\frac{2kt+1}{kt}\left(\frac{\alpha^*}{2kt+1} - a^\dagger\right)\left(\frac{\alpha}{2kt+1} - a\right) - \frac{2|\alpha|^2}{2kt+1} + 2a^\dagger a\right] : \\ &= \frac{1}{2kt} : \exp\left[-\frac{1}{kt}(\alpha^* - a^\dagger)(\alpha - a)\right] :. \quad (\text{B8}) \end{aligned}$$

此即验证了 (40) 式.

参考文献

- [1] Chakrabarti R, Yogesh V 2018 *Physica A* **490** 886
- [2] Wei C P, Xie F S, Zhang H L, Hu L Y 2013 *Int. J. Theor. Phys.* **52** 798
- [3] Agarwal G S 1971 *Phys. Rev. A* **3** 828
- [4] Hu L Y, Fan H Y 2009 *Opt. Commun.* **282** 4379
- [5] Takahashi K 1986 *J. Phys. Soc. Jpn.* **55** 762
- [6] Fan H Y 1991 *Phys. Lett. A* **161** 1
- [7] Meng X G, Wang J S, Liang B L 2009 *Chin. Phys. B* **18** 01534
- [8] Fan H Y 2010 *Commun. Theor. Phys.* **53** 344
- [9] Fan H Y, Hu L Y 2009 *Commun. Theor. Phys.* **51** 729
- [10] Fan H Y, Lu H L, Fan Y 2006 *Ann. Phys.* **321** 480
- [11] Kurchan J, Leboeuf P, Saraceno M 1989 *Phys. Rev. A* **40** 6800
- [12] Fan H Y, Zaidi H R 1987 *Phys. Lett. A* **124** 303
- [13] Fan H Y, Wang J S 2007 *Commun. Theor. Phys.* **47** 431
- [14] Fan H Y, Fan Y 1997 *Commun. Theor. Phys.* **27** 105
- [15] Yan S L 2007 *Acta Phys. Sin.* **56** 1994 (in Chinese) [颜森林 2007 物理学报 **56** 1994]
- [16] Lan D D, Guo X M, Peng C S, Ji Y L, Liu X L, Li P, Guo Y Q 2017 *Acta Phys. Sin.* **66** 120502 (in Chinese) [兰豆豆, 郭晓敏, 彭春生, 姬玉林, 刘香莲, 李璞, 郭龔强 2017 物理学报 **66** 120502]
- [17] Weinbub J, Ferry D K 2018 *Appl. Phys. Rev.* **5** 041104
- [18] Fan H Y, Liang Z F 2015 *Acta Phys. Sin.* **64** 050301 (in Chinese) [范洪义, 梁祖峰 2015 物理学报 **64** 050301]
- [19] Wigner E P 1932 *Phys. Rev.* **40** 749
- [20] Fan H Y 1992 *J. Phys. A* **25** 3443
- [21] Yuan H C, Xu X X 2012 *Acta Phys. Sin.* **61** 064205 (in Chinese) [袁洪春, 徐学翔 2012 物理学报 **61** 064205]
- [22] Fan H Y, Yang Y L 2006 *Phys. Lett. A* **353** 439
- [23] Zhang K, Fan C Y, Fan H Y 2019 *Int. J. Theor. Phys.* **58** 1687
- [24] Buot F, Jensen K 1990 *Phys. Rev. B* **42** 9429
- [25] Chountasis S, Vourdas A 1998 *Phys. Rev. A* **58** 1794
- [26] Fan H Y, Hu L Y, Yuan H C 2010 *Chin. Phys. B* **19** 060305

Time evolution law of Wigner operator in diffusion channel*

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Abstract

As is well known, the evolution of quantum state can be replaced by its Wigner function's time evolution. The Wigner function of a quantum state is the same as the density matrix of a quantum state, because they both contain many messages, such as the probability distribution and phases. Thus, the important information about the quantum state in the evolution process can be obtained more quickly and effectively by studying the Wigner function of a quantum state. In this paper, based on the classical diffusion equation, the diffusion equation of the quantum state density operator is derived by using the P representation of the density operator. Furthermore, by introducing the Weyl ordering symbol of the quantum operator, the corresponding Weyl quantization scheme is given. In addition, the evolution equation of Wigner operator in diffusion channel is established by using another phase space representation of density operator—Wigner function, and the solution form of Wigner operator is given. In this paper, we derive the evolution law of Wigner operator in quantum diffusion channel for the first time, that is, the form of Wigner operator at any time in the evolution process. Based on this conclusion, the evolution of coherent states through quantum diffusion channels is discussed.

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