J. Eur. Opt. Society-Rapid Publ. 2023, **19**, 38
© The Author(s), published by EDP Sciences, 2023
https://doi.org/10.1051/jeos/2023035
Available online at: https://jeos.edpsciences.org

**Journal of the European Optical Society-Rapid Publications**

RESEARCH ARTICLE OPEN ACCESS

# Dynamical system of optical soliton parameters by variational principle (super-Gaussian and super-sech Pulses)

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Received 3 June 2023 / Accepted 6 July 2023

**Abstract.** The parameter dynamics of super-sech and super-Gaussian pulses for the perturbed nonlinear Schrödinger's equation with power-law nonlinearity is obtained in this article. The variational principle successfully recovers this dynamical system. The details of the variational principle with the implementation of the Euler–Lagrange's equation to the nonlinear Schrödinger's equation with power-law of nonlinearity described in this paper have not been previously reported.

**Keywords:** Solitons, Variational principle, Perturbation, Euler–Lagrange.

## 1 Introduction

The dynamics of optical solitons is a long standing study that has now extended over half-a-century. Various aspects of soliton science have been reported. Notably, most of the papers are from the integrability aspects of a variety of models that arose from wide range of self-phase modulation (SPM) structures. A few papers are from additional, sparingly visible, topics such as conservation laws, quasimonochromatic solitons with the usage of perturbation theory, stochastic perturbation and the corresponding mean free velocity of the soliton and others.

One of the most viable topics that serves as an important foundation stone in optical soliton dynamics is the recovery of the soliton parameter dynamics such as the amplitude, width, center position, phase constant and similar such parameters. This can be achieved in several different ways. A few such mathematical approaches are the soliton perturbation theory, collective variables approach and the moment method. However, for example, soliton perturbation theory has its shortcomings. It fails to recover the variation of the phase constant as well as the variation of the center position of the soliton. The variational principle (VP) overcomes this hurdle. This has been succesfully and widely applied to various areas of Physics and Engineering such as Condensed Matter Physics, Fluid Dynamics and Fiber Optics including dispersion-managed solitons [1–20].

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The advantages and necessity of obtaining the dynamical system of soliton parameters are multifold. The study of soliton features in optics can be further enhanced through the utilization of these parameter dynamics. Four wave mixing effects, collision-induced timing jitter, and various other phenomena are among those that are included. Therefore, the parameter dynamics with the existence of perturbation terms is being studied by applying the VP to the nonlinear Schrödinger's equation (NLSE). Super-sech and super-Gaussian pulses are the two types of pulses being examined in this context. This would give a generalized flavor to the study of soliton parameters. The details of the VP with the implementation of the Euler–Lagrange's equation to NLSE with power-law of nonlinearity described in this paper have not been previously reported. A quick and succinct introduction is followed by the presentation of results.

## 2 Unperturbed NLSE with power-law nonlinearity

The governing model of such equation is written as:

$$iq_t + aq_{xx} + b|q|^{2n}q = 0, \tag{1}$$

where the coefficients $b$ and $a$ are utilized to denote SPM and chromatic dispersion in sequence. The function $q = q(x, t)$ represents the wave profile in a complex-valued form, where $i = \sqrt{-1}$. Equation (1) contains the linear temporal evolution, represented by the first term.

### 2.1 Variational principle

The Lagrangian (L) is associated with equation (1) is written as:

$$L = \frac{1}{2}\int_{-\infty}^{\infty}\left[i\left(qq_t^* - q^*q_t\right) - 2a|q_x|^2 + \frac{2b}{n+1}|q|^{2n+2}\right]\mathrm{d}x. \tag{2}$$

One obtains $q^*$ by complex-conjugating $q$. In equation (1), the assumed pulse $q = q(x, t)$ is presented as:

$$q(x,t) = A(t)f[B(t)\{x - \bar{x}(t)\}]\exp\left[-i\kappa(t)\{x - \bar{x}(t)\} + i\theta_0(t)\right]. \tag{3}$$

We use the symbols $\theta_0(t)$, $\kappa(t)$, $\bar{x}(t)$, $B(t)$, and $A(t)$ to denote the soliton phase, soliton frequency, center position of the soliton, pulse width, and soliton amplitude, respectively. Setting

$$s = B(t)[x - \bar{x}(t)], \tag{4}$$

then the pulse hypothesis (3) becomes

$$q(x,t) = A(t)f(s)\exp\left[-i\frac{\kappa(t)}{B(t)}s + i\theta_0(t)\right]. \tag{5}$$

Through the application of the provided equation

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{s}{B(t)}\frac{\mathrm{d}B(t)}{\mathrm{d}t} - B(t)\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}, \tag{6}$$

we conclude that:

$$q_x = A(t)B(t)\left[\frac{\mathrm{d}f(s)}{\mathrm{d}s} - i\frac{\kappa(t)}{B(t)}f(s)\right]\exp\left[-i\frac{\kappa(t)}{B(t)}s + i\theta_0(t)\right], \tag{7}$$

and

$$q_t = \left[\frac{\mathrm{d}A(t)}{\mathrm{d}t}f(s) + \frac{A(t)}{B(t)}s\frac{\mathrm{d}f(s)}{\mathrm{d}s}\frac{\mathrm{d}B(t)}{\mathrm{d}t} - A(t)B(t)\frac{\mathrm{d}f(s)}{\mathrm{d}s}\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} - i\frac{A(t)}{B(t)}sf(s)\frac{\mathrm{d}\kappa(t)}{\mathrm{d}t}\right.$$
$$\left. + iA(t)f(s)\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t} + iA(t)\kappa(t)f(s)\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}\right]\exp\left[-i\frac{\kappa(t)}{B(t)}s + i\theta_0(t)\right]. \tag{8}$$

Substituting (5)–(8) into (2) and using the formula $\mathrm{d}s = B(t)\mathrm{d}x$, the Lagrangian (2) reduces to

$$L = \frac{A^2(t)}{B(t)}\left(\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t} + \kappa(t)\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} - a\kappa^2(t)\right)I_{0,2,0} - aA^2(t)B(t)I_{0,0,2} + \frac{bA^{2n+2}(t)}{(n+1)B(t)}I_{0,2n+2,0}, \tag{9}$$

where

$$I_{a,b,c} = \int_{-\infty}^{\infty} s^a f^b(s) \left(\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right)^c \mathrm{d}s, \tag{10}$$

and non-negative integers are the only values that $a$, $b$, and $c$ can assume.

The integrals of motion can be obtained from the pulse form (5), which can be derived, as presented below

$$E = \int_{-\infty}^{\infty} |q|^2 \mathrm{d}x = \frac{A^2(t)}{B(t)} I_{0,2,0}, \tag{11}$$

$$M = i\int_{-\infty}^{\infty} \left(q^* q_x - q q_x^*\right) \mathrm{d}x = \frac{2A^2(t)\kappa(t)}{B(t)} I_{0,2,0}. \tag{12}$$

The mathematical representation of the Hamiltonian takes the form of

$$H = \int_{-\infty}^{\infty} \left[a|q_x|^2 - \frac{b}{n+1}|q|^{2n+2}\right] \mathrm{d}x = \frac{A^2(t)}{B(t)}\left[aB^2(t)I_{0,0,2} + a\kappa^2(t)I_{0,2,0} - \frac{bA^{2n}(t)}{(n+1)} I_{0,2n+2,0}\right]. \tag{13}$$

## 2.2 Parameter dynamics of the NLSE

Introducing the following Euler-Lagrange (EL) equation [4, 8] in this subsection leads to the derivation of the dynamical system:

$$\frac{\partial L}{\partial p} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial p_t}\right) = 0, \tag{14}$$

where the soliton parameters $A(t)$, $B(t)$, $\bar{x}(t)$, $\kappa(t)$ and $\theta_0(t)$ are represented by the variable $p$, where $p$ denotes one of them. The dynamic system below is derived by substituting (9) into (14):

$$\left[\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t} + \kappa(t)\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} - a\kappa^2(t)\right] I_{0,2,0} - aB^2(t)I_{0,0,2} + bA^{2n}(t)I_{0,2n+2,0} = 0, \tag{15}$$

$$-\left[\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t} + \kappa(t)\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} - a\kappa^2(t)\right] I_{0,2,0} - aB^2(t)I_{0,0,2} - \frac{b}{n+1}A^{2n}(t)I_{0,2n+2,0} = 0, \tag{16}$$

$$-A(t)\kappa(t)\frac{\mathrm{d}B(t)}{\mathrm{d}t} + 2B(t)\kappa(t)\frac{\mathrm{d}A(t)}{\mathrm{d}t} + A(t)B(t)\frac{\mathrm{d}\kappa(t)}{\mathrm{d}t} = 0, \tag{17}$$

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = 2a\kappa(t), \tag{18}$$

and

$$-A(t)\frac{\mathrm{d}B(t)}{\mathrm{d}t} + 2B(t)\frac{\mathrm{d}A(t)}{\mathrm{d}t} = 0. \tag{19}$$

For the pulse form given by (5), the equations (15)–(19) provide the general forms of the soliton parameter dynamics of equation (1). The dynamic system (15)–(19) can be expressed in a simplified and reduced form as:

$$\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t} = -a\kappa^2(t) - \frac{(n+2)aB^2(t)}{n}\frac{I_{0,0,2}}{I_{0,2,0}}, \tag{20}$$

$$A^{2n}(t) = \frac{2(n+1)aB^2(t)}{nb}\frac{I_{0,0,2}}{I_{0,2n+2,0}}, \tag{21}$$

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = 2a\kappa(t), \tag{22}$$

$$\frac{\mathrm{d}\kappa(t)}{\mathrm{d}t}=0, \tag{23}$$

and

$$A(t)=K\sqrt{B(t)}, \tag{24}$$

where the square roots of the energy are proportional to the constant $K$ in (24). From (21) and (24), we have:

$$B^{n-2}(t)=\frac{2a(n+1)}{nbK^{2n}}\frac{I_{0,0,2}}{I_{0,2n+2,0}}. \tag{25}$$

### 2.3 Super-Gaussian pulses

Assuming $m>0$, the super Gaussian pulse function can be written as $f(s)=e^{-s^{2m}}$. Then, one can obtain the integrals of motion as:

$$E=\frac{A^2(t)}{mB(t)}2^{-\frac{1}{2m}}\Gamma\left(\frac{1}{2m}\right), \tag{26}$$

$$M=\frac{A^2(t)\kappa(t)}{mB(t)}2^{1-\frac{1}{2m}}\Gamma\left(\frac{1}{2m}\right), \tag{27}$$

and the Hamiltonian is given by:

$$H=\frac{A^2(t)}{mB(t)}\left[\frac{amB^2(t)(2m-1)}{2}2^{1/2m}\Gamma\left(1-\frac{1}{2m}\right)+a\kappa^2(t)2^{-1/2m}\Gamma\left(\frac{1}{2m}\right)-\frac{bA^{2n}(t)}{(n+1)^{1+\frac{1}{2m}}}2^{-1/2m}\Gamma\left(\frac{1}{2m}\right)\right]. \tag{28}$$

For $u>0$, the gamma function is defined as $\Gamma(u)$. This compels the parameter $m$ to be bounded below as given by

$$m>\frac{1}{2}. \tag{29}$$

The pulse parameters can be obtained from the evolution equations (20)–(25), which can be expressed in a reduced form as:

$$\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t}=-a\kappa^2(t)-\frac{amB^2(t)(n+2)(2m-1)}{2n}2^{1/m}\frac{\Gamma\left(1-\frac{1}{2m}\right)}{\Gamma\left(\frac{1}{2m}\right)}, \tag{30}$$

$$A^{2n}(t)=\frac{amB^2(t)(2n+2)^{1+\frac{1}{2m}}(2m-1)}{2nb}2^{1/2m}\frac{\Gamma\left(1-\frac{1}{2m}\right)}{\Gamma\left(\frac{1}{2m}\right)}, \tag{31}$$

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}=2a\kappa(t), \tag{32}$$

$$\frac{\mathrm{d}\kappa(t)}{\mathrm{d}t}=0, \tag{33}$$

$$A(t)=K\sqrt{B(t)}, \tag{34}$$

and

$$B^{n-2}(t)=\frac{m(2n+2)^{1+\frac{1}{2m}}(2m-1)}{2nbK^{2n}}2^{1/2m}\frac{\Gamma\left(1-\frac{1}{2m}\right)}{\Gamma\left(\frac{1}{2m}\right)}. \tag{35}$$

Figures 1 and 2 provide a few plots of super-Gaussian pulse and super-sech pulse with the governing model (1), respectively. These plots offer a visual depiction of the waveform characteristics and provide valuable insights into the behavior of the pulses under investigation. The parameter vales chosen are: $K=1$, $\kappa(t)=1$, $a=1$, $\bar{x}(t)=2t$, $b=1$, $n=1{,}5$ and $m=2{,}5$.

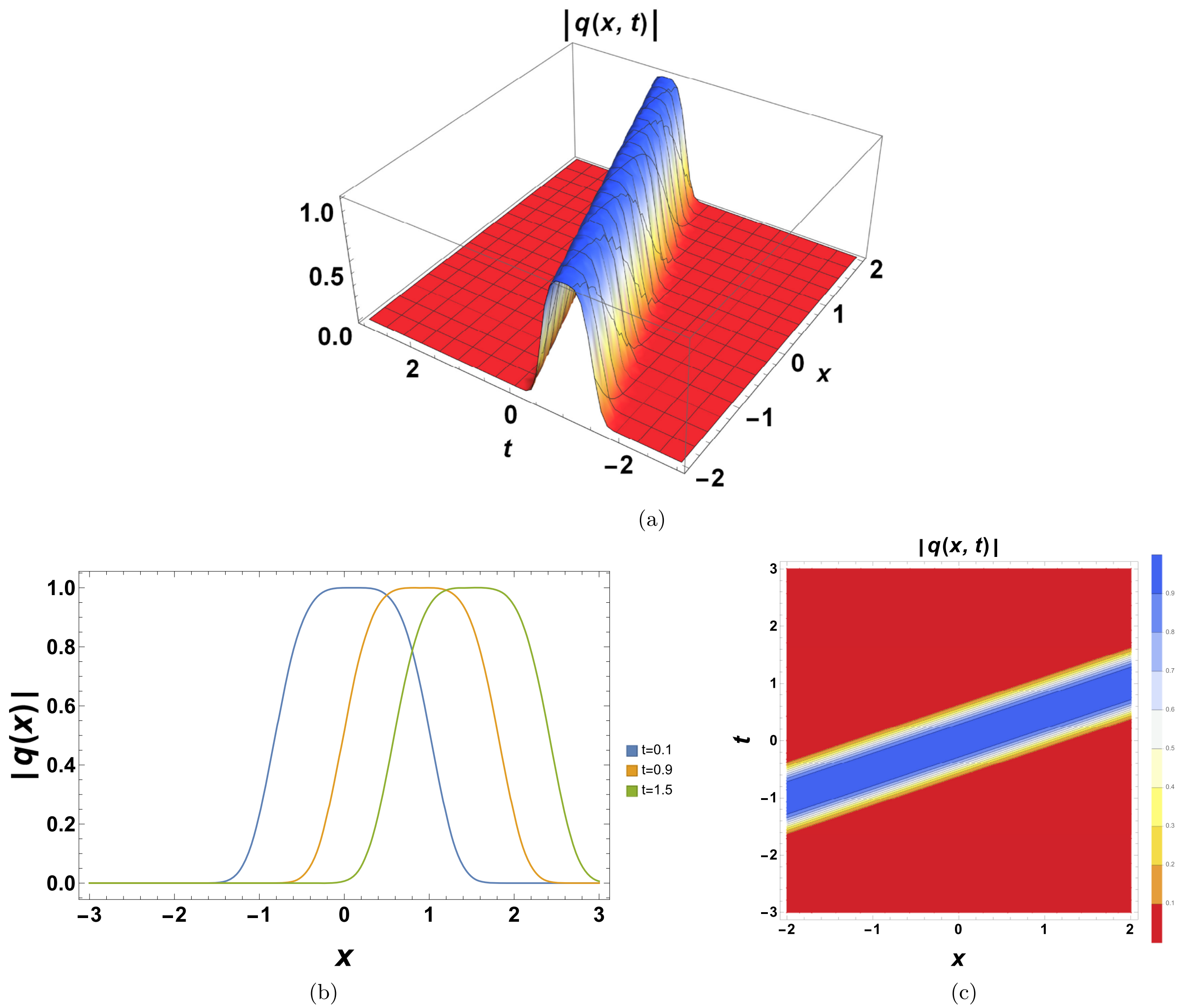


**Figure 1.** Profile of a super-Gaussian pulse. (a) Surface plot. (b) 2D plots moving in time. (c) Contour plot.

### 2.4 Super-sech pulses

For super-sech pulses, we set $f(s) = \mathrm{sech}^{2\mathrm{m}}\,\mathrm{s}$, $m > 0$. Then, one can address the equations governing the integrals of motion that are expressed as:

$$E = \frac{\sqrt{\pi}A^2(t)}{B(t)} \frac{\Gamma(2m)}{\Gamma\left(2m + \frac{1}{2}\right)}, \tag{36}$$

$$M = \frac{2\sqrt{\pi}A^2(t)\kappa(t)}{B(t)} \frac{\Gamma(2m)}{\Gamma\left(2m + \frac{1}{2}\right)}, \tag{37}$$

and we can express the Hamiltonian as:

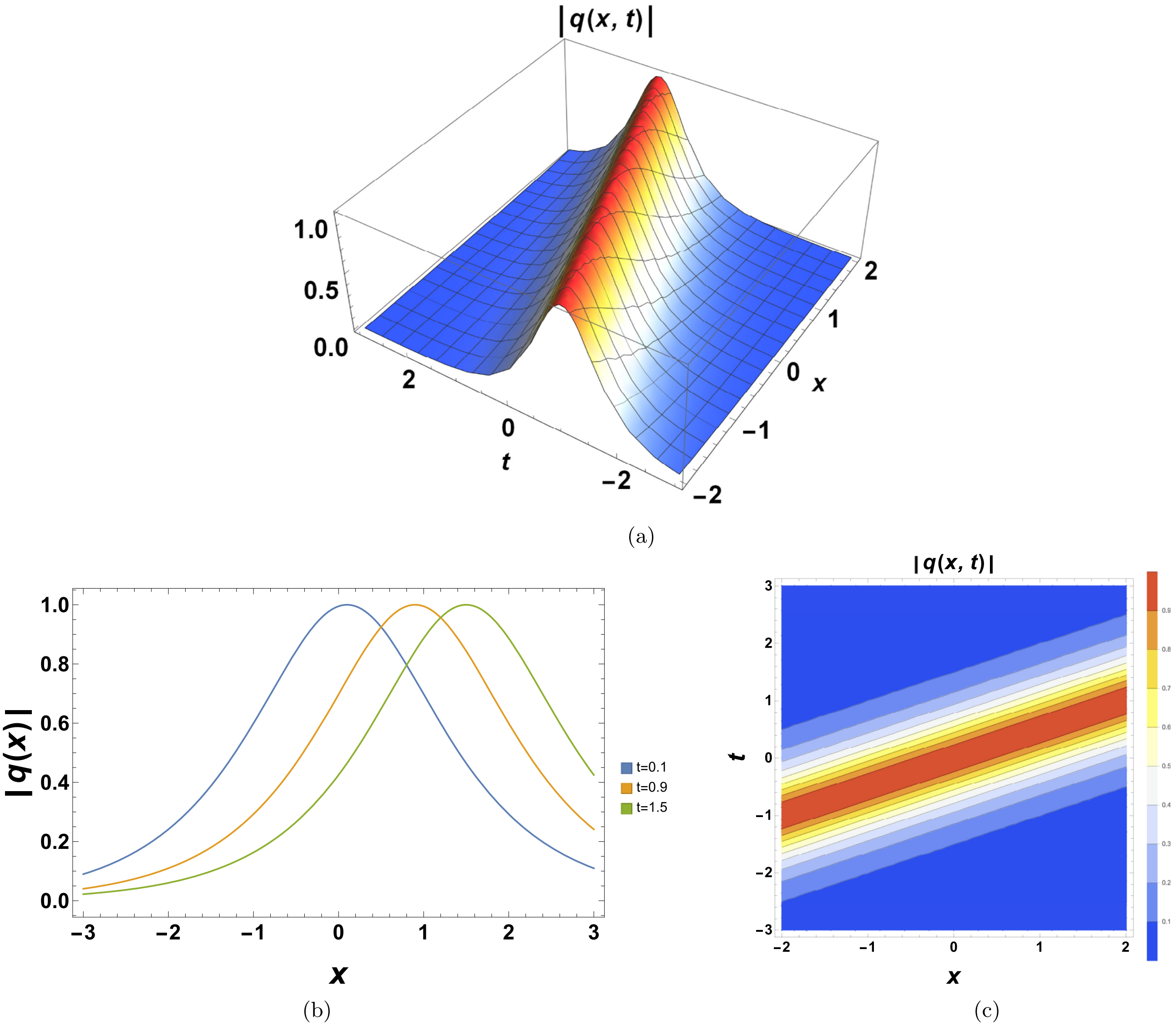


**Figure 2.** Profile of a super-sech pulse. (a) Surface plot. (b) 2D plots moving in time. (c) Contour plot.

$$
\begin{aligned}
H = -4m^2 a A^2(t) B(t) \Bigg[ & \frac{4m\sqrt{\pi}}{4m+1} \frac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)} - \frac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)} \\
& - \frac{2^{2+2m}+1}{2+2m} {}_2F_1(2+2m, 2+4m; 3+2m; -1) \Bigg] \\
& + \frac{A^2(t)}{B(t)} \left[ \sqrt{\pi} a \kappa^2(t) \frac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)} - \frac{\sqrt{\pi} b A^{2n}(t)}{(n+1)} \frac{\Gamma(2(n+1)m)}{\Gamma\left(2(n+1)m+\frac{1}{2}\right)} \right].
\end{aligned} \tag{38}
$$

Here, the generalized form of Gauss' hypergeometric function is expressed as:

$$
{}_pF_q(a_1, \dots a_p; b_1, \dots b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \tag{39}
$$

and the symbol for the Pochhammer is:

$$
(p)_n = \begin{cases} p(p+1)\cdots(p+n-1) & n > 0, \\ 1 & n = 0\,. \end{cases} \tag{40}
$$

The pulse parameters are governed by the evolution equations (20)–(25), which can be expressed in a simpler form as:

$$\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t} = -a\kappa^2(t) + \frac{4m^2(n+2)aB^2(t)\Gamma\left(2m+\frac{1}{2}\right)}{n\sqrt{\pi}\Gamma(2m)}\left(\frac{4m\sqrt{\pi}}{4m+1}\frac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)} - \frac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)} - \frac{2^{2+2m}+1}{2+2m}{}_2F_1(2+2m,2+4m;3+2m;-1)\right), \tag{41}$$

$$A^{2n}(t) = -\frac{4m^2(n+1)aB^2(t)\Gamma\left(2(n+1)m+\frac{1}{2}\right)}{nb\sqrt{\pi}\Gamma(2(n+1)m)}\left(\frac{4m\sqrt{\pi}}{4m+1}\frac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)} - \frac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)} - \frac{2^{2+2m}+1}{2+2m}{}_2F_1(2+2m,2+4m;3+2m;-1)\right). \tag{42}$$

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = 2a\kappa(t), \tag{43}$$

$$\frac{\mathrm{d}\kappa(t)}{\mathrm{d}t} = 0, \tag{44}$$

$$A(t) = K\sqrt{B(t)}, \tag{45}$$

and

$$B^{n-2}(t) = -\frac{4m^2(n+1)\Gamma\left(2(n+1)m+\frac{1}{2}\right)}{nbK^{2n}\sqrt{\pi}\Gamma(2(n+1)m)}\left[\frac{4m\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)} - \frac{2^{4m-1}(2\mathrm{m}+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)} - \frac{2^{2+2m}+1}{2+2m}{}_2F_1(2+2m,2+4m;3+2m;-1)\right]. \tag{46}$$

## 3 Perturbed NLSE with power-law nonlinearity

The equation is described by the following governing model:

$$iq_t + aq_{xx} + b|q|^{2n}q = i\epsilon R[q,q^*], \tag{47}$$

where $R[q,q^*]$ is given by:

$$R = \delta|q|^{2m}q + \alpha q_x + \beta q_{xx} + \lambda\left(|q|^{2m}q\right)_x + \theta\left(|q|^{2m}\right)_x q + \sigma|q|^{2m}q_x - i\xi\left(q^2q_x^*\right)_x - i\eta q_x^2 q^* - i\zeta q^*\left(q^2\right)_{xx} - i\mu\left(|q|^{2m}\right)_x q + (\sigma_1 q + \sigma_2 q_x)\int_{-\infty}^{x}|q|^{2m}\mathrm{d}s, \tag{48}$$

and $\epsilon$, $\delta$, $\alpha$, $\beta$, $\lambda$, $\theta$, $\sigma$, $\xi$, $\eta$, $\zeta$, $\mu$, $\sigma_1$ and $\sigma_2$ are constants, where $\epsilon$ is from quasimonochromaticity. From (5) and (47), we have

$$R = \Bigg\{\delta A^{2m+1}(t)f^{2m+1}(s) + \alpha A(t)B(t)\frac{\mathrm{d}f(s)}{\mathrm{d}s} + \beta A(t)B^2(t)\frac{\mathrm{d}^2f(s)}{\mathrm{d}s^2} - \beta A(t)\kappa^2(t)f(s)$$

$$+[(2m+1)\lambda + 2m\theta + \sigma]A^{2m+1}(t)B(t)f^{2m}(s)\frac{\mathrm{d}f(s)}{\mathrm{d}s} + [2\xi - 2\eta - 8\zeta]A^3(t)B(t)\kappa(t)f^2(s)\frac{\mathrm{d}f(s)}{\mathrm{d}s}$$

$$+A^{2m+1}(t)\sigma_1 f(s)\int_{-\infty}^{x}f^{2m}(s)\mathrm{d}s + \sigma_2 A^{2m+1}(t)B(t)\frac{\mathrm{d}f(s)}{\mathrm{d}s}\int_{-\infty}^{x}f^{2m}(s)\mathrm{d}s$$

$$-i\Big[\alpha A(t)\kappa(t)\mathrm{f}(s) + 2\beta A(t)B(t)\tfrac{\mathrm{d}f(s)}{\mathrm{d}s} + [\lambda+\sigma]A^{2m+1}(t)\kappa(t)f^{2m+1}(s)$$

$$+[2\xi+\eta+2\zeta]A^3(t)B^2(t)f(s)\left(\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right)^2+[\xi+2\zeta]A^3(t)B^2(t)f^2(s)\frac{\mathrm{d}^2f(s)}{\mathrm{d}s^2}$$

$$+[\xi-4\zeta-\eta]A^3(t)\kappa^2(t)f^3(s)+2m\mu A^{2m+1}(t)B(t)f^{2m}(s)\frac{\mathrm{d}f(s)}{\mathrm{d}s}$$

$$+\sigma_2A^{2m+1}(t)\kappa(t)f(s)\int_{-\infty}^{x}f^{2m}(s)\mathrm{d}s\Bigg]\Bigg\}\exp\left[-i\frac{\kappa(t)}{B(t)}s+i\theta_0(t)\right]. \tag{49}$$

### 3.1 Parameter dynamics of the perturbed NLSE

In this subsection, we derive the dynamical system of equation (47) by introducing the following Euler Lagrange (EL) equation:

$$\frac{\partial L}{\partial p}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial p_t}\right)=i\epsilon\int_{-\infty}^{\infty}\left(R\frac{\partial q^*}{\partial p}-R^*\frac{\partial q}{\partial p}\right)\mathrm{d}x, \tag{50}$$

where L is given by (9) and $p$ is one of these same five parameters $A(t)$, $B(t)$, $\bar{x}(t)$, $\kappa(t)$ and $\theta_0(t)$, respectively, while $R^*$ is the complex-conjugate of $R$. Now, we have the following dynamic system:

$$\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t}+\kappa(t)\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}-a\kappa^2(t)=aB^2(t)\frac{I_{0,0,2}}{I_{0,2,0}}-bA^{2n}(t)\frac{I_{0,2n+2,0}}{I_{0,2,0}}$$

$$+\epsilon\kappa(t)\left[\alpha+\sigma_2A^{2m}(t)\int_{-\infty}^{x}f^{2m}(s)\mathrm{d}s\right]+\epsilon(\lambda+\sigma)\kappa(t)A^{2m}(t)\frac{I_{0,2m+2,0}}{I_{0,2,0}}$$

$$+\epsilon(\xi+\eta+4\zeta)A^2(t)\kappa^2(t)\frac{I_{0,4,0}}{I_{0,2,0}}+\epsilon(2\xi+\eta+2\zeta)A^2(t)B^2(t)\frac{I_{0,2,2}}{I_{0,2,0}}, \tag{51}$$

$$-\left[\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t}+\kappa(t)\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}-a\kappa^2(t)\right]=aB^2(t)\frac{I_{0,0,2}}{I_{0,2,0}}+\frac{b}{n+1}A^{2n}(t)\frac{I_{0,2n+2,0}}{I_{0,2,0}}-\epsilon(\xi+2\zeta)A^2(t)B^2(t)\frac{I_{0,2,2}}{I_{0,2,0}}, \tag{52}$$

$$-2B(t)\kappa(t)\frac{\mathrm{d}A(t)}{\mathrm{d}t}-A(t)B(t)\frac{\mathrm{d}\kappa(t)}{\mathrm{d}t}+A(t)\kappa(t)\frac{\mathrm{d}B(t)}{\mathrm{d}t}=2\epsilon\delta\kappa(t)B(t)A^{2m+1}(t)\frac{I_{0,2m+2,0}}{I_{0,2,0}}-2\epsilon\beta A(t)B^3(t)[\kappa(t)+2]\frac{I_{0,0,2}}{I_{0,2,0}}$$

$$-2\epsilon\beta B(t)A(t)\kappa^3(t)-4\epsilon\mu mA^{2m+1}(t)B^3(t)\frac{I_{0,2m,2}}{I_{0,2,0}}+2\epsilon\sigma_1B(t)A^{2m+1}(t)\kappa(t)\int_{-\infty}^{x}f^{2m}(s)\mathrm{d}s, \tag{53}$$

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}=2a\kappa(t), \tag{54}$$

and

$$-2B(t)\frac{\mathrm{d}A(t)}{\mathrm{d}t}+A(t)\frac{\mathrm{d}B(t)}{\mathrm{d}t}=2\epsilon\delta B(t)A^{2m+1}(t)\frac{I_{0,2m+2,0}}{I_{0,2,0}}-2\epsilon\beta A(t)B^3(t)\frac{I_{0,0,2}}{I_{0,2,0}}-2\epsilon\beta B(t)A(t)\kappa^2(t). \tag{55}$$

The general forms of the soliton parameters dynamics of equation (47) for the pulse form given by (5) are represented by equations (51)–(55). A simplified version of the dynamic system (51)–(55) is:

$$\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t}=\frac{-\kappa\epsilon A^{2m}\left[\epsilon(\xi+2\zeta)A^2I_{0,2,2}-aI_{0,0,2}\right]\left[I_{0,2m+2,0}(\lambda+\sigma)+I_{0,2,0}\sigma_2\int_{-\infty}^{x}f^{2m}(s)\mathrm{d}s\right]}{\left[2aI_{0,0,2}+A^2\epsilon(\xi+\eta)I_{0,2,2}\right]I_{0,2,0}}$$

$$+\frac{bA^{2n}\left\{[(\xi+2\zeta)n-\eta-\xi]\epsilon A^2I_{0,2,2}-a(n+2)I_{0,0,2}\right\}I_{0,2n+2,0}}{(n+1)\left[2aI_{0,0,2}+A^2\epsilon(\xi+\eta)I_{0,2,2}\right]I_{0,2,0}}$$

$$-\frac{\begin{pmatrix}\kappa\left\{\epsilon^2A^2(\xi+2\zeta)\left[\kappa A^2(\xi+\eta+4\zeta)+\alpha\right]+\kappa a\epsilon A^2(\xi+\eta)\right\}I_{0,2,2}\\ -a\kappa\left[\kappa A^2\epsilon(\xi+\eta+4\zeta)+\alpha\epsilon-2a\kappa\right]I_{0,0,2}\end{pmatrix}}{2aI_{0,0,2}+A^2\epsilon(\xi+\eta)I_{0,2,2}}, \tag{56}$$

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = 2\,a\kappa(t), \tag{57}$$

$$\frac{\mathrm{d}\kappa(t)}{\mathrm{d}t} = \frac{2\,\epsilon}{A}\left[A^{2\,m+1}\left(2\,m\mu\,B^2\frac{I_{0,2\,m,2}}{I_{0,2,0}} - \sigma_1\kappa\int_{-\infty}^{x} f^{2m}(s)\mathrm{d}s\right) + 2\,\beta\,AB^2\frac{I_{0,0,2}}{I_{0,2,0}}\right], \tag{58}$$

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = \frac{2B}{A}\left(\frac{\mathrm{d}A(t)}{\mathrm{d}t} - A\beta\,\kappa^2\epsilon - \epsilon\,\beta\,AB^2\frac{I_{0,0,2}}{I_{0,2,0}} + \epsilon\,\delta\,A^{2\,m+1}\frac{I_{0,2\,m+2,0}}{I_{0,2,0}}\right), \tag{59}$$

$$B^2(t) = \frac{nbA^{2\,n}I_{0,2\,n+2,0} - \epsilon\kappa(n+1)\left\{(\lambda+\sigma)A^{2\,m}I_{0,2\,m+2,0} + \begin{bmatrix}\alpha + \kappa A^2(\xi+\eta+4\,\zeta)\\ +\sigma_2 A^{2\,m}\int_{-\infty}^{x} f^{2m}(s)\mathrm{d}s\end{bmatrix} I_{0,2,0}\right\}}{(n+1)\left[2aI_{0,0,2} + \epsilon A^2(\xi+\eta)I_{0,2,2}\right]}, \tag{60}$$

where $A = A(t)$, $B = B(t)$ and $\kappa = \kappa(t)$.

## 3.2 Super-Gaussian pulses

The dynamical system (56)–(60) is reduced to a simpler form for super-Gaussian pulses, which is:

$$\begin{aligned}\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t} = {} & \frac{-\kappa\epsilon\,A^{2\,m}\begin{bmatrix}\epsilon\,(\xi+2\,\zeta)A^2 2^{(1-3m)/m}\\ -a2^{(1-2m)/2m}\end{bmatrix}\begin{bmatrix}(\lambda+\sigma)m(2m)^{1/2m}\\ -2\sigma_2(m+1)^{1/2m}\Gamma\left(\frac{1}{2m}, 2mx^{2m}\right)\end{bmatrix}}{\left[a2^{1/2m} + A^2\epsilon\,(\xi+\eta)2^{(1-3m)/m}\right](m+1)^{1/2m}m(2m)^{1/2m}} \\ & + \frac{bA^{2\,n}\left\{\epsilon A^2[(\xi+2\,\zeta)n - \eta - \xi])2^{(1-3m)/m} - a(n+2)2^{(1-2m)/2m}\right\}}{(n+1)^{(1+2m)/2m}\left[a2^{1/2m} + \epsilon\,A^2(\xi+\eta)2^{(1-3m)/m}\right]} \\ & - \frac{\begin{pmatrix}\kappa\left\{\epsilon^2A^2(\xi+2\,\zeta)\left[\kappa\,A^2(\xi+\eta+4\,\zeta)+\alpha\right] + \kappa a\epsilon A^2(\xi+\eta)\right\}2^{(1-3m)/m}\\ -\left[\kappa A^2\,\epsilon(\xi+\eta+4\,\zeta) + \alpha\epsilon - 2\,a\kappa\right]a\kappa 2^{(1-2m)/2m}\end{pmatrix}}{a2^{1/2m} + \epsilon A^2\,(\xi+\eta)2^{(1-3m)/m}},\end{aligned} \tag{61}$$

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = 2\,a\kappa(t), \tag{62}$$

$$\frac{\mathrm{d}\kappa(t)}{\mathrm{d}t} = 2\epsilon\begin{bmatrix}\dfrac{A^{2\,m}\mu B^2m^2(2m-1)(m+1)^{(1-4m)/2m}2^{1/m}\Gamma\left(1-\frac{1}{2m}\right)}{\Gamma\left(\frac{1}{2m}\right)}\\ +\dfrac{2\kappa\sigma_1\Gamma\left(\frac{1}{2m}, 2mx^{2m}\right)A^{2\,m}}{m(2m)^{1/2m}} + \dfrac{\beta B^2m(2m-1)2^{1/m}\Gamma\left(1-\frac{1}{2m}\right)}{\Gamma\left(\frac{1}{2m}\right)}\end{bmatrix}, \tag{63}$$

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = 2B\begin{pmatrix}\dfrac{1}{A}\dfrac{\mathrm{d}A(t)}{\mathrm{d}t} - \epsilon\beta\,\kappa^2 + \dfrac{\epsilon\,\delta\,A^{2\,m}}{(m+1)^{1/2m}}\\ -\dfrac{\epsilon\,\beta\,B^2(2m-1)2^{1/2m}m\Gamma\left(1-\frac{1}{2m}\right)}{2^{(2m-1)/2m}\Gamma\left(\frac{1}{2m}\right)}\end{pmatrix}, \tag{64}$$

$$B^2(t)=\frac{\left\{nbA^{2n}-\epsilon\kappa(n+1)^{(1+2m)/2m}2^{1/m}\left[\begin{array}{c}\alpha+\kappa A^2(\xi+\eta+4\zeta)\\ +\dfrac{(\lambda+\sigma)A^{2m}}{(m+1)^{1/2m}}-\dfrac{2A^{2m}\sigma_2\Gamma\left(\frac{1}{2m},2mx^{2m}\right)}{m(2m)^{1/2m}}\end{array}\right]\right\}\Gamma\left(\frac{1}{2m}\right)}{m(2m-1)(n+1)^{(1+2m)/2m}2^{1/2m}\left[a2^{1/2m}+\epsilon A^2(\xi+\eta)2^{(1-3m)/m}\right]\Gamma\left(1-\frac{1}{2m}\right)}. \tag{65}$$

The equations involve the incomplete gamma function, which is represented by $\Gamma(a,\,x)$.

### 3.3 Super-sech pulses

The dynamical system (56)–(60) simplifies to a specific form when considering super-sech pulses, as described below

$$\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t}=\left[\frac{(\lambda+\sigma)\Gamma(2(2m+1)m)}{\Gamma\left(2(2m+1)m+\frac{1}{2}\right)}+\frac{\Gamma(2m)\sigma_2\,\mathrm{sech}^{4m}(x)}{4m\Gamma\left(2m+\frac{1}{2}\right)}\,{}_2F_1\left(\frac{1}{2},2m;2m+1;\,\mathrm{sech}^2(x)\right)\right]$$

$$\times\frac{-\kappa\epsilon A^{4m}\left\{\dfrac{4\epsilon(\xi+2\zeta)A^2m^2\sqrt{\pi}\Gamma(4m)}{(8m+1)\Gamma\left(4m+\frac{1}{2}\right)}+4am^2\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}\Gamma(2m)}{4m+1\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right]\right\}}{\left\{-8am^2\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right]+\dfrac{4\epsilon A^2(\xi+\eta)\mathrm{m}^2\sqrt{\pi}\Gamma(4m)}{(8m+1)\Gamma\left(4m+\frac{1}{2}\right)}\right\}\dfrac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}}$$

$$+\frac{bA^{2n}\left\{\dfrac{4[(\xi+2\zeta)n-\eta-\xi]\epsilon A^2m^2\sqrt{\pi}\Gamma(4m)}{(8m+1)\Gamma\left(4m+\frac{1}{2}\right)}+4a(n+2)m^2\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right]\right\}\dfrac{\Gamma(2(n+1)m)}{\Gamma\left(2(n+1)m+\frac{1}{2}\right)}}{(n+1)\left\{-8am^2\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}}{4m+1}\dfrac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right]+\dfrac{4A^2\epsilon(\xi+\eta)m^2\sqrt{\pi}\Gamma(4m)}{(8m+1)\Gamma\left(4m+\frac{1}{2}\right)}\right\}\dfrac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}}$$

$$-\frac{4\kappa\left\{\epsilon^2A^2(\xi+2\zeta)\left[\kappa A^2(\xi+\eta+4\zeta)+\alpha\right]+\kappa a\epsilon A^2(\xi+\eta)\right\}m^2\sqrt{\pi}\Gamma(4m)}{\left\{-8\,am^2\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right]+\dfrac{4A^2\epsilon\,(\xi+\eta)m^2\sqrt{\pi}\Gamma(4m)}{(8m+1)\Gamma\left(4m+\frac{1}{2}\right)}\right\}(8m+1)\Gamma\left(4m+\frac{1}{2}\right)}$$

$$+\frac{4m^2a\kappa\left[\kappa A^2\epsilon(\xi+\eta+4\zeta)+\alpha\epsilon-2a\kappa\right]\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}\Gamma(2\mathrm{m})}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right]}{\left\{-8am^2\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right]+\dfrac{4A^2\epsilon(\xi+\eta)m^2\sqrt{\pi}\Gamma(4m)}{(8m+1)\Gamma\left(4m+\frac{1}{2}\right)}\right\}}, \tag{66}$$

$$\frac{d\bar{x}(t)}{dt} = 2\,a\kappa(t), \tag{67}$$

$$\begin{aligned}\frac{d\kappa(t)}{dt} = 2\epsilon A^{4\,m}&\left[\begin{array}{c}\dfrac{16\,m\mu B^2 m^2\Gamma(2(2m+1)m)\Gamma\left(2m+\frac{1}{2}\right)}{[4(2m+1)m+1]\Gamma\left(2(2m+1)m+\frac{1}{2}\right)\Gamma(2m)}\\ -\dfrac{2\epsilon\sigma_1\kappa\,\mathrm{sech}^{4m}(x)}{4mA}\,{}_2F_1\left(\dfrac{1}{2},2m;2m+1;\,\mathrm{sech}^2(x)\right)\end{array}\right]\\ -\frac{16\epsilon\beta B^2 m^2\Gamma\left(2m+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(2m)}&\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}}{4m+1}\dfrac{\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right],\end{aligned} \tag{68}$$

$$\begin{aligned}\frac{dB(t)}{dt} = \frac{2B}{A}\left[\frac{dA(t)}{dt}-\epsilon A\beta\kappa^2\right]&+\frac{2\epsilon B\delta A^{4\,m}\Gamma(2(2m+1)m)\Gamma\left(2m+\frac{1}{2}\right)}{\Gamma(2m)\Gamma\left(2(2m+1)m+\frac{1}{2}\right)}\\ +\frac{8\epsilon\beta B^3 m^2\Gamma\left(2m+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(2m)}&\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right],\end{aligned} \tag{69}$$

$$B^2(t) = \frac{\left(\begin{array}{c}\dfrac{nbA^{2\,n}\sqrt{\pi}\Gamma(2(n+1)m)}{\Gamma\left(2(n+1)m+\frac{1}{2}\right)}-\epsilon\kappa(n+1)\dfrac{(\lambda+\sigma)A^{4m}\sqrt{\pi}\Gamma(2(2m+1)m)}{\Gamma\left(2(2m+1)m+\frac{1}{2}\right)}\\ -\epsilon\kappa(n+1)\left(\alpha+\kappa A^2(\xi+\eta+4\zeta)\right)\dfrac{\sqrt{\pi}\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\\ -\epsilon\kappa(n+1)\sigma_2 A^{4m}\dfrac{\mathrm{sech}^{4m}(x)}{4m}\,{}_2F_1\left(\dfrac{1}{2},2m;2m+1;\,\mathrm{sech}^2(x)\right)\dfrac{\sqrt{\pi}\Gamma(2m)}{\Gamma\left(2m+\frac{1}{2}\right)}\end{array}\right)}{(n+1)\left\{-8am^2\left[\begin{array}{c}\dfrac{4m\sqrt{\pi}\Gamma(2m)}{(4m+1)\Gamma\left(2m+\frac{1}{2}\right)}-\dfrac{2^{4m-1}(2m+1)\Gamma^2(2m)}{(4m+1)\Gamma(4m)}\\ -\dfrac{2^{2+2m}+1}{2+2m}\,{}_2F_1(2+2m,2+4m;3+2m;-1)\end{array}\right]+\dfrac{4\epsilon A^2(\xi+\eta)m^2\sqrt{\pi}\Gamma(4m)}{(8m+1)\Gamma\left(4m+\frac{1}{2}\right)}\right\}}. \tag{70}$$

## 4 Conclusions

Our study recovers the dynamical system of soliton parameters for super-sech and super-Gaussian pulses, as described in this paper. The details of the VP with the implementation of the Euler–Lagrange's equation to the NLSE with power-law of nonlinearity indicated in the current work have not been previously reported. These parameter variations, namely the dynamical system opens up with an avalanche of opportunities to study optical soliton sciences further along. This foundation stone of results pave way to further future investigations in this chapter. Later, the dynamical system would be revealed for additional forms of SPM that have not yet been considered. The studies would later be extended to birefringent fibers and DWDM topology. These would give an increased perspective to carry out the analysis further along. This would also be applicable to various additional devices and other forms of waveguides, including optical metamaterials, magneto-optic waveguides, optical couplers, gap solitons and many others. The results of these studies will be reported soon after we align them with the pre-existing ones [21–25]. All of these activities are currently underway.

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