Matter and Radiation at Extremes

RESEARCH ARTICLE

scitation.org/journal/mre

High-order corrections to the radiation-free dynamics of an electron in the strongly radiation-dominated regime

Cite as: Matter Radiat. Extremes 8, 014402 (2023); doi: 10.1063/5.0117504
Submitted: 1 August 2022 • Accepted: 15 November 2022 •
Published Online: 7 December 2022

View Online Export Citation CrossMark

A. S. Samsonov,[a] E. N. Nerush, and I. Yu. Kostyukov

AFFILIATIONS

Institute of Applied Physics of the Russian Academy of Sciences, 46 Ulyanov St., Nizhny Novgorod 603950, Russia

[a] **Author to whom correspondence should be addressed:** asams@ipfran.ru

ABSTRACT

A system of reduced equations is proposed for electron motion in the strongly radiation-dominated regime for an arbitrary electromagnetic field configuration. The approach developed here is used to analyze various scenarios of electron dynamics in this regime: motion in rotating electric and magnetic fields and longitudinal acceleration in a plane wave and in a plasma wakefield. The results obtained show that this approach is able to describe features of electron dynamics that are essential in certain scenarios, but cannot be captured in the framework of the original radiation-free approximation [Samsonov *et al.*, Phys. Rev. A **98**, 053858 (2018) and A. Gonoskov and M. Marklund, Phys. Plasmas **25**, 093109 (2018)]. The results are verified by numerical integration of the nonreduced equations of motion with account taken of radiation reaction in both semiclassical and fully quantum cases.

© *2022 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).* https://doi.org/10.1063/5.0117504

I. INTRODUCTION

With upcoming laser facilities such as ELI,[1] SULF,[2] SEL,[3] and XCELS,[4] investigation of laser–matter interactions in the regime of extreme laser intensity will become feasible. Radiation reaction is expected to accompany such interactions, although its direct impact is usually quite hard to predict. It has been known for more than a century that a charged particle experiences a recoil force when radiating, but a consistent model describing this phenomenon in both theoretical and numerical studies has yet to be firmly established. Recently conducted experiments aimed at determining the correct model of radiation friction are still subject to a certain level of ambiguity[5,6] and thus have not been able to solve this problem. The problem is becoming more and more acute with the growing number of studies discovering possible new effects caused by radiation reaction. It is clear that these effects vary greatly and include, among many others, alterations in particle acceleration mechanisms,[7–16] highly efficient laser pulse absorption,[17] relativistic transparency reduction,[18,19] the inverse Faraday effect,[20–22] particle polarization,[23–31] initiation of quantum electrodynamic (QED) cascades.[32–47] The signatures of these effects are expected to be most prominent in the so-called radiation-dominated regime, i.e., the regime in which radiative losses of charged particles are comparable to the energy gain in the electromagnetic (EM) field. Estimates show that the field amplitudes needed for realization of this regime can be achieved experimentally, either at future laser facilities such as ELI, SEL, and XCELS, or at future accelerators such as FACET-II.[48]

Via QED, one can describe radiation reaction self-consistently and calculate the probability of radiation of a photon with a given energy. While a full QED description currently provides the most accurate description of radiation reaction, it is not usually applicable to practical problems involving complex light–matter interactions, since via QED one calculates scattering probabilities between some stationary (commonly Volkov) electron states. To describe a dynamic problem where these states evolve owing to the evolution of the EM fields, nonstationary Dirac equations have to be solved, which is usually either unfeasible or impractical. However, under some conditions, this is not necessary, since the problem can be significantly simplified. The first main parameter that defines such a condition is the dimensionless EM field amplitude a_0:

© Author(s) 2023

$$a_0 = \frac{eE_0}{mc\omega}, \tag{1}$$

where m and $e > 0$ are respectively the electron mass and charge, and E_0 and ω are respectively the characteristic strength and frequency of the EM field. In the regime $a_0 \gg 1$, the characteristic radiation formation length λ_f can in most cases be estimated as $\lambda/a_0 \ll \lambda$, where $\lambda = 2\pi c/\omega$, i.e., individual acts of radiation occur almost instantaneously compared with the scale of variation of the EM fields, and thus these fields can be assumed constant on a radiation formation length. In this locally constant field approximation (LCFA[49–51]), the total radiation probabilities and the shapes of emission spectra depend only on the QED parameter χ,

$$\chi = \frac{\gamma}{E_S}\sqrt{(\mathbf{E} + \mathbf{v}\times\mathbf{B})^2 - (\mathbf{vE})^2}, \tag{2}$$

where γ and $\mathbf{v}$ are respectively the electron Lorentz factor and velocity normalized to c, $\mathbf{E}$ and $\mathbf{B}$ are respectively the electric and magnetic fields, $E_S = m^2c^3/e\hbar$ is the critical Sauter–Schwinger field,[50] and $\hbar$ is Planck's constant. These probabilities can be calculated analytically in either classical ($\chi \ll 1$) or quantum ($\chi \gg 1$) regimes:

$$W_{\rm rad} \approx \alpha\frac{mc^2}{\gamma\hbar} \times \begin{cases} 1.4\chi, & \chi \ll 1, \\ 0.7\chi^{2/3}, & \chi \gg 1, \end{cases} \tag{3}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant. Note that a number of different approaches have been proposed for calculation of radiation probabilities when the LCFA is no longer valid.[52–56] In LCFA, the characteristic distance traveled by an ultrarelativistic electron between two consecutive photon emissions, λ_W, can be estimated as $c/W_{\rm rad}$, which in both classical and quantum cases is at least $1/\alpha \approx 137$ times longer than the radiation formation length. Note, however, that the above-mentioned estimate for the radiation formation length actually depends on the frequency of the emitted radiation[57] and can be inaccurate for $\chi \gtrsim 10$. Furthermore, the ratio between the mean free path λ_W and the EM field wavelength can be estimated as

$$\frac{\lambda_W}{\lambda} \approx \frac{1}{\alpha a_0} \times \begin{cases} 1, & \chi \ll 1, \\ \chi^{1/3}, & \chi \gg 1. \end{cases} \tag{4}$$

So, as $\chi \lesssim 10$ for most experiments that are likely to be performed in the near future, and for $a_0 \gg 137$, the hierarchy of the characteristic scales of the problem is as follows:

$$\lambda_f \ll \lambda_W \ll \lambda, \tag{5}$$

which means that the electron moves classically between short but frequent acts of photon emission.

In that case, we can approximate the effect of radiation recoil as an additional continuous force acting on a particle, i.e., the equations of motion take the form

$$\frac{d\mathbf{p}}{dt} = -\mathbf{E} - \mathbf{v}\times\mathbf{B} - F_{\rm rr}\mathbf{v}, \tag{6}$$

$$\frac{d\gamma}{dt} = -\mathbf{vE} - F_{\rm rr}v^2, \tag{7}$$

where the electron momentum $\mathbf{p}$ is normalized to mc, the time t is normalized to $1/\omega$, and the electric and magnetic fields are normalized to $mc\omega/e$. In Eqs. (6) and (7), $F_{\rm rr}$ is the total radiation power normalized to $mc^2\omega$ and is given by

$$F_{\rm rr} = \frac{\alpha a_S}{3\sqrt{3}\pi}\int_0^\infty \frac{4u^3 + 5u^2 + 4u}{(1+u)^4} K_{2/3}\left(\frac{2u}{3\chi}\right) du, \tag{8}$$

where $a_S = eE_S/mc\omega \equiv mc^2/\hbar\omega$ is the normalized Sauter–Schwinger field. In the limiting cases, this expression simplifies to

$$F_{\rm rr} \approx \alpha a_S \times \begin{cases} 0.67\chi^2, & \chi \ll 1, \\ 0.37\chi^{2/3}, & \chi \gg 1. \end{cases} \tag{9}$$

This approach to the description of the electron dynamics with account of radiation reaction is commonly referred to as semiclassical.[58–62] In the quantum regime, radiated photons can carry away a significant portion of the electron energy, and as this radiation is stochastic, electrons contained in a small phase volume can significantly diverge in phase space after some time. Equations (6) and (7) essentially describe the zeroth moment, i.e., the trajectory of the center of mass of the electron distribution function, while effects caused by the probabilistic nature of radiation, such as straggling and quenching,[62–65] lead to diffusion of the distribution function and thus cannot be captured using this approach. In that case, a more accurate description requires equations for higher moments of the distribution function. Such an approach was applied to calculate the mean particle energy and energy spread in different fields configurations in Refs. 61, 66, and 67. If, on the contrary, $\chi \lesssim 1$, then the recoil from a single-photon radiation is small, and the approximation of continuous recoil is sufficient to describe the electron dynamics.

Another important consideration in studying the effect of radiation reaction is its dependence on the internal degree of freedom of the electron, i.e., spin. Strictly speaking, the quasiclassical limit of the Dirac equation leads to equations of motion where both the orbital motion of the electron and the evolution of its spin are coupled. In particular, one should add the Stern–Gerlach force[68] to the equation for the electron momentum and describe the spin dynamics via the Thomas–Bargmann–Michel–Telegdi (T-BMT)[69,70] equation. Note that although the latter is strictly valid only in homogeneous EM fields, it can still be used in heterogeneous fields if the Stern–Gerlach force can be neglected.[71] The ratio between the Lorentz force and Stern–Gerlach force can be estimated to be of the order of $\hbar\omega/mc^2$, and thus for optical frequencies the latter can be neglected with a large margin of accuracy. In that case, spin dynamics is decoupled from electron orbital motion and can be calculated after the electron trajectory has been found. Radiation reaction can again couple spin dynamics and electron orbital motion, since radiation probabilities depend on the spin of the electron (and the polarization of the emitted photon). Note that the order-of-magnitude estimates made above where radiation probabilities are averaged over the initial and summed over the final polarization states of the electron remain valid. However, in certain scenarios, the assumption that electrons are generally not polarized may no longer hold, since the radiation probabilities of spin-up and spin-down electrons are different. Resolving electron polarization

© Author(s) 2023

can lead to effects such as a significant increase in pair production during QED cascade development,[72] production of polarized high-energy particles,[28,73] and spatially inhomogeneous polarization.[31] In this paper, such effects caused by spin dynamics are not covered.

Previous studies have shown that the problem can be simplified even further. In particular, to some extent, radiation reaction can be accounted for implicitly, i.e., without specifying an expression for radiation reaction in the equations of motion.[74,75] This is done by noticing that in constant homogeneous EM fields, electron motion is stable if the electron does not experience transverse acceleration. As the radiation probability depends on the parameter χ, which is essentially proportional to the transverse acceleration, the direction of such motion is called the radiation-free direction (RFD). As this direction corresponds to vanishing of the transverse acceleration, and radiation recoil is directed against the electron velocity, to find the RFD, one does not need to specify any expression for the radiation probability at all. The timescale τ_v on which the electron velocity approaches the RFD in constant fields is of the order of $\gamma mc/eE_0$. If the EM fields are varying with characteristic frequency ω, then the RFD defined by the local and instantaneous field configuration changes on the same timescale. In the absence of radiation reaction, one can estimate that $\gamma \sim a_0$, and thus, by the time the electron velocity approaches the RFD, the latter itself changes, and so the geometric relation between the electron velocity and the RFD is arbitrary. This is not the case in the strongly radiation-dominated regime, however, when, by definition, $\gamma \ll a_0$, and thus the EM field orients the electron velocity much faster than the field itself changes, and so the electron velocity quickly aligns to the RFD defined by the local and instantaneous electric and magnetic fields. Thus, to approximately determine the electron trajectory, one can assume that at each time instant, the electron velocity coincides with the RFD. While this approach allows one to describe the dynamics of the electron in the strongly radiation-dominated regime without specifying an expression for the radiation power, it is quite limited for a couple of reasons. First, the electron velocity converges to the RFD sufficiently fast only at extremely large intensities $I \gtrsim 10^{25}$ W/cm^2. Second, this approach does not allow one to find the electron energy and radiation losses as the RFD is approached, since the particle energy is assumed to be indefinitely large, albeit much smaller than the field amplitude at the same time. Despite its apparent drawbacks, this approach has recently been applied successfully to describe electron motion in an astrophysical environment.[76] In this paper, we extend this radiation-free approach to overcome its inherent problems and to describe the dynamics of an electron in the strongly radiation-dominated regime more precisely.

The remainder of the paper is organized as follows. In Sec. II, we reintroduce the concept of radiation-free dynamics and extend it by application of perturbation theory. In Sec. III, we consider several EM field configurations where the reduced equations of motion obtained can be explicitly solved. In Sec. IV, we discuss the domain of applicability of the proposed approach and draw conclusions.

II. RADIATION-FREE APPROACH

Let us start by introducing a radiation-free approach to the description of electron dynamics, loosely following the original papers.[74,75] The equations governing electron dynamics in an EM field with account taken of radiation reaction can be written in terms of the electron velocity $\mathbf{v}$ and Lorentz factor γ:

$$\frac{d\gamma}{dt} = -\mathbf{v}\mathbf{E} - F_{\mathrm{rr}}v^2, \tag{10}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\gamma}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B} - \mathbf{v}(\mathbf{v}\mathbf{E}) + \frac{F_{\mathrm{rr}}\mathbf{v}}{\gamma^2}\right). \tag{11}$$

Since radiation reaction is sufficient to significantly alter dynamics only for ultrarelativistic particles ($\gamma \gg 1$), the last term in Eq. (11) can be omitted. There exists a formal stationary solution $\mathbf{v}_0$ of these equations, corresponding to vanishing of the transverse force acting on the electron and in turn vanishing of radiation reaction. Because of that property, this solution is called the radiation-free direction (RFD):

$$\mathbf{E} + \mathbf{v}_0\times\mathbf{B} - \mathbf{v}_0(\mathbf{v}_0\mathbf{E}) = 0. \tag{12}$$

Note, first, that there always exists a solution to this equation that can be calculated algebraically[74] or geometrically[75] and, second, that $|\mathbf{v}_0| = 1$, as can be shown by taking the scalar product of Eq. (12) with $\mathbf{v}_0$, which means that this solution is not entirely physical, i.e., an electron is unable to move in the EM fields without experiencing transverse acceleration. To understand how this solution relates to the actual solution of the equations of motion in the strongly radiation-dominated regime, let us consider the following. By definition, in the strongly radiation-dominated regime, the energy of an electron is significantly smaller than that of a hypothetical electron that is in the same EM fields but does not experience radiation reaction. One can roughly estimate that the characteristic energy of the latter electron is of the order of the dimensionless electric field amplitude E. Therefore, for a real electron in the radiation-dominated regime, we can assume that $\gamma \ll E$. Under this assumption, Eq. (11) then states that the EM fields orient the electron velocity much faster than they themselves change. So, on the timescale of velocity orientation, the EM fields can be assumed constant and homogeneous. In that case, the electron velocity tends asymptotically to the RFD. Neglecting the time taken for the electron velocity to approach the RFD, one can construct an asymptotic trajectory that in some sense serves as an attractor for real electron trajectories:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_0(\mathbf{E}(\mathbf{r},t), \mathbf{B}(\mathbf{r},t)). \tag{13}$$

As mentioned in Sec. I, although Eqs. (12) and (13) describe electron dynamics in the strongly radiation-dominated regime with radiation reaction being accounted for implicitly, there are two drawbacks to this approach. First, these asymptotic trajectories describe real particle trajectories well only at extremely large intensities $I \gtrsim 10^{25}$ W/cm^2. This is because, for most field configurations, the characteristic time at which the electron velocity approaches the RFD is underestimated by above reasoning. Second, this approach does not allow one to find the electron energy and radiation losses on approaching this asymptotic trajectory, since the particle energy is assumed to be indefinitely large (albeit much smaller than the field amplitude at the same time).

© Author(s) 2023

To deal with these problems, we develop a perturbation theory, assuming that the electron velocity deviates from the RFD but that this deviation is small, i.e.,

$$\mathbf{v} = \left(1 - \frac{\delta^2}{2}\right)\mathbf{v}_0 + \mathbf{v}_1, \tag{14}$$

where $\mathbf{v}_1 \perp \mathbf{v}_0$, and δ can be found from the condition that $|\mathbf{v}|^2 = 1 - \gamma^{-2}$, from which we get

$$\delta^2 \approx v_1^2 + \gamma^{-2}. \tag{15}$$

Substituting this into Eqs. (10) and (11), utilizing Eq. (12) and keeping only terms of first order in v_1 (see the expansion up to terms of second order in Appendix A), we obtain a set of general equations governing electron dynamics in the strongly radiation-dominated regime:

$$\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = \frac{\mathbf{F}_1}{\gamma} - \frac{\mathrm{d}\mathbf{v}_0}{\mathrm{d}t} - \mathbf{v}_0\left(\mathbf{v}_1\frac{\mathrm{d}\mathbf{v}_0}{\mathrm{d}t}\right), \tag{16}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\mathbf{v}_0\mathbf{E}\left(1 - \frac{v_1^2}{2} - \frac{1}{2\gamma^2}\right) - \mathbf{v}_1\mathbf{E} - F_{\mathrm{rr}}(\chi), \tag{17}$$

$$\mathbf{F}_1 = (\mathbf{v}_0\mathbf{E})\mathbf{v}_1 + (\mathbf{v}_0\mathbf{B})[\mathbf{v}_0\times\mathbf{v}_1] + \mathcal{O}(\delta^2), \tag{18}$$

$$\chi = \frac{\gamma|\mathbf{F}_1|}{a_S}. \tag{19}$$

Note that although χ is proportional to a small term v_1, it can be arbitrarily large owing to the factor γ, and therefore the term F_{rr} should be kept at all expansion orders, as a consequence of which the equations of motion remain nonlinear. This does not contradict the fact that the last term in Eq. (11) is smaller than the first three, which are equal to $\mathbf{F}_1$, and thus it can be omitted. Indeed, $|F_{\mathrm{rr}}\mathbf{v}/\gamma^2| \lesssim \alpha a_S\chi^2/\gamma^2 = \alpha|\mathbf{F}_1|^2/a_S \sim \alpha E v_1|\mathbf{F}_1|/a_S \lll |\mathbf{F}_1|$, since $E \ll a_S$, $\alpha \ll 1$, $v_1 \lesssim 1$. The full time derivatives should be considered as derivatives of the vector field $\mathbf{v}_0$ along the real electron trajectory $\mathbf{r}(t)$, i.e.,

$$\frac{\mathrm{d}\mathbf{v}_0}{\mathrm{d}t} = \frac{\partial\mathbf{v}_0}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}_0. \tag{20}$$

Let us consider the equation for the magnitude of the vector $\mathbf{v}_1$:

$$\frac{1}{2}\frac{\mathrm{d}v_1^2}{\mathrm{d}t} = -\mathbf{v}_1\frac{\mathrm{d}\mathbf{v}_0}{\mathrm{d}t} + \frac{v_1^2(\mathbf{v}_0\mathbf{E})}{\gamma}. \tag{21}$$

In a constant EM fields $(\mathrm{d}\mathbf{v}_0/\mathrm{d}t = 0)$, one can estimate the characteristic timescale on which the electron velocity approaches the RFD:

$$\tau_v = \frac{\gamma}{|\mathbf{v}_0\mathbf{E}|} \sim \frac{\gamma}{a_0}. \tag{22}$$

However, for varying EM fields, the sign of the first term in Eq. (21) can be arbitrary and its magnitude can be as large as v_1, and so the condition $\gamma \ll a_0$ alone is not enough to justify the description of the electron dynamics by Eq. (13) in an arbitrary field configuration. Instead, one should use the set of Eqs. (16) and (17), where the variation of the RFD is taken into account. Moreover, these equations allow one to find the electron energy and radiative losses.

The procedure to obtain the reduced equations of motion can be explained in few simple steps. First, it is shown that there exists a preferred RFD that the electron velocity approaches in constant EM fields. By decomposing the electron velocity in a new basis in which one axis coincides with the RFD, the equations of motion can be split. Motion along the RFD is essentially described via the particle energy, while the equations for the transverse velocity can be expanded in series, which clearly converge, since the magnitude of the velocity is strictly smaller than unity. Although the final set of equations remain nonlinear and cannot be solved explicitly in an arbitrary field configuration, the examples considered below show that this approach can be superior to solving the nonreduced Newton equations. It is worth noting that recently, in Ref. 77, a similar decomposition of the velocity vector was used to explore equilibrium solutions of Eqs. (16) and (17).

III. EXAMPLE PROBLEMS

Below, we consider several examples of field configurations in which Eqs. (16) and (17) can be solved explicitly.

A. Generalized Zeldovich problem

The equations of electron motion with a radiation reaction force can be integrated analytically for a rotating uniform electric field, as was first demonstrated by Zeldovich.[78] Recently Zeldovich's solution has been extended to a configuration with rotating electric and magnetic fields that are parallel to each other.[37] Let us analyze the latter configuration within our approach. We assume that the electric and magnetic fields are uniform and parallel, and rotate with velocity $\mathbf{\Omega}$. The RFD in this configuration is antiparallel to the electric field: $\mathbf{v}_0 = -\mathbf{E}/E = -\mathbf{e}$. Let us consider a stationary solution in which the deviation vector $\mathbf{v}_1$ rotates synchronously with the electric and magnetic fields. In this case, all the time derivatives can be replaced by the cross product $\mathbf{\Omega}\times$. Equation (16) then becomes

$$\mathbf{\Omega}\times\mathbf{v}_1 = -\frac{E}{\gamma}\mathbf{v}_1 + \frac{B}{\gamma}\mathbf{e}\times\mathbf{v}_1 + \mathbf{\Omega}\times\mathbf{e} + v_1\mathbf{e}. \tag{23}$$

Keeping in mind that $\mathbf{v}_1 \perp \mathbf{v}_0$, we can express $\mathbf{v}_1$ as follows:

$$\mathbf{v}_1 = v_\perp\mathbf{\Omega}\times\mathbf{e} + v_x\mathbf{\Omega}. \tag{24}$$

Equation (23) can then be split into a set of linear equations, the solution of which is easily found:

$$v_x = \frac{\gamma B}{E^2 + B^2}, \tag{25}$$

$$v_\perp = \frac{\gamma E}{E^2 + B^2}. \tag{26}$$

© Author(s) 2023

This stationary solution corresponds to constant radiative losses and constant energy. Consequently, the final relation between the electron energy and the EM fields can be obtained from Eq. (17):

$$E = F_{\rm rr}(\chi) = F_{\rm rr}\left(\frac{\gamma^2}{a_S}\right). \tag{27}$$

This result coincides exactly with the result obtained in Ref. 37 and, in the special case $B = 0$, with Zeldovich's original solution.[78] A comparison of the solution obtained here with numerical solutions of the nonreduced equations of motion (10) and (11) for both semiclassical and quantum approaches to radiation reaction is presented in Fig. 1. The magnitude of the EM fields has been chosen in such a way that the average value of the χ parameter of an electron is around 5. This has been done deliberately to show where the assumptions made in our approach start to break down. In particular, our analytical solution and the quasiclassical numerical solutions assume that the χ parameter does not deviate significantly between particles, and thus it is assumed that $\langle F_{\rm rr}(\chi)\rangle \approx F_{\rm rr}(\langle\chi\rangle)$, where the angular brackets denote averaging over the particle distribution function. However, as can be seen from Fig. 1, the parameters of individual electrons with the same initial conditions spread quite significantly, owing to the stochastic nature of radiation in the quantum regime, as mentioned in Sec. I. Because of this and the nonlinearity of $F_{\rm rr}$, in fact $\langle F_{\rm rr}(\chi)\rangle < F_{\rm rr}(\langle\chi\rangle)$, which explains the discrepancy in Fig. 1 between the values averaged over quantum solutions and those obtained analytically or via numerical solution in the quasiclassical approximation.

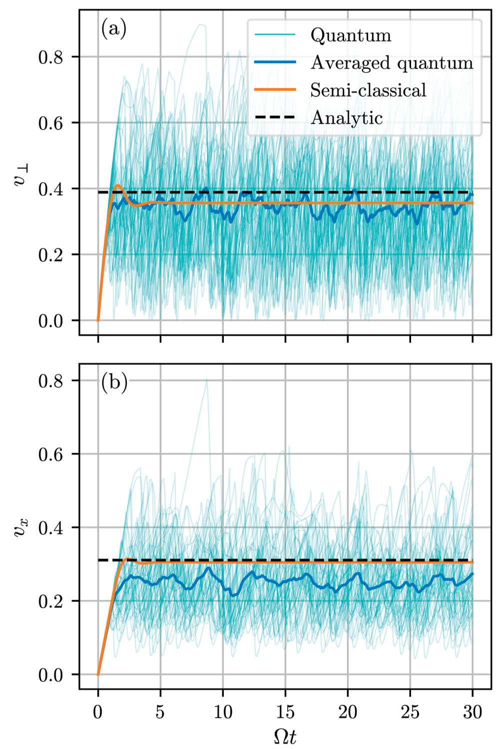

FIG. 1. Electron dynamics in an electric field with dimensionless amplitude $eE/mc\Omega = 2500$ and a parallel magnetic field with dimensionless amplitude $eB/mc\Omega = 2000$ rotating with angular frequency Ω, corresponding to the wavelength $\lambda = 1\ \mu$m: (a) component of electron velocity transverse to the electric field; (b) component of electron velocity along the angular velocity vector $\mathbf{\Omega}$. Orange and cyan lines correspond to numerical solution of the nonreduced equations of motion (10) and (11) with radiation reaction taken into account via semiclassical and quantum approaches, respectively. Blue lines correspond to the average value of 100 "quantum" solutions. Black dashed lines correspond to the analytical solution (25) and (26).

B. Monochromatic linearly polarized plane wave

Some interesting results can be obtained if our approach is applied to electron motion in a plane wave. In this configuration, the RFD coincides with the direction of the Poynting vector

$$\mathbf{v}_0 = \frac{\mathbf{E}\times\mathbf{B}}{E^2}, \tag{28}$$

where both $\mathbf{E}$ and $\mathbf{B}$ are functions of the phase $\varphi = x - t$. For simplicity, let us assume that initially the deviation vector $\mathbf{v}_1$ is parallel to the electric field, in which case it will remain so at any time instant. It is also more convenient to write down equations in terms of momentum $\mathbf{p}_1 = \gamma\mathbf{v}_1$ and phase φ. Since χ oscillates with constant amplitude in a plane wave without account of radiation reaction and radiation leads only to decrease χ, eventually any electron will reach the classical regime when $\chi \ll 1$, and so we will only consider electron dynamics in that regime. In this case, Eqs. (16) and (17) take the form

$$\frac{d\gamma}{d\varphi} = -2\frac{p\gamma E}{1+p^2} - 2A_{\rm rr}E^2(1+p^2), \tag{29}$$

$$\frac{dp}{d\varphi} = -E - 2A_{\rm rr}E^2\frac{p}{\gamma}(1+p^2), \tag{30}$$

where $A_{\rm rr} = \alpha/6E_S$. Note that in a plane wave configuration, $\mathbf{F}_1$ as defined in Eq. (18) contains only terms of second and higher orders of smallness in v_1, the derivation of which can be found in Appendix A. Let us start solving Eqs. (29) and (30) without account of radiation reaction for a linearly polarized monochromatic plane wave, i.e., assuming $A_{\rm rr} = 0$ and $E = a_0\cos\varphi$. In this case, it is easy to obtain the following solution:

$$p_{\rm pw} = -a_0 \sin\varphi, \tag{31}$$

$$\gamma_{\rm pw} = \gamma_0(1 + p_{\rm pw}{}^2), \tag{32}$$

where it has been assumed that initially the electron momentum has only a component along the direction of plane wave propagation. Note that the exact solution of the electron equations of motion in a linearly polarized plane wave (see, e.g., Ref. 79) coincides exactly with the solution given by Eqs. (31) and (32) in the limit $\gamma_0 \gg 1$. Thus, our method, which is essentially a series expansion, can be applied even for certain problems where radiation reaction is not taken into account.

When $A_{\rm rr} \neq 0$, it can be shown from Eqs. (29) and (30) that there is an asymmetry in the particle motion in the accelerating and decelerating phases. This leads to a nonzero energy gain in a

© Author(s) 2023

single period. Under several additional assumptions, we can obtain the following solution (see the detailed step-by-step solution in Appendix B):

$$\gamma \approx \gamma_0\left(1 + a_0^2 \sin^2\varphi\right)\left(1 + \frac{A_{\rm rr}a_0^2}{\gamma_0}\varphi\right), \quad (33)$$

$$v \approx -\frac{a_0 \sin\varphi}{\gamma}. \quad (34)$$

Rewriting the solution in terms of laboratory time t yields the following result

$$\langle\chi\rangle,\ \langle v_1\rangle,\ \langle\gamma^{-1}\rangle \propto (A_{\rm rr}t)^{-1/3}. \quad (35)$$

Note that the average behavior of the dependences in Eq. (35) asymptotically coincides with that extracted from the exact solution derived in Ref. 80.

In obtaining the above solution, we have assumed that at the initial moment, the transverse momentum of the particle is equal to zero and the longitudinal momentum is sufficiently large and positive. To obtain a solution with arbitrary initial conditions, one can perform a Lorentz boost to an auxiliary reference frame where the above assumptions are satisfied and then transform the solution obtained there back to an initial reference frame. The solution shown in Fig. 2 was obtained in such a way, with the auxiliary reference frame taken to be moving with a velocity corresponding to a Lorentz factor 1000 along the negative x axis in the laboratory reference frame, and so the initial longitudinal electron momentum in the auxiliary reference frame was approximately $5mc$.

The obtained solution is not only nonperiodic, but also features quite unexpected behavior: instead of slowing down the electron, radiation reaction actually allows it to gain infinite energy (in infinite time, obviously). Although this behavior has been reported before[78,80–84] and has been confirmed by numerical solution of the nonreduced equations of motion (10) and (11) (see Fig. 2), it does not appear to be widely acknowledged. Simple reasoning can explain this seemingly controversial phenomenon. For this, it is more convenient to resort to a quantum description of radiation reaction. In a relativistically strong plane wave ($E \gg 1$), the formation length of the radiation can be estimated as $\lambda/E \ll \lambda$, which can be interpreted as indicating that an electron moves classically between short acts of photon emission. Without radiation reaction, the light-front momentum $\gamma - p_x$ is a constant of motion, where p_x is the electron momentum along the direction of plane wave propagation [see the red line in Fig. 2(b)]. The radiation probability depends on the QED parameter χ, which in the plane wave configuration is given by

$$\chi = \frac{E(\varphi)}{E_S}(\gamma - p_x). \quad (36)$$

As the radiation formation length is much smaller than the wavelength, the EM fields can be assumed constant during a single act of photon emission, and it follows from energy and momentum conservation that the parameter χ of the electron strictly decreases after this act of emission [see the distinct jumps corresponding to emission of individual photons in the blue line in Fig. 2(b)]. We can therefore conclude that owing to radiation reaction, the light-front momentum $\gamma - p_x$ tends asymptotically to zero, which can be satisfied only when p_x (and correspondingly γ) grows indefinitely.

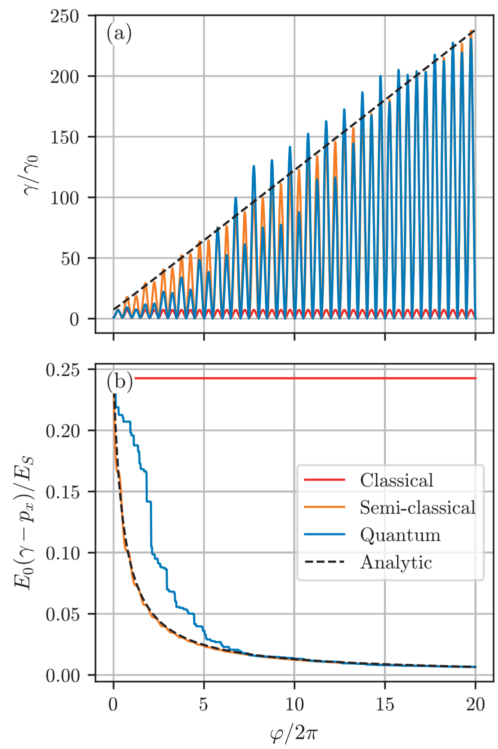

FIG. 2. Dynamics of an electron with initial momentum $p_x = -100\ mc$ in a plane wave with amplitude $a_0 = 500$ and wavelength $\lambda = 1\ \mu$m propagating along the x axis: (a) energy of electron normalized to its initial value; (b) maximum value of QED parameter χ: $a_0(\gamma - p_x)/E_S$. Red lines correspond to the classical solution without radiation reaction. Orange and blue lines correspond to numerical solution of the nonreduced equations of motion (10) and (11) with radiation reaction taken into account via semiclassical and quantum approaches, respectively. Black dashed lines correspond to the analytical solution (33) and (34). Note that for visual clarity, the black dashed line only depicts the amplitude of the oscillations of γ in (a).

C. Plasma accelerator

Finally, let us consider a toy model of a plasma accelerator and derive conditions for a known stable solution in a radiation-dominated regime[9,16] using our approach. For this, we assume that the EM fields are a uniform accelerating field $\mathbf{z}_0 E_{\rm acc}$ and a linear focusing field $\mathbf{y}E_{\rm foc}$ in which the electron undergoes betatron oscillations. To find a solution that corresponds to the averaged radiation losses being constant over many betatron periods, we will assume that any function of the QED parameter χ is a strictly periodic function of time, and thus the average of any function of χ is also constant, i.e.,

$$\frac{{\rm d}\langle\chi^2\rangle}{{\rm d}t} = 0, \quad (37)$$

© Author(s) 2023

where χ^2 is used for convenience. In the field configuration under consideration, from Eq. (19), we get

$$\chi = \frac{\gamma E v_1}{a_S}. \tag{38}$$

Hereinafter, we will assume that $\gamma \approx \langle \gamma \rangle$, i.e., oscillations of the particle energy are much smaller than the energy itself, and so we can extract γ from the angular brackets. Also, as the particle is accelerated, γ grows with time, while χ remains constant on average. This means that the oscillation amplitude of v_1 decreases, and therefore at later times we can safely assume that the electric field experienced by the electron is mostly accelerating, so $E \approx E_{acc} = \text{const}$. Both of these assumptions are reliably confirmed by numerical simulations and remain valid in the final solution that we obtain below. Utilizing these assumptions and expanding Eq. (37) yields the following:

$$\langle F_{rr} \rangle \langle v_1^2 \rangle + \gamma \left\langle \mathbf{v}_1 \frac{d\mathbf{v}_0}{dt} \right\rangle = 0. \tag{39}$$

Differentiating this expression again, we obtain the following result:

$$\frac{\langle F_{rr} \rangle}{\gamma} \left(3\langle F_{rr} \rangle - 2E_{acc}\right) \langle v_1^2 \rangle + \gamma \left\langle \mathbf{v}_1 \frac{d^2\mathbf{v}_0}{dt^2} - \left(\frac{d\mathbf{v}_0}{dt}\right)^2 \right\rangle = 0. \tag{40}$$

To deal with the last two terms in this expression, let us write the equation for the electron trajectory:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_0 + \mathbf{v}_1, \tag{41}$$

where

$$\mathbf{v}_0 = -\frac{-\mathbf{z}_0 E_{acc} + \mathbf{y} E_{foc}}{E} \approx \mathbf{z}_0 - \mathbf{y}\frac{E_{foc}}{E_{acc}} \equiv \mathbf{z}_0 - \kappa \mathbf{y}. \tag{42}$$

If we assume that the betatron oscillations are harmonic, i.e.,

$$y = y_0 \cos \omega t, \tag{43}$$

then from the y component of Eq. (41) we get

$$v_{1,y} = y_0(\kappa \cos \omega t - \omega \sin \omega t). \tag{44}$$

To calculate the average of the last two terms in Eq. (40), we note that $d\mathbf{v}_0/dt = -\kappa d\mathbf{y}/dt$:

$$\left\langle \mathbf{v}_1 \frac{d^2\mathbf{v}_0}{dt^2} - \left(\frac{d\mathbf{v}_0}{dt}\right)^2 \right\rangle = y_0^2 \omega^2 \kappa \left(\kappa \langle \cos^2 \omega t - \sin^2 \omega t \rangle - \omega \langle \sin \omega t \cos \omega t \rangle \right) = 0. \tag{45}$$

So, finally, in a model accelerator, we get

$$\langle F_{rr} \rangle = \frac{2}{3} E_{acc}. \tag{46}$$

Thus, on average, the particle is accelerated only at one-third of the classical rate of acceleration.[9,16] Figure 3 shows that the solution obtained here coincides quite well with the numerical solution of the nonreduced equations of motion (10) and (11). It should be noted that we have not used a specific expression for the power of radiative losses, although a more rigorous derivation of the relation (46) shows that the result actually depends on the scaling law of the radiation power with the parameter χ: for example, according to Ref. 16, in the fully quantum regime when $\chi \gg 1$ and $F_{rr} \propto \chi^{2/3}$, the relation (46) should in fact be slightly different, specifically,

$$\langle F_{rr} \rangle = \frac{12}{19} E_{acc}, \tag{47}$$

although this is only 5% different from (46). However, reasoning using our approach cannot exactly reproduce this minor difference, owing to the approximations used throughout, in particular the neglect of terms proportional to $1/\gamma^2$ when calculating χ.

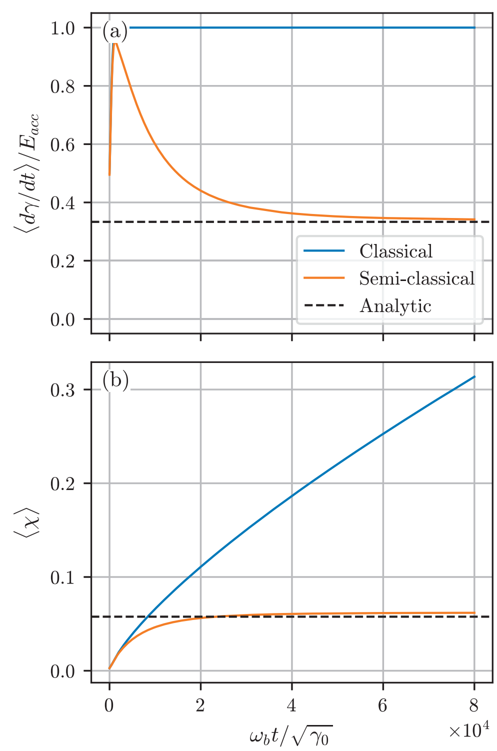

FIG. 3. Electron dynamics in a model accelerator with E_{acc} = 30 TV/m and E_{foc} growing linearly from 0 to 30 TV/m at displacement 0.1 μm: (a) average rate of acceleration; (b) average value of QED parameter χ. Time is normalized to the initial value of the inverse betatron frequency $\omega_b/\sqrt{\gamma_0}$ of the electron. Blue lines correspond to the solution of the nonreduced equations of motion (10) and (11) without account of radiation reaction Orange lines correspond to the solution with radiation reaction taken into account via a semiclassical approach. The black dashed line corresponds to the analytical solution (46).

IV. DISCUSSION AND CONCLUSION

In this paper, we have developed an approach to tackle the problem of single-electron dynamics in arbitrary EM fields in the

© Author(s) 2023

strongly radiation-dominated regime. We have shown that the electron velocity approaches a certain direction, moving along which an electron does not experience transverse acceleration and thus does not radiate. If we assume that the electron velocity deviates slightly from this radiation-free direction (RFD), then the equations of motion can be simplified. In certain EM field configurations, this simplification is enough to allow an analytical solution of the electron equations of motion to be obtained. Remarkably, in a plane wave example, the solution obtained is valid even if radiation reaction is not taken into account. This shows that the domain of applicability of our method is wider than was initially expected. This can be partially attributed to the fact that our approach is based on an expansion of the equations of motion in terms of the electron velocity. Since the magnitude of this velocity vector is smaller than unity, series expansions in terms of velocity should converge. The rates at which these series converge depend on how close the zeroth order is to the real value. We have shown that in the strongly radiation-dominated regime, the RFD can be chosen as a zeroth-order approximation of the direction of the electron velocity. However, a plane wave example shows that the same expansion can be valid even without account of radiation reaction in certain field configurations.

It should be noted that the approach developed here is valid when the continuous radiation recoil approximation is justified. The validity of this approximation is mostly determined by the value of the QED parameter χ. In particular, in a sufficiently quantum regime, when $\chi \gg 1$, electron dynamics can become stochastic, and thus the electron distribution function can evolve in a complex way. In that case, equations for higher moments of the distribution function can provide a more accurate description, but this lies outside the scope of this paper.

In conclusion, we have proposed a general approach for theoretical investigation of single-particle dynamics in the strongly radiation-dominated regime. Most importantly, the method developed here allows one to obtain qualitatively new results compared with the radiation-free approach originally developed in Refs. 74 and 75. We have demonstrated the applicability of our method in different EM field configurations. In particular, we have reproduced the generalized Zeldovich solution in rotating parallel electric and magnetic fields,[37,78] the damping of the average rate of electron acceleration in a model plasma accelerator in the radiation-dominated regime,[9,16] and a peculiar feature of the electron motion in a strong plane wave, namely, unlimited longitudinal acceleration.[80–84] Utilizing this approach to explore plasma behavior in a radiation-dominated regime is planned for future work.

ACKNOWLEDGMENTS

This work has been supported by the Ministry of Science and Higher Education of the Russian Federation (Agreement No. 075-15-2020-906, Center of Excellence "Center of Photonics"). We are grateful to Anton Golovanov for fruitful discussions.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

A. S. Samsonov: Conceptualization (equal); Formal analysis (lead). **E. N. Nerush**: Conceptualization (equal). **I. Yu. Kostyukov**: Conceptualization (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: DERIVING REDUCED EQUATIONS OF MOTION UP TO TERMS OF THE SECOND ORDER OF SMALLNESS

To obtain the reduced equations of motion, we consider the following representation of the electron velocity:

$$\mathbf{v} = \mathbf{v}_0\left(1 - \frac{v_1^2}{2} - \frac{1}{2\gamma^2}\right) + \mathbf{v}_1. \tag{A1}$$

Let us substitute this into Eq. (10) and expand it, keeping only terms up to second order (v_1^2) and larger:

$$\begin{aligned}\left(1 - \frac{v_1^2}{2} - \frac{1}{2\gamma^2}\right)\frac{d\mathbf{v}_0}{dt} - \frac{\mathbf{v}_0}{2}\left(\frac{dv_1^2}{dt} + \frac{d\gamma^{-2}}{dt}\right) + \frac{d\mathbf{v}_1}{dt}\\ = -\frac{1}{\gamma}\Bigg\{-\left(\frac{v_1^2}{2} + \frac{1}{2\gamma^2}\right)\mathbf{v}_0 \times \mathbf{B} + \mathbf{v}_1 \times \mathbf{B}\\ + \left(v_1^2 + \frac{1}{\gamma^2}\right)\mathbf{v}_0(\mathbf{v}_0\mathbf{E}) - \mathbf{v}_0(\mathbf{v}_1\mathbf{E}) - \mathbf{v}_1(\mathbf{v}_0\mathbf{E}) - \mathbf{v}_1(\mathbf{v}_1\mathbf{E})\Bigg\}.\end{aligned} \tag{A2}$$

To eliminate the term dv_1^2/dt from this equation, we take the scalar product of the equation with $\mathbf{v}_1$, keeping in mind that $\mathbf{v}_1\mathbf{v}_0 = 0$ and neglecting higher-order terms:

$$\frac{1}{2}\frac{dv_1^2}{dt} = -\mathbf{v}_1\frac{d\mathbf{v}_0}{dt} + \frac{v_1^2(\mathbf{v}_0\mathbf{E})}{\gamma}. \tag{A3}$$

We expand the term $d\gamma^{-2}/dt$:

$$\frac{d\gamma^{-2}}{dt} = -\frac{2}{\gamma^3}\frac{d\gamma}{dt} \approx \frac{2\mathbf{v}_0\mathbf{E}}{\gamma^3}. \tag{A4}$$

Substituting Eqs. (A3) and (A4) back into Eq. (A2), we get

$$\frac{d\mathbf{v}_1}{dt} = -\frac{\mathbf{F}_1}{\gamma} - \left(1 - \frac{v_1^2}{2} - \frac{1}{2\gamma^2}\right)\frac{d\mathbf{v}_0}{dt} - \mathbf{v}_0\left(\mathbf{v}_1\frac{d\mathbf{v}_0}{dt}\right), \tag{A5}$$

$$\begin{aligned}\mathbf{F}_1 = -\left(\frac{v_1^2}{2} + \frac{1}{2\gamma^2}\right)\mathbf{v}_0 \times \mathbf{B} + \mathbf{v}_1 \times \mathbf{B}\\ - \frac{\mathbf{v}_0(\mathbf{v}_0\mathbf{E})}{\gamma^2} - \mathbf{v}_0(\mathbf{v}_1\mathbf{E}) - \mathbf{v}_1(\mathbf{v}_0\mathbf{E}) - \mathbf{v}_1(\mathbf{v}_1\mathbf{E}).\end{aligned} \tag{A6}$$

© Author(s) 2023

Let us separately examine the vector $\mathbf{v}_1 \times \mathbf{B}$:

$$\mathbf{v}_1 \times \mathbf{B} = -(\mathbf{v}_0\mathbf{B})[\mathbf{v}_0 \times \mathbf{v}_1] + \mathbf{v}_0(\mathbf{v}_0[\mathbf{v}_1 \times \mathbf{B}]). \tag{A7}$$

We now take the scalar product of Eq. (12) with $\mathbf{v}_1$:

$$\mathbf{v}_1\mathbf{E} + \mathbf{v}_1[\mathbf{v}_0 \times \mathbf{B}] = 0. \tag{A8}$$

Performing a cyclic permutation of the scalar triple product, we get

$$\mathbf{v}_0[\mathbf{v}_1 \times \mathbf{B}] = \mathbf{v}_1\mathbf{E}. \tag{A9}$$

Substituting this relation into Eq. (A7), we obtain

$$\mathbf{v}_1 \times \mathbf{B} = -(\mathbf{v}_0\mathbf{B})[\mathbf{v}_0 \times \mathbf{v}_1] + \mathbf{v}_0(\mathbf{v}_1\mathbf{E}). \tag{A10}$$

Finally, substituting this into Eq. (A6), we get

$$\mathbf{F}_1 = -(\mathbf{v}_0\mathbf{B})[\mathbf{v}_0 \times \mathbf{v}_1] - (\mathbf{v}_0\mathbf{E})\mathbf{v}_1 - \left(\frac{v_1^2}{2} + \frac{1}{2\gamma^2}\right)\mathbf{v}_0 \times \mathbf{B} + \frac{\mathbf{v}_0(\mathbf{v}_0\mathbf{E})}{\gamma^2} - (\mathbf{v}_1\mathbf{E})\mathbf{v}_1. \tag{A11}$$

APPENDIX B: APPROXIMATE ELECTRON MOTION IN A PLANE MONOCHROMATIC WAVE

Let us consider the following plane wave configuration: $\mathbf{E} = E(\varphi)\mathbf{y}_0$, $\mathbf{B} = E(\varphi)\mathbf{z}_0$, $\mathbf{v}_0 = \mathbf{E} \times \mathbf{B}/E^2$, $\varphi = t - x$. For the sake of simplicity, we take the initial electron velocity to lie in the xy plane, in which case the y component of the electron velocity will always remain zero. The reduced equations of motion (16) and (17) in the considered configuration are as follows:

$$\frac{dv}{dt} = \frac{E}{2\gamma}\left(v^2 - \frac{1}{\gamma^2}\right), \tag{B1}$$

$$\frac{d\gamma}{dt} = -vE - F_{rr}(\chi), \tag{B2}$$

$$\frac{d\varphi}{dt} = \frac{1}{2}\left(v^2 + \frac{1}{\gamma^2}\right), \tag{B3}$$

where $v = v_1$. Changing the integration variable from t to φ, we get

$$\frac{dv}{d\varphi} = \frac{E}{2\gamma}\left(v^2 - \frac{1}{\gamma^2}\right)\frac{2}{v^2 + 1/\gamma^2} \tag{B4}$$

$$\frac{d\gamma}{d\varphi} = -[vE + F_{rr}(\chi)]\frac{2}{v^2 + 1/\gamma^2}. \tag{B5}$$

The expression for the QED parameter χ is as follows:

$$\chi = \frac{\gamma E}{E_S}\frac{1}{2}\left(v^2 + \frac{1}{\gamma^2}\right). \tag{B6}$$

As χ decreases owing to radiation, any electron will eventually reach the classical regime when $\chi \ll 1$. In that case,

$$F_{rr}(\chi) = \frac{2}{3}\alpha E_S\chi^2 = A_{rr}\gamma^2E^2\left(v^2 + \frac{1}{\gamma^2}\right)^2. \tag{B7}$$

Equations (B4) and (B5) can then be rewritten as

$$\frac{dv}{d\varphi} = \frac{E}{\gamma}\frac{(v\gamma)^2 - 1}{(v\gamma)^2 + 1}, \tag{B8}$$

$$\frac{d\gamma}{d\varphi} = -2\frac{(v\gamma)\gamma E}{(v\gamma)^2 + 1} - 2A_{rr}E^2[(v\gamma)^2 + 1]. \tag{B9}$$

Introducing the momentum $p = v\gamma$, we obtain the final set of equations:

$$\frac{d\gamma}{d\varphi} = -2\frac{p\gamma E}{1 + p^2} - 2A_{rr}E^2(1 + p^2), \tag{B10}$$

$$\frac{dp}{d\varphi} = -E - 2A_{rr}E^2\frac{p}{\gamma}(1 + p^2). \tag{B11}$$

Let us start by solving Eqs. (B10) and (B11) without account of radiation reaction for a linearly polarized monochromatic plane wave, i.e., assuming $A_{rr} = 0$ and $E = a_0 \cos\varphi$:

$$\frac{d\gamma_{pw}}{d\varphi} = -2\frac{p_{pw}\gamma_{pw}a_0 \cos\varphi}{1 + p_{pw}^2}, \tag{B12}$$

$$\frac{dp_{pw}}{d\varphi} = -a_0 \cos\varphi. \tag{B13}$$

Equation (B11) has the solution

$$p_{pw} = -a_0 \sin\varphi. \tag{B14}$$

Substituting this into Eq. (B10), we get

$$\frac{d\gamma_{pw}}{d\varphi} = \gamma_{pw}\frac{2a_0^2 \sin\varphi \cos\varphi}{1 + a_0^2 \sin^2\varphi}, \tag{B15}$$

$$\gamma_{pw} = \gamma_0(1 + a_0^2 \sin^2\varphi) = \gamma_0(1 + p_{pw}^2). \tag{B16}$$

To find the solution when $A \neq 0$, let us assume

$$p = p_{pw} + u, \tag{(B17)}$$

$$\gamma = \gamma_{pw}\Gamma, \tag{(B18)}$$

where $u \ll p_{pw}$. Equations (B10) and (B11) transform to

$$\frac{d\Gamma}{d\varphi} = -2\frac{A_{rr}a_0^2\cos^2\varphi}{\gamma_0} - 2u\Gamma a_0 \cos\varphi\frac{1 - a_0^2\cos^2\varphi}{(1 + a_0^2\cos^2\varphi)^2}, \tag{B19}$$

© Author(s) 2023

$$\frac{du}{d\varphi} = 2A_{rr}a_0^3\frac{\cos^2\varphi\,\sin\varphi}{\gamma_0\Gamma}. \tag{B20}$$

Assuming that Γ changes only slightly during a single period, it can be factored outside the integration, i.e.,

$$u \approx \frac{2A_{rr}a_0^3}{\gamma_0\Gamma}\int_0^{\varphi}\cos^2\varphi'\,\sin\varphi'\,d\varphi' = \frac{2A_{rr}a_0^3}{3\gamma_0\Gamma}(1-\cos^3\varphi). \tag{B21}$$

Substituting this into Eq. (B19) gives

$$\frac{d\Gamma}{d\varphi} = -2\frac{A_{rr}a_0^2\cos^2\varphi}{\gamma_0} - \frac{4A_{rr}a_0^4}{3\gamma_0}\frac{\cos\varphi(1-\cos^3\varphi)(1-a_0^2\sin^2\varphi)}{(1+a_0^2\sin^2\varphi)^2}. \tag{B22}$$

Since we assume that Γ changes on a timescale much longer than a single wave period, we can replace the right-hand side of this equation by its average value, i.e.,

$$\frac{d\Gamma}{d\varphi} \approx \frac{1}{2\pi}\int_0^{2\pi}\left[-2\frac{A_{rr}a_0^2\cos^2\varphi}{\gamma_0} - \frac{4A_{rr}a_0^4}{3\gamma_0}\frac{\cos\varphi(1-\cos^3\varphi)(1-a_0^2\sin^2\varphi)}{(1+a_0^2\sin^2\varphi)^2}\right]d\varphi = \frac{A_{rr}}{\gamma_0}\left(a_0^2+4-4\sqrt{1+a_0^2}\right) \overset{a_0\gg1}{\approx} \frac{A_{rr}a_0^2}{\gamma_0}. \tag{B23}$$

Thus,

$$\Gamma \approx 1 + \frac{A_{rr}a_0^2}{\gamma_0}\varphi. \tag{B24}$$

The final solution is then

$$\gamma \approx \gamma_0(1+a_0^2\sin^2\varphi)\left(1+\frac{A_{rr}a_0^2}{\gamma_0}\varphi\right), \tag{B25}$$

$$v \approx -\frac{a_0\sin\varphi}{\gamma_0(1+a_0^2\sin^2\varphi)(1+A_{rr}a_0^2\gamma_0\varphi)}. \tag{B26}$$

To express the solution in terms of the laboratory time instead of the wave phase, we solve the following equation:

$$\frac{d\varphi}{dt} = \frac{1}{2}\left(v^2+\frac{1}{\gamma^2}\right) = \frac{1}{2\gamma_0^2(1+a_0^2\sin^2\varphi)\left(1+\frac{A_{rr}a_0^2}{\gamma_0}\varphi\right)^2}, \tag{B27}$$

$$t = 2\gamma_0^2\int(1+a_0^2\sin^2\varphi)\left(1+\frac{A_{rr}a_0^2}{\gamma_0}\varphi\right)^2 d\varphi. \tag{B28}$$

Averaging over a wave period, we get the approximate relation

$$\langle t\rangle \approx a_0^6A_{rr}^2\varphi^3. \tag{B29}$$

REFERENCES

[1] See http://www.eli-laser.eu. for the extreme light infrastructure (ELI) official website.

[2] Z. Gan, L. Yu, C. Wang, Y. Liu, Y. Xu, W. Li, S. Li, L. Yu, X. Wang, X. Liu *et al.*, "The Shanghai Superintense Ultrafast Laser Facility (SULF) project," in *Progress in Ultrafast Intense Laser Science XVI* (Springer, 2021), pp. 199–217.

[3] B. Shao, Y. Li, Y. Peng, P. Wang, J. Qian, Y. Leng, and R. Li, "Broad-bandwidth high-temporal-contrast carrier-envelope-phase-stabilized laser seed for 100 PW lasers," Opt. Lett. **45**, 2215–2218 (2020).

[4] See http://www.xcels.iapras.ru for XCELS.

[5] J. M. Cole, K. T. Behm, E. Gerstmayr, T. G. Blackburn, J. C. Wood, C. D. Baird, M. J. Duff, C. Harvey, A. Ilderton, A. S. Joglekar, K. Krushelnick, S. Kuschel, M. Marklund, P. McKenna, C. D. Murphy, K. Poder, C. P. Ridgers, G. M. Samarin, G. Sarri, D. R. Symes, A. G. R. Thomas, J. Warwick, M. Zepf, Z. Najmudin, and S. P. D. Mangles, "Experimental evidence of radiation reaction in the collision of a high-intensity laser pulse with a laser-wakefield accelerated electron beam," Phys. Rev. X **8**, 011020 (2018).

[6] K. Poder, M. Tamburini, G. Sarri, A. Di Piazza, S. Kuschel, C. D. Baird, K. Behm, S. Bohlen, J. M. Cole, D. J. Corvan, M. Duff, E. Gerstmayr, C. H. Keitel, K. Krushelnick, S. P. D. Mangles, P. McKenna, C. D. Murphy, Z. Najmudin, C. P. Ridgers, G. M. Samarin, D. R. Symes, A. G. R. Thomas, J. Warwick, and M. Zepf, "Experimental signatures of the quantum nature of radiation reaction in the field of an ultraintense laser," Phys. Rev. X **8**, 031004 (2018).

[7] M. Tamburini, F. Pegoraro, A. Di Piazza, C. H. Keitel, and A. Macchi, "Radiation reaction effects on radiation pressure acceleration," New J. Phys. **12**, 123005 (2010).

[8] M. Tamburini, T. V. Liseykina, F. Pegoraro, and A. Macchi, "Radiation-pressure-dominant acceleration: Polarization and radiation reaction effects and energy increase in three-dimensional simulations," Phys. Rev. E **85**, 016407 (2012).

[9] I. Y. Kostyukov, E. N. Nerush, and A. G. Litvak, "Radiative damping in plasma-based accelerators," Phys. Rev. Spec. Top.–Accel. Beams **15**, 111001 (2012).

[10] R. Capdessus, E. d'Humières, and V. T. Tikhonchuk, "Modeling of radiation losses in ultrahigh power laser-matter interaction," Phys. Rev. E **86**, 036401 (2012).

[11] R. Capdessus and P. McKenna, "Influence of radiation reaction force on ultraintense laser-driven ion acceleration," Phys. Rev. E **91**, 053105 (2015).

[12] E. N. Nerush and I. Y. Kostyukov, "Laser-driven hole boring and gamma-ray emission in high-density plasmas," Plasma Phys. Controlled Fusion **57**, 035007 (2015).

[13] E. G. Gelfer, A. M. Fedotov, and S. Weber, "Theory and simulations of radiation friction induced enhancement of laser-driven longitudinal fields," Plasma Phys. Controlled Fusion **60**, 064005 (2018).

[14] E. Gelfer, N. Elkina, and A. Fedotov, "Unexpected impact of radiation friction: Enhancing production of longitudinal plasma waves," Sci. Rep. **8**, 6478 (2018).

© Author(s) 2023

[15]E. G. Gelfer, A. M. Fedotov, and S. Weber, "Radiation induced acceleration of ions in a laser irradiated transparent foil," New J. Phys. **23**, 095002 (2021).
[16]A. A. Golovanov, E. N. Nerush, and I. Y. Kostyukov, "Radiation reaction-dominated regime of wakefield acceleration," New J. Phys. **24**, 033011 (2022).
[17]T. Grismayer, M. Vranic, J. L. Martins, R. A. Fonseca, and L. O. Silva, "Laser absorption via quantum electrodynamics cascades in counter propagating laser pulses," Phys. Plasmas **23**, 056706 (2016).
[18]P. Zhang, C. P. Ridgers, and A. G. R. Thomas, "The effect of nonlinear quantum electrodynamics on relativistic transparency and laser absorption in ultra-relativistic plasmas," New J. Phys. **17**, 043051 (2015).
[19]M. Serebryakov, A. Samsonov, E. Nerush, and I. Y. Kostyukov, "Opacity of relativistically underdense plasmas for extremely intense laser pulses," arXiv:2210.01606 (2022).
[20]T. V. Liseykina, S. V. Popruzhenko, and A. Macchi, "Inverse Faraday effect driven by radiation friction," New J. Phys. **18**, 072001 (2016).
[21]T. V. Liseykina, A. Macchi, and S. V. Popruzhenko, "Quantum effects on radiation friction driven magnetic field generation," Eur. Phys. J. Plus **136**, 170 (2021).
[22]A. S. Samsonov, E. N. Nerush, and I. Y. Kostyukov, "Effect of electron–positron plasma production on the generation of a magnetic field in laser-plasma interactions," Quantum Electron. **51**, 861–865 (2021).
[23]D. Del Sorbo, D. Seipt, T. G. Blackburn, A. G. R. Thomas, C. D. Murphy, J. G. Kirk, and C. P. Ridgers, "Spin polarization of electrons by ultraintense lasers," Phys. Rev. A **96**, 043407 (2017).
[24]D. Del Sorbo, D. Seipt, A. G. R. Thomas, and C. P. Ridgers, "Electron spin polarization in realistic trajectories around the magnetic node of two counter-propagating, circularly polarized, ultra-intense lasers," Plasma Phys. Controlled Fusion **60**, 064003 (2018).
[25]Y.-Y. Chen, P.-L. He, R. Shaisultanov, K. Z. Hatsagortsyan, and C. H. Keitel, "Polarized positron beams via intense two-color laser pulses," Phys. Rev. Lett. **123**, 174801 (2019).
[26]D. Seipt, D. Del Sorbo, C. P. Ridgers, and A. G. R. Thomas, "Ultrafast polarization of an electron beam in an intense bichromatic laser field," Phys. Rev. A **100**, 061402 (2019).
[27]Y. Wu, L. Ji, X. Geng, Q. Yu, N. Wang, B. Feng, Z. Guo, W. Wang, C. Qin, X. Yan, L. Zhang, J. Thomas, A. Hützen, M. Büscher, T. P. Rakitzis, A. Pukhov, B. Shen, and R. Li, "Polarized electron-beam acceleration driven by vortex laser pulses," New J. Phys. **21**, 073052 (2019).
[28]Y.-F. Li, R. Shaisultanov, K. Z. Hatsagortsyan, F. Wan, C. H. Keitel, and J.-X. Li, "Ultrarelativistic electron-beam polarization in single-shot interaction with an ultraintense laser pulse," Phys. Rev. Lett. **122**, 154801 (2019).
[29]Y.-F. Li, Y.-Y. Chen, W.-M. Wang, and H.-S. Hu, "Production of highly polarized positron beams via helicity transfer from polarized electrons in a strong laser field," Phys. Rev. Lett. **125**, 044802 (2020).
[30]F. Wan, R. Shaisultanov, Y.-F. Li, K. Z. Hatsagortsyan, C. H. Keitel, and J.-X. Li, "Ultrarelativistic polarized positron jets via collision of electron and ultraintense laser beams," Phys. Lett. B **800**, 135120 (2020).
[31]Z. Gong, K. Z. Hatsagortsyan, and C. H. Keitel, "Retrieving transient magnetic fields of ultrarelativistic laser plasma via ejected electron polarization," Phys. Rev. Lett. **127**, 165002 (2021).
[32]E. N. Nerush and I. Yu. Kostyukov, "Radiation emission by extreme relativistic electrons and pair production by hard photons in a strong plasma wakefield," Phys. Rev. E **75**, 057401 (2007).
[33]A. R. Bell and J. G. Kirk, "Possibility of prolific pair production with high-power lasers," Phys. Rev. Lett. **101**, 200403 (2008).
[34]E. N. Nerush, I. Y. Kostyukov, A. M. Fedotov, N. B. Narozhny, N. V. Elkina, and H. Ruhl, "Laser field absorption in self-generated electron-positron pair plasma," Phys. Rev. Lett. **106**, 035001 (2011).
[35]C. P. Ridgers, C. S. Brady, R. Duclous, J. G. Kirk, K. Bennett, T. D. Arber, A. P. L. Robinson, and A. R. Bell, "Dense electron-positron plasmas and ultraintense γ rays from laser-irradiated solids," Phys. Rev. Lett. **108**, 165006 (2012).
[36]N. B. Narozhny and A. M. Fedotov, "Quantum-electrodynamic cascades in intense laser fields," Phys.-Usp. **58**, 95 (2015).
[37]I. Y. Kostyukov and E. N. Nerush, "Production and dynamics of positrons in ultrahigh intensity laser-foil interactions," Phys. Plasmas **23**, 093119 (2016).
[38]T. Grismayer, M. Vranic, J. L. Martins, R. A. Fonseca, and L. O. Silva, "Seeded QED cascades in counterpropagating laser pulses," Phys. Rev. E **95**, 023210 (2017).
[39]M. Jirka, O. Klimo, M. Vranic, S. Weber, and G. Korn, "QED cascade with 10 PW-class lasers," Sci. Rep. **7**, 15302 (2017).
[40]W. Luo, W.-Y. Liu, T. Yuan, M. Chen, J.-Y. Yu, F.-Y. Li, D. Del Sorbo, C. P. Ridgers, and Z.-M. Sheng, "QED cascade saturation in extreme high fields," Sci. Rep. **8**, 8400 (2018).
[41]T. Yuan, J. Y. Yu, W. Y. Liu, S. M. Weng, X. H. Yuan, W. Luo, M. Chen, Z. M. Sheng, and J. Zhang, "Spatiotemporal distributions of pair production and cascade in solid targets irradiated by ultra-relativistic lasers with different polarizations," Plasma Phys. Controlled Fusion **60**, 065003 (2018).
[42]D. Del Sorbo, D. R. Blackman, R. Capdessus, K. Small, C. Slade-Lowther, W. Luo, M. J. Duff, A. P. L. Robinson, P. McKenna, Z.-M. Sheng *et al.*, "Efficient ion acceleration and dense electron–positron plasma creation in ultra-high intensity laser-solid interactions," New J. Phys. **20**, 033014 (2018).
[43]Y. Lu, T.-P. Yu, L.-X. Hu, Z.-Y. Ge, W.-Q. Wang, J.-X. Liu, K. Liu, Y. Yin, and F.-Q. Shao, "Enhanced copious electron–positron pair production via electron injection from a mass-limited foil," Plasma Phys. Controlled Fusion **60**, 125008 (2018).
[44]W. Luo, S.-D. Wu, W.-Y. Liu, Y.-Y. Ma, F.-Y. Li, T. Yuan, J.-Y. Yu, M. Chen, and Z.-M. Sheng, "Enhanced electron-positron pair production by two obliquely incident lasers interacting with a solid target," Plasma Phys. Controlled Fusion **60**, 095006 (2018).
[45]E. S. Efimenko, A. V. Bashinov, A. A. Gonoskov, S. I. Bastrakov, A. A. Muraviev, I. B. Meyerov, A. V. Kim, and A. M. Sergeev, "Laser-driven plasma pinching in e^-e^+ cascade," Phys. Rev. E **99**, 031201 (2019).
[46]A. S. Samsonov, E. N. Nerush, and I. Y. Kostyukov, "Laser-driven vacuum breakdown waves," Sci. Rep. **9**, 11133 (2019).
[47]A. S. Samsonov, I. Y. Kostyukov, and E. N. Nerush, "Hydrodynamical model of QED cascade expansion in an extremely strong laser pulse," Matter Radiat. Extremes **6**, 034401 (2021).
[48]Technical Design Report No. SLAC-R-1072 for the FACET-II Project at SLAC, National Accelerator Laboratory, 2016.
[49]A. Nikishov and V. Ritus, "Quantum processes in the field of a plane electromagnetic wave and in a constant field I," Sov. Phys. JETP **19**, 529–541 (1964).
[50]V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Butterworth-Heinemann, 1982).
[51]V. Ritus, "Quantum effects of the interaction of elementary particles with an intense electromagnetic field," J. Sov. Laser Res. **6**, 497 (1985).
[52]M. K. Khokonov and H. Nitta, "Standard radiation spectrum of relativistic electrons: Beyond the synchrotron approximation," Phys. Rev. Lett. **89**, 094801 (2002).
[53]A. Ilderton, B. King, and D. Seipt, "Extended locally constant field approximation for nonlinear Compton scattering," Phys. Rev. A **99**, 042121 (2019).
[54]T. Heinzl, B. King, and A. MacLeod, "Locally monochromatic approximation to QED in intense laser fields," Phys. Rev. A **102**, 063110 (2020).
[55].E. G. Gelfer, A. M. Fedotov, A. A. Mironov, and S. Weber, "Nonlinear Compton scattering in time-dependent electric fields beyond the locally constant crossed field approximation," Phys. Rev. D **106**, 056013 (2022).
[56]T. Podszus and A. Di Piazza, "High-energy behavior of strong-field QED in an intense plane wave," Phys. Rev. D **99**, 076004 (2019).
[57]I. I. Artemenko, E. N. Nerush, and I. Yu. Kostyukov, "Quasiclassical approach to synergic synchrotron-cherenkov radiation in polarized vacuum," New J. Phys **22**, 093072 (2020).
[58]J. G. Kirk, A. R. Bell, and I. Arka, "Pair production in counter-propagating laser beams," Plasma Phys. Controlled Fusion **51**, 085008 (2009).
[59]S. Bulanov, C. Schroeder, E. Esarey, and W. Leemans, "Electromagnetic cascade in high-energy electron, positron, and photon interactions with intense laser pulses," Phys. Rev. A **87**, 062110 (2013).
[60]T. Z. Esirkepov, S. S. Bulanov, J. K. Koga, M. Kando, K. Kondo, N. N. Rosanov, G. Korn, and S. V. Bulanov, "Attractors and chaos of electron dynamics in electromagnetic standing wave," Phys. Lett. A **379**, 2044 (2015).

© Author(s) 2023

[61] F. Niel, C. Riconda, F. Amiranoff, R. Duclous, and M. Grech, “From quantum to classical modeling of radiation reaction: A focus on stochasticity effects,” Phys. Rev. E **97**, 043209 (2018).
[62] A. Gonoskov, T. Blackburn, M. Marklund, and S. Bulanov, “Charged particle motion and radiation in strong electromagnetic fields,” Rev. Mod. Phys **94**, 045001 (2022).
[63] C. S. Shen and D. White, “Energy straggling and radiation reaction for magnetic bremsstrahlung,” Phys. Rev. Lett. **28**, 455 (1972).
[64] R. Duclous, J. G. Kirk, and A. R. Bell, “Monte Carlo calculations of pair production in high-intensity laser–plasma interactions,” Plasma Phys. Controlled Fusion **53**, 015009 (2010).
[65] C. N. Harvey, A. Gonoskov, A. Ilderton, and M. Marklund, “Quantum quenching of radiation losses in short laser pulses,” Phys. Rev. Lett. **118**, 105004 (2017).
[66] N. Neitz and A. Di Piazza, “Stochasticity effects in quantum radiation reaction,” Phys. Rev. Lett. **111**, 054802 (2013).
[67] C. P. Ridgers, T. G. Blackburn, D. Del Sorbo, L. E. Bradley, C. Slade-Lowther, C. D. Baird, S. P. D. Mangles, P. McKenna, M. Marklund, C. D. Murphy, and A. G. R. Thomas, “Signatures of quantum effects on radiation reaction in laser–electron-beam collisions,” J. Plasma Phys. **83**, 715830502 (2017).
[68] W. Gerlach and O. Stern, “Der experimentelle Nachweis der Richtungsquantelung im Magnetfeld,” Z. Phys. **9**, 349–352 (1922).
[69] L. H. Thomas, “The motion of the spinning electron,” Nature **117**, 514 (1926).
[70] V. Bargmann, L. Michel, and V. L. Telegdi, “Precession of the polarization of particles moving in a homogeneous electromagnetic field,” Phys. Rev. Lett. **2**, 435 (1959).
[71] S. R. Mane, Yu. M. Shatunov, and K. Yokoya, “Spin-polarized charged particle beams in high-energy accelerators,” Rep. Prog. Phys **68**, 1997 (2005).
[72] D. Seipt, C. P. Ridgers, D. Del Sorbo, and A. G. R. Thomas, “Polarized QED cascades,” New J. Phys. **23**, 053025 (2021).
[73] M. Wen, M. Tamburini, and C. H. Keitel, “Polarized laser-wakefield-accelerated kiloampere electron beams,” Phys. Rev. Lett. **122**, 214801 (2019).
[74] A. S. Samsonov, E. N. Nerush, and I. Y. Kostyukov, “Asymptotic electron motion in the strongly-radiation-dominated regime,” Phys. Rev. A **98**, 053858 (2018).
[75] A. Gonoskov and M. Marklund, “Radiation-dominated particle and plasma dynamics,” Phys. Plasmas **25**, 093109 (2018).
[76] P. Jérôme, “Particle acceleration and radiation reaction in a strongly magnetised rotating dipole,” Astron. Astrophys **666**, A5 (2022).
[77] Y. Cai, S. E. Gralla, and V. Paschalidis, “Dynamics of ultrarelativistic charged particles with strong radiation reaction. I. Aristotelian equilibrium state,” arXiv:2209.07469 (2022).
[78] Y. B. Zel’dovich, “Interaction of free electrons with electromagnetic radiation,” Sov. Phys. Usp **18**, 79 (1975).
[79] L. D. Landau, *The Classical Theory of Fields* (Elsevier, 2013), Vol. 2.
[80] A. D. Piazza, “Exact solution of the Landau-Lifshitz equation in a plane wave,” Lett. Math. Phys. **83**, 305–313 (2008).
[81] J. E. Gunn and J. P. Ostriker, “On the motion and radiation of charged particles in strong electromagnetic waves. I. Motion in plane and spherical waves,” Astrophys. J. **165**, 523 (1971).
[82] M. Grewing, E. Schrüfer, and H. Heintzmann, “Acceleration of charged particles and radiation reaction in strong plane and spherical waves. II,” Z. Phys. A: Hadrons Nucl. **260**, 375–384 (1973).
[83] K. Thielheim, “Particle acceleration in extremely strong electromagnetic wave fields,” in *Proceedings of International Conference on Particle Accelerators* (IEEE, 1993), pp. 276–278.
[84] R. Ekman, T. Heinzl, and A. Ilderton, “Exact solutions in radiation reaction and the radiation-free direction,” New J. Phys. **23**, 055001 (2021).

© Author(s) 2023