PHOTONICS Research

Stronger Hardy-like proof of quantum contextuality

Wen-Rong QI,^{1,2,†} JIE ZHOU,^{3,†} LING-JUN KONG,^{4,5,†} ZHEN-PENG XU,⁶ HUI-XIAN MENG,³ RUI LIU,¹ ZHOU-XIANG WANG,¹ CHENGHOU TU,¹ YONGNAN LI,^{1,8} ADÁN CABELLO,⁷ JING-LING CHEN,^{3,9} (D) AND HUI-TIAN WANG^{4,10} (D)

¹Key Laboratory of Weak-Light Nonlinear Photonics and School of Physics, Nankai University, Tianjin 300071, China ²School of Physics, Henan Normal University, Xinxiang 453007, China

³Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, China

⁴National Laboratory of Solid State Microstructures, Nanjing University, Nanjing 210093, China

⁵Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China

⁶Naturwissenschaftlich-Technische Fakultät, Universität Siegen, 57068 Siegen, Germany

⁷Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain

8e-mail: liyongnan@nankai.edu.cn

[°]e-mail: chenjl@nankai.edu.cn

¹⁰e-mail: htwang@nju.edu.cn

Received 30 December 2021; revised 17 March 2022; accepted 4 May 2022; posted 9 May 2022 (Doc. ID 452704); published 14 June 2022

A Hardy-like proof of quantum contextuality is a compelling way to see the conflict between quantum theory and noncontextual hidden variables (NCHVs), as the latter predict that a particular probability must be zero, while quantum theory predicts a nonzero value. For the existing Hardy-like proofs, the success probability tends to 1/2 when the number of measurement settings n goes to infinity. It means the conflict between the existing Hardy-like proof and NCHV theory is weak, which is not conducive to experimental observation. Here we advance the study of a stronger Hardy-like proof of quantum contextuality, whose success probability is always higher than the previous ones generated from a certain n-cycle graph. Furthermore, the success probability tends to 1 when n goes to infinity. We perform the experimental test of the Hardy-like proof in the simplest case of n = 7 by using a four-dimensional quantum system encoded in the polarization and orbital angular momentum of single photons. The experimental result agrees with the theoretical prediction within experimental errors. In addition, by starting from our Hardy-like proof, one can establish the stronger noncontextuality inequality, for which the quantum-classical ratio is higher with the same n, which provides a new method to construct some optimal noncontextuality inequalities. Our results offer a way for optimizing and enriching exclusivity graphs, helping to explore more abundant quantum properties. © 2022 Chinese Laser Press

https://doi.org/10.1364/PRJ.452704

1. INTRODUCTION

Ideal measurements yield the same outcome when repeated, even when other compatible measurements have been performed in between. Some predictions of quantum theory for ideal measurements cannot be explained under the assumption that ideal measurements reveal preexisting outcomes that are independent of the context (i.e., the set of compatible measurements that have been jointly measured), this is known as quantum contextuality or Kochen–Specker (KS) contextuality [1]. Quantum contextuality is an intrinsic signature of nonclassicality as its classical simulation requires memory [2] and thermodynamical cost [3]. Moreover, it has been proven that quantum contextuality is necessary for fault-tolerant quantum computation via magic state distillation and measurement-based quantum computation [4]. Quantum contextuality also plays a fundamental role in some quantum key distribution protocols [5] and is crucial for understanding the underlying physics behind the limitation of quantum correlations [6]. So far, quantum contextuality has been experimentally observed in trapped ions [7], photons [8], ensembles of molecular nuclear spins in the solid state [9], and superconducting systems [10].

KS contextuality is the first theory about contextuality. The KS-type proof [1,11,12] serves as a no-go theorem, indicating that it is impossible for the noncontextual hidden variable (NCHV) models to describe quantum mechanics. For instance, for Peres-33 rays [11], it is impossible to define any consistent assignment of 0s and 1s to the states in the set. In other words, the standard KS sets cannot be colored in a consistent way in classical theory. Beyond all doubt, the KS-type proof is an elegant argument for contextuality, which is similar to the Greenberger-Horne-Zeilinger (GHZ) proof for the Bell nonlocality. Throughout research history, many well-known scientists have done excellent work on KS contextuality. The first proof scheme of KS includes 117 ray quantities in threedimensional space, and that with less rays is found step by step [11–14]. Those that we are familiar with are Peres-33 rays [11], CEG-18 rays [12], and so on. There are some that also have other famous research, for example, the Peres-Mermin square-a considerable simplification of the original KS argument by Mermin and Peres using nine observables that are organized in a 3×3 square [15,16]. Later, noncontextuality inequality has been proposed as an experiment-friendly approach to reveal quantum contextuality and has been tested by many experiments [7,12,14,17,18]. We will talk about noncontextuality inequality in detail in the next paragraph. In general, a KS-type proof must be transferred to a noncontextuality inequality to be experimentally validated, and experimental tests of contextuality based on standard KS-type proofs were seldom performed in the literature, which may be due to this reason: noncontextuality inequalities are straightforward for the Klyachko-Can-Binicioğlu-Shumovsky (KCBS) cases [19], but in the KS case, the meaning is less clear because value assignments are logically inconsistent, and it is not clear how to compare them with quantum prediction.

The graph-theoretic approach to quantum correlations bridges a fundamental connection between graph theory and quantum contextuality [20,21]. The central idea is that, for an arbitrary exclusivity graph, [Note: Any graph G(V, E) is made up by vertices and edges, where V is a set of vertices and E is the set of edges. The exclusivity graph described that the measurement events e_i are represented as the vertices *i* of the graph, and there is an edge between vertex *i* and vertex *j* if events e_i and e_j are mutually exclusive events, and the corresponding measurement vectors $|v_i\rangle$ and $|v_i\rangle$ are mutually orthogonal.] one can always associate it with a noncontextuality inequality, for which the classical bound has a definite meaning as the independence number of the graph, and the maximum quantum violation of the inequality is the Lovász number describing the Shannon capacity of the graph. The noncontextuality inequality is a correlation inequality that any theory should satisfy under the assumption of outcome noncontextuality for ideal measurements. The typical examples are the KCBS inequality [12] and its extensions [22] and some state-independent-contextuality (SIC) inequalities [14,23]. The noncontextuality inequalities have these advantages: (i) theoretically, quantum contextuality can be revealed in a direct way by the violations of noncontextuality inequalities; (ii) compared with the KS-type proofs, the violations of noncontextuality inequalities are more feasible to observe quantum contextuality in the experiments.

A natural result yielded from the graph-theoretic approach is the so-called "Hardy-like proof" [24], which for quantum contextuality is analogous to Hardy's proof for Bell nonlocality [25] and is also a particularly compelling way to reveal contextuality. A Hardy-like proof of contextuality may be seen as a particular violation of the noncontextuality inequality, which is more experimentally friendly than noncontextuality inequalities sometimes. The first Hardy-like proof of contextuality was proposed in Ref. [24] by studying the *n*-cycle graphs. In such a proof, the NCHV theory predicts that the success probability P_{SUC} of a particular event must be zero, while quantum theory predicts a nonzero value. For the simple Hardy-like proof of the five-cycle graph, $P_{SUC} = 1/9$ has been experimentally observed [26]. Remarkably, for the *n*-cycle graphs, P_{SUC} tends to 1/2 when the number of measurement settings *n* goes to infinity [24]. The conflict between the simple Hardy-like proof and NCHV theory has not reached the limit. A natural question is whether there is a Hardy-like proof for a certain *n*-vertex graph, in which P_{SUC} tends to 1 when $n \to \infty$, thus providing the stronger quantum contextuality.

In this work, a Hardy-like proof was presented for the graphs with n = 4m + 3 (m = 1, 2, ...) vertices, in which the success probability tends to 1 when the number of measurement settings goes to infinity. For a fixed *n*, the new Hardy-like proof is stronger than the previous one for the *n*-cycle graph. To illustrate our idea, we experimentally test the Hardy-like proof with $P_{SUC} = 1/4$ in the simplest case of n = 7, by using a four-dimensional quantum system encoded in the polarization and orbital angular momentum (OAM) of single photons. For some symmetric graphs, by starting from the Hardy-like proofs, one can establish some stronger noncontextuality inequalities, for which the quantum-classical ratio is higher or optimal, and this provides a novel way to construct some useful noncontextuality inequalities.

2. STRONGER HARDY-LIKE PROOF

Considering n = 4m + 3 (m = 1, 2, ...) ideal measurements represented in quantum theory by the rank-one projectors and with possible outcomes 0 and 1, some of these measurements are jointly measurable, while some of the corresponding events are mutually exclusive. The *n*-vertex exclusivity graph is shown in Fig. 1. Supposing that there is a state $|\phi\rangle$ ($\rho = |\phi\rangle\langle\phi|$), its probabilities for the triangles in Fig. 1 satisfy the following Hardy-like constraints:



Fig. 1. Exclusivity graph of the *n* measurements (with n = 7, 11, 15, 19, ...) used for the Hardy-like proof of contextuality. Each vertex represents a measurement. Vertices connected by an edge are jointly measurable. Each of the outer vertices belongs to two triangles. In total, there are (n - 1)/2 triangles.

$$\sum_{j=1,2,\frac{n+1}{2}} P(\tau_j \rho | \rho) = 1,$$

$$\sum_{j=2,3,\frac{n+3}{2}} P(\tau_j \rho | \rho) = 1,$$

$$\vdots$$

$$\sum_{j=1,\frac{n-1}{2},n-1} P(\tau_j \rho | \rho) = 1,$$
(1)

i.e., the sum of the probabilities of each triangle is 1. $(\tau_i \rho | \rho)$ is the event corresponding to outcome 1 obtained for the measurement τ_i in the state $|\phi\rangle$, no matter which other compatible measurements are performed. Under these assumptions, any NCHV theory predicts that the outcome of the measurement corresponding to the vertex in the center of Fig. 1 must be zero. That is to say, the NCHV theory predicts $P(\tau_n \rho | \rho) = 0$. This is due to the fact that, in the NCHV theory, Eq. (1) implies that at least one of the events that are mutually exclusive with $(\tau_n \rho | \rho)$ has happened for any given hidden variable. Thus, the event $(\tau_n \rho | \rho)$ can never happen whatever the hidden variable is, i.e., $P(\tau_n \rho | \rho) = 0$. In addition, for the odd number $\left(\frac{n-1}{2} = 2m + 1\right)$ outer vertices in the polygon, not all of those values can be zero; hence, at least one of the inner values has to be 1, and vertex n must be a value of 0. See Appendix A for a detailed proof for the NCHV value of the Hardy-like paradox.

In contrast, quantum theory predicts that for a fourdimensional quantum state, when n measurements satisfy Eq. (1), the success probability could be

$$P_{\rm SUC} \equiv P(\tau_n \rho | \rho) = \cos^2\left(\frac{2\pi}{n-1}\right).$$
 (2)

Clearly, the probability tends to 1 when $n \to \infty$ and is 1/4 for the simplest case of n = 7. Moreover, the probability is very close to 1 even for a relatively small n [e.g., for n = 83, $P(\tau_{83}\rho|\rho) \approx 0.994$]. The state and the measurement vectors for $P(\tau_n\rho|\rho)$ are listed in Table 3 (see Appendix B), and the detailed comparison of $P(\tau_n\rho|\rho)$ with the success probability obtained in the *n*-cycle Hardy-like proof [24] is shown in Table 4 (see Appendix C).

Table 1 lists the state $|\phi\rangle$ and the corresponding projectors $\tau_j = |\nu_j\rangle\langle\nu_j|$, and we have $P(\tau_n\rho|\rho) = 1/4$ for n = 7. The exclusivity relations between the projection measurements are given by the graph in Fig. 2. In addition, three annex points (denoted by $|\nu_8\rangle$, $|\nu_9\rangle$, $|\nu_{10}\rangle$) are added due to the need of experimental test for the compatibility conditions.



3. STRONGER NONCONTEXTUALITY INEQUALITY

8 C

Every Hardy-like proof of quantum contextuality can be considered as a violation of a noncontextuality inequality [24]. Hence, a Hardy-like proof can be used as the starting point for identifying new noncontextuality inequalities. An interesting issue is identifying situations in which few measurement settings can produce a high degree of contextuality. The issue is of great importance for applications in which the degree of contextuality has one-to-one correspondence with the quantum-versus-classical advantage [27]. To characterize the degree of quantum contextuality, it is defined as the ratio between the maximum quantum violation and the noncontextual bound [27]. When performing an operation by the Hardy-like proof presented above, we select the weights of the vertices in the symmetric graph (Fig. 1) for achieving the optimal degree of quantum contextuality. Thus, we obtain the following noncontextuality inequality as

$$I_n = 2\sum_{j=1}^{\frac{n-1}{2}} P(\tau_j \rho | \rho) + \sum_{j=\frac{n+1}{2}}^n P(\tau_j \rho | \rho) \stackrel{\text{NCHV}}{\leq} \frac{n-1}{2}.$$
 (3)

This inequality is violated by the quantum state in the fourdimensional quantum system. Notice that, in Fig. 1, black vertices have a weight of 2 while white vertices have a weight of 1 in Eq. (3). For n = 7, the noncontextual bound of I_n is 3, while the maximum quantum violation is $(1 + \sqrt{33})/2 \approx 3.372$. This maximum violation can be

Table 1. State $|\phi\rangle$ and the Measurement Vectors $|\nu_j\rangle$ (j = 1, 2, ..., 7), Where $|\nu_8\rangle$, $|\nu_9\rangle$, $|\nu_{10}\rangle$ Are Annex Measurement Vectors

$ \phi angle$	$ u_1 angle$	$ \nu_2\rangle$	$ \nu_3\rangle$	$ u_4 angle$	$ \nu_5\rangle$	$ \nu_6\rangle$	$ \nu_7\rangle$	$ u_8 angle$	$ \nu_9\rangle$	$ \nu_{10}\rangle$
$\frac{1}{2}$	1	0	0	0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{2}}$
$\frac{1}{2}$	0	1	0	0	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$	0
$\frac{1}{2}$	0	0	1	$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0	0
$\frac{1}{2}$	0	0	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

achieved by the state $|\psi\rangle = (\alpha, \alpha, \alpha, \beta)^T$, and the measurements correspond to the projectors onto the seven vectors of $|\nu_1\rangle = (1,0,0,0)^T$, $|\nu_2\rangle = (0,1,0,0)^T$, $|\nu_3\rangle = (0,0,1,0)^T$, $|\nu_4\rangle = (0,0,\gamma,\zeta)^T$, $|\nu_5\rangle = (\gamma, 0, 0, \zeta)^T$, $|\nu_6\rangle = (0, \gamma, 0, \zeta)^T$, $|\nu_7\rangle = (\mu, \mu, \mu, \varepsilon)^T$, where $\alpha = \sqrt{(11 + 5\sqrt{33})/132}$, $\beta = \sqrt{(33 - 5\sqrt{33})/44}$, $\gamma = \sqrt{(9 - \sqrt{33})/8}$, $\zeta = \sqrt{(\sqrt{33} - 1)/8}$, $\mu = \sqrt{2/3\gamma}$, $\varepsilon = -\sqrt{(\sqrt{33} - 5)/4}$ (see Appendix D for the cases of higher *n*).

Equation (3) has remarkable properties, as follows. (i) For any n, Eq. (3) leads to a higher degree of contextuality than the one obtained by the extended KCBS noncontextuality inequality (see Appendix E). The extended KCBS inequality is the default target inequality in the contextuality experiment [26] and is also the inequality that naturally follows from the Hardy-like proof in Ref. [24]. (ii) For the simplest case (n = 7), numerical computation shows that the inequality (3) leads to a higher degree of contextuality than other noncontextuality inequalities in form $\sum_{j=1}^{7} w_j \langle P_j \rangle \leq C$, where w_j is the weight of the rank-one projector P_j for the *j*th vertex, and *C* is the classical bound. For n = 7, the maximum degree of contextuality for the extended KCBS inequality is 1.106 [22], while for inequality $I_7 \leq 3$, it is 1.124. For some of the discussions, see Appendix F. This property reveals that searching for situations, in which a few measurement settings lead to a high degree of contextuality, may be beneficial for investigating symmetric graphs and attributing different weights to each of the classes of vertices.

4. EXPERIMENTAL RESULTS

The new Hardy-like proof and the corresponding inequality can reveal the contextuality in a new system because they provide better contextuality witnesses than the extended KCBS inequality under the same number of measurement settings. We will test the Hardy-like proof (for the simplest case of n = 7) in

a hybrid four-dimensional quantum system defined by the polarization and OAM of single photons. Testing contextuality is especially difficult as sequential measurements are not easy to implement. To overcome this difficulty, a simple method (which needs two measurements only) has been presented, in which the first measurement can be simulated by a demolition measurement followed by a preparation that depends on the outcome [28]. Of course, the necessary no-signaling conditions [29] also need to be satisfied. Based on this method, in order to test the Hardy-like proof in Eqs. (1) and (2) for n = 7, we should add mutually orthogonal vertices to form complete sets. Figure 2 shows that $\{1,2,4,8\}, \{1,3,6,9\}, \text{ and } \{2,3,5,10\}$ have formed three complete sets.

The experimental setup (Fig. 3) consists of two parts: the state preparation and the projection measurement. In the state preparation, photon pairs are produced via a type-II spontaneous parametric downconversion in a 10-mm-long periodically poled potassium titanyl phosphate (ppKTP) crystal pumped by a 405 nm cw TEM₀₀ diode laser, and one photon serves as a trigger. The produced photons can carry the discrete OAM of $l\hbar$ [30] (l is an arbitrary integer, and \hbar is the reduced Planck constant). Therefore, high-dimensional information can be encoded in the OAM of single photons [31]. As the original work on OAM [32], we consider an OAM carried by Laguerre–Gaussian (LG) mode of azimuthal l order and radial order p = 0 in our experiment. Equally weighted superpositions of LG modes with different OAMs ($l_1\hbar$ and $l_2\hbar$), and the relative phase ξ can be written as

$$LG_{l_1, l_2}(r, \varphi) = \frac{1}{\sqrt{2}} [LG_{l_1}(r, \varphi) + e^{i\xi} LG_{l_2}(r, \varphi)].$$
 (4)

We generate the ququart states based on the interferometric superposition of horizontally (H) and vertically (V) polarized photons carrying OAMs, which is similar to that in Ref. [33]. The collimated photons at 810 nm are split into two



Fig. 3. Experimental setup. In the state preparation part, a 405 nm cw laser pumps a type-II ppKTP crystal (not shown) to create photon pairs. One photon serves as a trigger. The other photon is projected into the horizontal polarization state with a polarizing beam splitter (PBS); the spatial light modulator (SLM) combines a Rochi grating (RG) through two 4f systems to generate the ququart subset of OAM. In the projection measurement part, two sets of *q*-plates (QPs, with different topological charges of $q_1 = 1/2$ and $q_2 = 1$) are sandwiched by two quarter-wave plates (QWPs), followed by a half-wave plate (HWP) and a PBS used to convert OAM mode into a fundamental mode that is coupled into a single mode fiber (SMF); the photons are detected by a single photon avalanche photodiode (SPAD).

paths by a beam splitter (BS). Two computer-generated holographic gratings (HG1 and HG2) displayed on a spatial light modulator (SLM) (1920 × 1080 pixels) diffract the H-polarized light into different diffraction orders. Then a Rochi grating (RG) is used to recombine the -1st diffraction order of HG1 and the +1st diffraction order of HG2 into the single one, as shown in Fig. 3. The weight and phase of two different OAMs can be controlled by adjusting the HGs. Thus, we use the twodimensional orthogonal polarizations and four-dimensional OAMs ($l = \pm 1$ and ± 3) to build four-dimensional subspace $\{|H, +1\rangle, |V, -1\rangle, |H, +3\rangle, |V, -3\rangle\}$. Finally, the prepared state can be expressed as follows:

$$|\phi\rangle = \frac{1}{2}(|H, +1\rangle + |V, -1\rangle + |H, +3\rangle + |V, -3\rangle).$$
(5)

In the projection measurement, we perform the projection measurements by two q-plates (QPs) with topological charges of $q_1 = 1/2$ and $q_2 = 1$, respectively. The function of the QP can be described as $|L, l\rangle \stackrel{\text{QP}}{\rightarrow} |R, l + 2q\rangle$, $|R, l\rangle \stackrel{\text{QP}}{\rightarrow} |L, l - 2q\rangle$ [34], where L and R denote left- and right-handed circular polarizations, respectively. Two cascaded QPs sandwiched between two quarter-wave plates (QWPs) are used to convert $|\nu_j\rangle$ into a TEM₀₀ mode, which is easily coupled to a single photon avalanche photodiode (SPAD) through the single mode fiber (SMF).

In order to measure the probability for n = 7 in Eq. (2), we need to introduce two additional vertices (a, b) to form a complete set $\{5,7,a,b\}$, where $|\nu_a\rangle = \frac{1}{\sqrt{2}}(0,1,-1,0)^T$ and $|\nu_b\rangle = \frac{1}{2}(-1,1,1,-1)^T$. Here we describe the observables by $o_j = |\nu_j\rangle\langle\nu_j|$. For the sake of simplicity, $P(\tau_j\rho|\rho)$ is replaced by $P(1|o_j)$. We first perform the verification of no-signaling between two measurements. We use the equations given in Ref. [22] to characterize the influence,

$$\delta(_,0|o_j, o_k) = |P(0|o_k) - P(0,0|o_j, o_k) - P(1,0|o_j, o_k)|,$$

$$\delta(_,1|o_j, o_k) = |P(1|o_k) - P(0,1|o_j, o_k) - P(1,1|o_j, o_k)|,$$

which represent the statistical effects of the first measurement on the second measurement, indicating that the result of projecting the state onto o_k is 0 and 1, respectively. In the same way, the statistical effects of the second measurement on the first should be

$$\delta(0, _|o_j, o_k) = |P(0|o_j) - P(0, 0|o_j, o_k) - P(0, 1|o_j, o_k)|,$$

$$\delta(1, _|o_j, o_k) = |P(1|o_j) - P(1, 0|o_j, o_k) - P(1, 1|o_j, o_k)|.$$

To obtain $P(0,1|o_j, o_k)$, we need to prepare the orthogonal state $|\nu_j^{\perp}\rangle$ based on the Lüder rule [35]. Our experiment meets the no-signaling condition within the experimental accuracy, $\delta(_,0|o_j, o_k) \approx \delta(0,_|o_j, o_k) = 0$ and $\delta(_,1|o_j, o_k) \approx \delta(1,_|o_j, o_k) = 0$ (see Appendix G). The projection probability and the joint measurement probability should be measured in their corresponding complete sets.

The measurement results for the Hardy-like proof of contextuality agree well with the predictions of quantum theory, as listed in Table 2. The errors are always inevitable, due to the limited number of experiments, unavoidable defects of

Projectors	Experiment	Ideal
01	0.248 ± 0.005	0.25
02	0.246 ± 0.004	0.25
2 04	0.495 ± 0.008	0.50
<i>o</i> ₁	0.247 ± 0.006	0.25
03	0.247 ± 0.005	0.25
06	0.495 ± 0.007	0.50
02	0.247 ± 0.005	0.25
03	0.246 ± 0.006	0.25
05	0.495 ± 0.005	0.50
07	0.248 ± 0.006	0.25

"The column "Ideal" denotes the predictions of quantum theory for an ideal experiment. All of the errors are calculated from photon-counting statistics.



Fig. 4. Quantum violation of Eq. (3) for n = 7. NCHV (=3) represents the classical bound of the NCHV theory. The maximum violation is QT1 \approx 3.372 by using the optimal measurement settings, and the non-maximum violation is QT2 = 3.250 by using the measurements adopted in the Hardy-like proof.

the devices, and the fluctuation caused by the long-lasting experiments. From Table 2, our results also meet the three conditions for verifying the Hardy-like contextuality in Ref. [26]:

(i) $P(1|o_1) + P(1|o_2) + P(1|o_4) = 0.989 \pm 0.010$, $P(1|o_1) + P(1|o_3) + P(1|o_6) = 0.989 \pm 0.010$, $P(1|o_2) + P(1|o_3) + P(1|o_5) = 0.988 \pm 0.009$ are equal to 1 within the experimental errors.

(ii) $P(1|o_7) = 0.248 \pm 0.006$ is nonzero and agrees well with the value predicted by quantum theory in Eq. (2).

(iii) The quantum violation of the noncontextuality inequality [Eq. (3)] from the results in Table 2 is calculated to be $I_7 \approx 3.214 \pm 0.018$ in experiment, which is $I_7 = 3.250$ in theory (Fig. 4).

5. CONCLUSION

Stronger quantum contextuality is particularly significant as it reveals a sharper contradiction between the NCHV theory and the quantum theory. In this work, we have advanced the study of a stronger Hardy-like proof of quantum contextuality. We have theoretically presented a new Hardy-like proof, which is stronger than the previous one given in Ref. [24]. For the simplest case (n = 7), the success probability equals 1/4, and it tends to 1 when $n \to \infty$. We have also performed the experimental test for the new Hardy-like proof by the four-dimensional quantum system encoded in the polarization and OAM of single photons. The experimental result gives $P_{\rm SUC} = 0.248 \pm 0.006$, which agrees with the theoretical prediction within the experimental errors. Importantly, the stronger Hardy-like proof can yield stronger noncontextuality inequality and has an advantage over the extended KCBS noncontextuality inequality for the *n*-cycle graphs. Our results not only advance the study of the Hardy-like proof for quantum correlations but also open a new approach to observe quantum contextuality in new systems. That also paves a way for further research on fundamental quantum resources.

APPENDIX A: PROOF FOR THE NCHV VALUE FOR THE HARDY-LIKE PARADOX

For simplicity, we use P_i to represent $P(\tau_n \rho | \rho)$.

For the NCHV case, once the hidden variable is given, each ray, which can be viewed as a dichotomic projective measurement, can be either associated with the value 0 or 1. To put it more explicitly,

$$P_{i} = \sum_{\mathbf{y} \in \Lambda} p(\mathbf{y}) \mathscr{D}_{i}(\mathbf{y}), \qquad (A1)$$

where Λ is the set of hidden variables, $\mathcal{D}_i(y) \in \{0,1\}$, and if i, j are connected in the graph,

$$\wp_i(\mathfrak{y})\wp_j(\mathfrak{y}) = 0. \tag{A2}$$

The conditions in Eq. (1) can be rewritten as

$$P_{1} + P_{2} + P_{\frac{n+1}{2}} = 1,$$

$$P_{2} + P_{3} + P_{\frac{n+3}{2}} = 1,$$

$$\vdots$$

$$P_{1} + P_{\frac{n+1}{2}} + P_{n-1} = 1.$$
(A3)

In the NCHV case, it implies that, for any hidden variable y, we have

$$\begin{split} \wp_{1}(y) + \wp_{2}(y) + \wp_{\frac{n+1}{2}}(y) &= 1, \\ \wp_{2}(y) + \wp_{3}(y) + \wp_{\frac{n+3}{2}}(y) &= 1, \\ &\vdots \\ \wp_{1}(y) + \wp_{\frac{n-1}{2}}(y) + \wp_{n-1}(y) &= 1. \end{split}$$
(A4)

By taking the sum of the above equations, we have

$$\begin{split} & [\wp_{1}(y) + \wp_{2}(y) + \wp_{\frac{n+1}{2}}(y)] \\ & + [\wp_{2}(y) + \wp_{3}(y) + \wp_{\frac{n+3}{2}}(y)] + \cdots \\ & + [\wp_{1}(y) + \wp_{\frac{n+1}{2}}(y) + \wp_{n-1}(y)] \\ & = 2[\wp_{1}(y) + \delta_{2}(y) + \cdots + \wp_{\frac{n-1}{2}}(y)] \\ & + [\wp_{\frac{n+1}{2}}(y) + \wp_{\frac{n+3}{2}}(y) + \cdots + \wp_{n-1}(y)] \\ & = \frac{n-1}{2}. \end{split}$$
(A5)

According to the relation in Eq. (A2), we can know that

$$\wp_1(\mathfrak{y}) + \wp_2(\mathfrak{y}) + \dots + \wp_{\frac{n-1}{2}}(\mathfrak{y}) \le \frac{\frac{n-1}{2} - 1}{2} = \frac{n-3}{4},$$
 (A6)

which implies that

$$\mathscr{D}_{\frac{n+1}{2}}(\mathfrak{y}) + \mathscr{D}_{\frac{n+3}{2}}(\mathfrak{y}) + \dots + \mathscr{D}_{n-1}(\mathfrak{y}) \ge 1.$$
 (A7)

That is, for any hidden variable y, there are always at least one $k \in \{\frac{n+1}{2}, ..., n-1\}$ such that $\mathscr{D}_k(y) = 1$. Again, because of the relation in Eq. (A2), we have $\mathscr{D}_n(y) = 0$. Consequently,

$$P_n = \sum_{\mathbf{y} \in \Lambda} p(\mathbf{y}) \mathcal{D}_n(\mathbf{y}) = 0.$$
 (A8)

APPENDIX B: MEASUREMENTS OF THE STATE FOR THE HARDY-LIKE PROOF OF QUANTUM CONTEXTUALITY WITH *n* MEASUREMENTS

For the *n*-vertex exclusivity graph (Fig. 1), based on whose symmetry and the mutually exclusive relationship of vertices, we can set the measurements $\tau_j = |\nu_j\rangle\langle\nu_j|$ (j = 1, 2, ..., n) form as shown in Table 3. Via calculation, we can get the parameters. Then, there are four-dimensional quantum state $|\phi\rangle$ and *n* measurements $\tau_j = |\nu_j\rangle\langle\nu_j|$ (j = 1, 2, ..., n), which are listed in Table 3. Based on that, one can have the success probability for the Hardy-like proof as

$$P_{\text{SUC}} = P(\tau_n \rho | \rho) = \cos^2 \left(\frac{2\pi}{n-1}\right).$$
(B1)

From Eq. (B1), we can find that the success probability tends to 1 when n goes to infinity.

Remark 1. The measurement vectors $|\nu_j\rangle$ can be obtained based on the symmetry of our exclusivity graph (Fig. 1). The state and the measurements listed in Table 3 for n = 7 are equivalent to the ones given in Table 1 under the unitary transformation.

APPENDIX C: COMPARISON BETWEEN THE SUCCESS PROBABILITIES PSUC AND PCBCB

The success probability is described by Eq. (B1) or Eq. (2). In Ref. [24], Cabello, Badzia, Cunha, and Bourennane (CBCB) have proposed a simple Hardy-like proof of contextuality, which is generated from the *n*-cycle graph. The corresponding success probability is given by

Table 3. $P(\tau_n \rho | \rho)$ Has the Maximum Quantum Value as $\cos^2[2\pi/(n-1)]^a$

-				
$ \nu_1\rangle$	1	0	$-\lambda \sin \theta$	$\lambda \cos \theta$
$ \nu_2\rangle$	$-\cos(2\theta)$	$\sin(2\theta)$	$-\lambda \sin \theta$	$\lambda \cos \theta$
$ \nu_3\rangle$	$\cos(4\theta)$	$-\sin(4\theta)$	$-\lambda \sin \theta$	$\lambda \cos \theta$
÷	÷	÷	÷	÷
$ \nu_{\frac{n-1}{2}}\rangle$	$\cos\left[(n-3)\theta\right]$	$-\sin[(n-3)\theta]$	$-\lambda \sin \theta$	$\lambda \cos \theta$
$ \nu_{\frac{n+1}{2}}\rangle$	$\lambda \sin \theta$	$\lambda \cos \theta$	1	0
$ \nu_{\frac{n+3}{2}}\rangle$	$-\lambda \sin (3\theta)$	$-\lambda \cos (3\theta)$	1	0
$ \nu_{\frac{n+5}{2}}\rangle$	$\lambda \sin(5\theta)$	$\lambda \cos(5\theta)$	1	0
:	÷	÷	÷	÷
$ u_{n-1} angle$	$\lambda \sin[(n-2)\theta]$	$\lambda \cos[(n-2)\theta]$	1	0
$ \nu_n\rangle$	0	0	0	1
$ \phi angle$	0	0	$-\sin(2\theta)$	$\cos(2\theta)$

"The *n* measurements τ_i and the state $|\phi\rangle$ are given as follow. Here τ_j is the projector onto the vector $|\nu_j\rangle$ (non-normalized) and $\rho = |\phi\rangle\langle\phi|, \theta = \pi/(n-1)$ and $\lambda = \sqrt{\cos(2\theta)}$.

n	7	11	15	19	23	27	31	35	39	43
P _{SUC}	0.250	0.655	0.812	0.883	0.921	0.943	0.957	0.966	0.973	0.978
$P_{\rm CBCB}$	0.200	0.297	0.347	0.377	0.397	0.412	0.423	0.432	0.438	0.444
n	47	51	55	59	63	67	71	75	79	83
P _{SUC}	0.981	0.984	0.987	0.988	0.990	0.991	0.992	0.993	0.994	0.994
$P_{\rm CBCB}$	0.449	0.453	0.456	0.459	0.461	0.464	0.466	0.468	0.469	0.471

Table 4. Success Probabilities $P_{SUC} = \cos^2[2\pi/(n-1)]$ and $P_{CBCB} = \cos^{\chi}(\pi/\chi)/[1 + \cos^{\chi}(\pi/\chi)]$

$$P_{\text{CBCB}} = \frac{\cos^{\chi}(\pi/\chi)}{1 + \cos^{\chi}(\pi/\chi)},$$
 (C1)

where $\chi = (n + 1)/2$ and n = 4m + 3 (m = 1, 2, ...).

Table 4 lists the numerical values of the success probabilities P_{SUC} and P_{CBCB} for different *n*, where n = 4m + 3 (m = 1, 2, ..., 20). From Table 4, we have also plotted the curves of these two success probabilities versus *n* in Fig. 5.

From Table 4 and Fig. 5, we can find that the success probability P_{SUC} is higher than P_{CBCB} . In Ref. [24], the probability P_{CBCB} tends to 1/2 when $n \to \infty$. However, in our work, the probability P_{SUC} has exceeded 1/2 for n = 11, i.e., $P_{SUC} = \cos^2(\pi/5) \approx 0.655 > 1/2$. In addition, the success probability P_{SUC} tends to be 1 when *n* goes to infinity. Interestingly, for a finite number of settings, P_{SUC} is already very close to 1. For example, $P_{SUC} = \cos^2(\pi/41) \approx 0.994$ for n = 83.

APPENDIX D: STATE AND MEASUREMENTS FOR THE MAXIMUM QUANTUM VALUE OF I_n

For the noncontextuality inequality with n = 4m + 3 (m = 1, 2, ...),

$$I_n = 2 \sum_{j=1}^{(n-1/2)} P(\tau_j \rho | \rho) + \sum_{j=(n+1)/2}^n P(\tau_j \rho | \rho) \stackrel{\text{NCHV}}{\leq} \alpha_n \stackrel{\text{QT}}{\leq} Q_n.$$
(D1)

The noncontextual bound is $\alpha_n = (n-1)/2$, and the maximum quantum violation Q_n can be achieved when we choose



Fig. 5. Success probabilities P_{SUC} (blue curve) and P_{CBCB} (red curve) versus *n*.

the four-dimensional quantum system state $|\psi\rangle$ (here $\rho = |\psi\rangle\langle\psi|$) and *n* measurements τ_j (j = 1, 2, ..., n), as shown in Table 5. The vectors in Tables 3 and 5 share the similar structure. By optimizing two parameters, θ and *x*, one can obtain the numerical value of quantum bound Q_n as shown in Table 6.

Remark 2. The maximum quantum violation Q_n in Eq. (D1) is obtained in the four-dimensional quantum system in Table 5. The state and the measurement vectors are obtained based on the symmetry of our compatibility graph. Except for the inequality for n = 7, the other cases are hard to find analytical results. We list the maximum quantum violation Q_n in Table 6.

APPENDIX E: COMPARISON OF DEGREE OF QUANTUM CONTEXTUALITY BETWEEN OUR INEQUALITY AND THE EXTENDED KCBS INEQUALITY

The degree of quantum contextuality is given by [27]

$$r_n = \frac{Q_n}{\alpha_n},$$
 (E1)

which is the ratio between the maximum quantum violation and the noncontextual bound.

The extended KCBS inequality reads [24]

$$I'_{n} = \sum_{j=1}^{n} P(0,1|j,j+1) \overset{\text{NCHV}}{\leqslant} \alpha'_{n} \overset{\text{QT}}{\leqslant} Q'_{n},$$
(E2)

where $n \ge 5$ and n is odd. Here the classical bound is

$$\alpha'_n = \frac{n-1}{2},$$
 (E3)

and the maximum quantum value can be realized in the threedimensional quantum system as

$$Q'_n = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)}.$$
 (E4)

Then one can obtain the degree of quantum contextuality for the extended KCBS inequality as

$$r'_{n} = \frac{Q'_{n}}{\alpha'_{n}} = \frac{2n\cos(\pi/n)}{(n-1)[1+\cos(\pi/n)]}.$$
 (E5)

In Table 7, we list r_n and r'_n with different *n* for the inequalities in Eqs. (D1) and (E2), respectively. From Table 7, we plot the curves of *r* versus *n* for the two inequalities in Eqs. (D1) and (E2), as shown in Fig. 6.

For any *n*, the inequality in Eq. (D1) leads to a degree of quantum contextuality higher than that obtained by the extended KCBS noncontextuality inequality in Eq. (E2).

Table 5.	Quantum State $ \psi angle$ and Measurements $\dot{\gamma}$	τ _i (j =	1, 2,, <i>n</i>), fe	or the	Maximum	Quantum	Value of	I_n^a
----------	---	------------------	------------	-----------------------	--------	---------	---------	----------	---------

		•		
$ u_1 angle$	1	0	$\lambda \sin \theta$	$\lambda \cos \theta$
$ \nu_2\rangle$	$\cos(t)$	$\sin(t)$	$\lambda \sin \theta$	$\lambda \cos \theta$
$ \nu_3\rangle$	$\cos(2t)$	$\sin(2t)$	$\lambda \sin \theta$	$\lambda \cos \theta$
:			:	÷
$ \nu_{\frac{n-1}{2}}\rangle$	$\cos(\frac{n-3}{2}t)$	$\sin(\frac{n-3}{2}t)$	$\lambda \sin \theta$	$\lambda \cos \theta$
$ \nu_{\frac{n+1}{2}}\rangle$	$\kappa\lambda \sin\theta \cos(\frac{t}{2})$	$\kappa\lambda \sin\theta \sin(\frac{t}{2})$	1	0
$ \nu_{\frac{n+3}{2}}\rangle$	$\kappa\lambda \sin\theta \cos(\frac{3t}{2})$	$\kappa\lambda \sin\theta \sin(\frac{3t}{2})$	1	0
$ \nu_{\frac{n+5}{2}}\rangle$	$\kappa\lambda \sin\theta \cos(\frac{5t}{2})$	$\kappa\lambda \sin\theta \sin(\frac{5t}{2})$	1	0
: 2				:
$ u_{n-1} angle$	$\kappa\lambda \sin\theta \cos(\frac{n-2}{2}t)$	$\kappa\lambda \sin\theta \sin(\frac{n-2}{2}t)$	1	0
$ \nu_n\rangle$	0	0	0	1
$ \psi angle$	0	0	$\cos x$	$\sin x$
				[a // 1)]

"Here τ_j are the projectors onto the vectors $|\nu_j\rangle$ (non-normalized) and $\rho = |\psi\rangle\langle\psi|$, $t = (n-3)\pi/(n-1)$, $k = -1/\cos(t/2)$, and $\lambda = \sqrt{\cos[2\pi/(n-1)]}$.

Table 6. Numerical Value of Quantum Bound^a

n	7	11	15	19	23	27	31
θ	$\frac{1}{2} \arccos[(\sqrt{33} - 7)/2]$	0.269	0.198	0.158	0.131	0.112	0.098
x	$\frac{1}{2} \arccos[-\sqrt{(23-4\sqrt{33})/11}]$	1.106	1.219	1.285	1.329	1.362	1.386
Q_n	$\frac{1}{2}(\sqrt{33}+1)$	5.730	7.849	9.903	11.932	13.950	15.962
n	35	39	43	63	83	103	
θ	0.087	0.079	0.071	-0.049	-0.037	0.030	
x	1.406	1.421	1.434	1.666	1.644	1.512	
Q_n	17.970	19.975	21.980	31.990	41.994	51.996	

"For the optimal values of the parameters θ and x, one can obtain the maximum quantum violation Q_n in Eq. (D1).

Table 7. Numerical Values of the Ratios r_n and r'_n with n for the Inequalities in Eqs. (D1) and (E2)

n	5	7	11	15	19	23	27	31	35	39	43	63	83	103
r_n	_	1.124	1.146	1.121	1.100	1.085	1.073	1.064	1.057	1.051	1.047	1.032	1.024	1.020
r'_n	1.118	1.106	1.077	1.060	1.048	1.041	1.035	1.031	1.027	1.025	1.022	1.016	1.011	1.010

Interestingly, the maximum degree of quantum contextuality $r'_5 = 1.118$ of the KCBS inequality with n = 5 is also lower than $r_7 = 1.124$ of our inequality with n = 7.



Fig. 6. Curve of the ratio r_n versus *n*. The red curve represents the relationship of r_n with *n* in our paper, while the blue curve represents the relationship of r'_n with *n* for the extended KCBS noncontextuality inequality.

APPENDIX F: OPTIMAL NONCONTEXTUALITY INEQUALITY WITH WEIGHTS OF c = 2 AND d = 1

The exclusivity graph in Fig. 1 of the main text is a family of symmetric *n*-vertex graphs. Since the symmetry, (n - 1)/2 vertices in the outer circle should possess the same weight, which is denoted by *c*. Similarly, the (n - 1)/2 vertices in the inner circle should possess the same weight, which is denoted by *d*. We then have a more general noncontextuality inequality as

$$I_{n} = c \sum_{i=1}^{(n-1)/2} P(\tau_{i}\rho|\rho) + d \sum_{i=(n+1)/2}^{n-1} P(\tau_{i}\rho|\rho) + P(\tau_{n}\rho|\rho) \sum_{i=1}^{NCHV} \alpha_{n} \leqslant Q_{n},$$
(F1)

where n = 4m + 3 (m = 1, 2, ...) and the weights $c, d \ge 0$. Notice that in Eq. (F1), for simplicity, three weights have been reduced, and the weight of $P(\tau_n \rho | \rho)$ is set to 1.

Now we would like to discuss on the optimization of Eq. (F1) within the framework related to the symmetric graphs. For the noncontextuality inequality [Eq. (F1)], a question is raised: what are the optimal values of c and d when the



Fig. 7. Different regions for α_n , where p_i are boundary points, $p_1 = (0,0), p_2 = (0,2/(n-1))$, and $p_3 = (2,1)$.

quantum-classical ratio Q_n/α_n is the highest? The answer will be c = 2 and d = 1 as the following proof.

For the proof, the state $|\psi\rangle$, and the *n* measurement vectors $|\nu_i\rangle$ shown in Table 3 in the main text, from the left-hand side of Eq. (F1), we can directly obtain the quantum value as

$$Q_{n}(c, d, x, \theta) = c \frac{n-1}{2} \frac{\lambda^{2} \sin^{2}(\theta + x)}{1+\lambda^{2}} + d \frac{n-1}{2} \frac{\cos^{2} x}{r^{2} \lambda^{2} \sin^{2} \theta + 1} + \sin^{2} x.$$
 (F2)

By analyzing the deterministic probability assignments for the NCHV case, we can have

$$\alpha_n(c,d) = \max\left\{\frac{n-1}{2}d, 1 + \frac{n-3}{4}c, d + \frac{n-3}{4}c\right\}.$$
 (F3)

So the quantum-classical ratio is

$$r_n(c,d,x,\theta) = \frac{Q_n(c,d,x,\theta)}{\alpha_n(c,d)}.$$
 (F4)

Notice that Q_n is a linear function of c and d for any given (x, θ) as shown in Eq. (F2) and α_n is also a linear function of c and d in different regions as illuminated in Fig. 7. Therefore, r_n is a monotone function of c and d in each region. This implies that the maximum value of r_n for given (x, θ) should appear at a certain boundary point within a region. A direct calculation shows that $r_n \leq 1$ for all the cases except for $(c, d) = p_3 = (2, 1)$. When c = 2 and d = 1, Eq. (F1) degenerates into Eq. (3) in the main text, and then the quantum violation of the Hardy-like proof implies that

$$Q_n^{\max} \ge \frac{(n-1)}{2} + \cos^2\left(\frac{2\pi}{n-1}\right),$$
 (F5)

which is easily obtained directly by adding Eqs. (1) and (2) in the main text. Since $\alpha_n = (n-1)/2$ in this case, $r_n^{\text{max}} > 1$. Thus, we prove that the maximum value of r_n can be obtained when (c, d) = (2, 1) only.

Remark 3. Here we would like to provide a detail proof for the case of n = 7. In this case, the quantum bound becomes

$$Q_7(c, d, x, \theta) = c \sin^2(\theta + x) + \frac{3d \cos^2(x)}{2 - \cos(2\theta)} + \sin^2(x),$$
(F6)

and the noncontextual bound is

$$\alpha_7(c,d) = \max\{3d, 1+c, d+c\}.$$
 (F7)

Thus, the quantum-classical ratio reads

$$r_7(c,d,x,\theta) = \frac{Q_7(c,d,x,\theta)}{\alpha_7(c,d)}.$$
 (F8)

• If assuming that 3d is maximum, we have $\alpha_7 = 3d$, and then

$$\begin{cases} 3d \ge 1 + c \quad \Rightarrow d \ge (1 + c)/3, \\ 3d \ge d + c \quad \Rightarrow d \ge c/2. \end{cases}$$

The quantum-classical ratio is

$$r_{7}(c,d,x,\theta) = \frac{Q_{7}(c,d,x,\theta)}{\alpha_{7}(c,d)} = \frac{c\sin^{2}(\theta+x) + \frac{3d\cos^{2}(x)}{2-\cos(2\theta)} + \sin^{2}(x)}{3d}$$
$$= \frac{c}{3d}\sin^{2}(\theta+x) + \frac{\cos^{2}(x)}{2-\cos(2\theta)} + \frac{1}{3d}\sin^{2}(x)$$
$$\leq \frac{2d}{3d}\sin^{2}(\theta+x) + \frac{\cos^{2}(x)}{2-\cos(2\theta)} + \frac{1}{3d}\sin^{2}(x)$$
$$= \frac{2}{3}\sin^{2}(\theta+x) + \frac{\cos^{2}(x)}{2-\cos(2\theta)} + \frac{1}{3d}\sin^{2}(x).$$
(F9)

From the above formula, we know that the larger d, the smaller r_7 .

1. Supposing $c/2 \ge (1 + c)/3$, then $c \ge 2$ and $d \ge c/2$. From Eq. (F9), $r_7(c, d, x, \theta)$ has the maximum value when d = c/2, and the quantum-classical ratio is

$$r_{7}(c, x, \theta) = \frac{Q_{7}(c, d, x, \theta)}{\alpha_{7}(c, d)}$$
$$= \frac{c}{3\frac{c}{2}} \sin^{2}(\theta + x) + \frac{\cos^{2}(x)}{2 - \cos(2\theta)} + \frac{1}{3\frac{c}{2}} \sin^{2}(x)$$
$$= \frac{2}{3} \sin^{2}(\theta + x) + \frac{\cos^{2}(x)}{2 - \cos(2\theta)} + \frac{2}{3c} \sin^{2}(x),$$
(F10)

implying that the larger *c*, the smaller r_7 . Then the maximum value of r_7 can be obtained when c = 2 and d = 1.

2. Supposing $(1+c)/3 \ge c/2$, then $c \le 2$ and $d \ge (1+c)/3$.

From Eq. (F9), $r_7(c, d, x, \theta)$ has the maximum value when d = (1 + c)/3, and the quantum-classical ratio is

$$r_{7}(c, x, \theta) = \frac{Q_{7}(c, d, x, \theta)}{\alpha_{7}(c, d)}$$

$$= \frac{c}{3\frac{1+c}{3}}\sin^{2}(\theta + x) + \frac{\cos^{2}(x)}{2 - \cos(2\theta)} + \frac{1}{3\frac{1+c}{3}}\sin^{2}(x)$$

$$= \frac{c}{1+c}\sin^{2}(\theta + x) + \frac{\cos^{2}(x)}{2 - \cos(2\theta)}$$

$$+ \frac{1}{1+c}\sin^{2}(x),$$
(F11)

implying that r_7 is a linear function of *c* for any given (x, θ) . Then the maximum value of r_7 for given (x, θ) can be obtained at the boundary point of *c*. From Eq. (F11), when $c \rightarrow 0$, we have

$$r'_{7}(x,\theta) = \frac{\cos^{2}(x)}{2 - \cos(2\theta)} + \sin^{2}(x).$$
 (F12)

In this case, r'_7 can get the maximum value of 1 when $\theta = 0$ and $x = \pi/2$.

When c = 2, we have

$$r_7''(x,\theta) = \frac{2}{3}\sin^2(\theta+x) + \frac{\cos^2(x)}{2-\cos(2\theta)} + \frac{1}{3}\sin^2(x).$$
 (F13)

In this case, r_7'' has the maximum value of 1.124 > 1 when $\theta = 0.446$ and x = 0.808. Then the maximum value of r_7 can be also obtained when c = 2 and d = 1.

• If assuming that 1 + c is maximum, we have $\alpha_7 = 1 + c$, and then

$$\begin{cases} 1+c \ge 3d \Rightarrow d \le (1+c)/3, \\ 1+c \ge d+c \Rightarrow d \le 1. \end{cases}$$

The quantum-classical ratio is

$$r_{7}(c, d, x, \theta) = \frac{Q_{7}(c, d, x, \theta)}{\alpha_{7}(c, d)}$$

$$= \frac{c \sin^{2}(\theta + x) + \frac{3d \cos^{2}(x)}{2 - \cos(2\theta)} + \sin^{2}(x)}{1 + c}$$

$$= \frac{c}{1 + c} \sin^{2}(\theta + x) + \frac{3d}{1 + c} \frac{\cos^{2}(x)}{2 - \cos(2\theta)}$$

$$+ \frac{1}{1 + c} \sin^{2}(x).$$
(F14)

From the above equation, we know that the larger d, the larger r_7 .

1. Supposing $1 \le (1 + c)/3$, then $c \ge 2$ and $d \le 1$.

From Eq. (F14), $r_7(c, d, x, \theta)$ has the maximum value when d = 1. Then we have the quantum-classical ratio,

$$r_7(c, x, \theta) = \frac{c \sin^2(\theta + x) + \frac{3 \cos^2(x)}{2 - \cos(2\theta)} + \sin^2(x)}{1 + c}.$$
 (F15)

Clearly, r_7 is a linear function of *c* for any given (x, θ) . Then the maximum value of r_7 for given (x, θ) can be obtained at the boundary point of *c*. From Eq. (F15), when $c \to +\infty$, $r_7 \to 1$. But through a direct calculation, when c = 2, we can get the quantum-classical ratio as Eq. (F13), and then the maximum value of r_7 can be obtained when c = 2 and d = 1.

2. Supposing $(1 + c)/3 \le 1$, then $c \le 2, d \le (1 + c)/3$. From Eq. (F14), $r_7(c, d, x, \theta)$ has the maximum value when d = (1 + c)/3. Then, we have the quantum-classical ratio,

$$r_{7}(c, x, \theta) = \frac{c \sin^{2}(\theta + x) + \frac{1 + c}{3} \frac{3 \cos^{2}(x)}{2 - \cos(2\theta)} + \sin^{2}(x)}{1 + c}$$
$$= \frac{c}{1 + c} \sin^{2}(\theta + x) + \frac{\cos^{2}(x)}{2 - \cos(2\theta)} + \frac{1}{1 + c} \sin^{2}(x)$$
(F16)

This is similar to Eq. (F11), and the maximum value of r_7 can be obtained when c = 2 and d = 1.

• If assuming that d + c is maximum, we have $\alpha_7 = d + c$, and then

$$\begin{cases} d+c \ge 3d \Rightarrow d \le c/2, \\ d+c \ge 1+c \Rightarrow d \ge 1. \end{cases}$$

1. If c/2 < 1, the above conditions are not valid. 2. If $c/2 \ge 1$, then $c \ge 2$ and $1 \le d \le c/2$.

The quantum-classical ratio is

$$r_{7}(c,d,x,\theta) = \frac{Q_{7}(c,d,x,\theta)}{\alpha_{7}(c,d)} = \frac{c\sin^{2}(\theta+x) + \frac{3d\cos^{2}(x)}{2-\cos(2\theta)} + \sin^{2}(x)}{d+c}$$

$$\leq \frac{c\sin^{2}(\theta+x) + \frac{3d\cos^{2}(x)}{2-\cos(2\theta)} + \sin^{2}(x)}{3d}$$

$$= \frac{c}{3d}\sin^{2}(\theta+x) + \frac{3d}{3d}\frac{\cos^{2}(x)}{2-\cos(2\theta)} + \frac{1}{3d}\sin^{2}(x)$$

$$= \frac{c}{3d}\sin^{2}(\theta+x) + \frac{\cos^{2}(x)}{2-\cos(2\theta)} + \frac{1}{3d}\sin^{2}(x).$$
(F17)

From the above formula, we know that the larger d, the smaller r_7 . Then r_7 can obtain the maximum value when d = 1. The quantum-classical ratio is

$$r_{7}(c, x, \theta) = \frac{Q_{7}(c, d, x, \theta)}{\alpha_{7}(c, d)}$$
$$= \frac{c \sin^{2}(\theta + x) + \frac{3 \cos^{2}(x)}{2 - \cos(2\theta)} + \sin^{2}(x)}{1 + c}.$$
 (F18)

This is the same as Eq. (F15), and the maximum value of r_7 can be obtained when c = 2 and d = 1.

Now we have finished the proof for the case of n = 7.

Remark 4. In addition, for the case of n = 7, we can also calculate the values of the inequality of the other seven vertices, and it shows that Eq. (3) in the main text leads to a higher degree of quantum contextuality than any other noncontextuality inequality in the form $\sum_{i=1}^{7} w_i P_i \leq \alpha_7$, where w_i is the weight of the rank-one projector P_i for the *i*th vertex and α_7 is the classical bound. More intuitively, the maximum degree of quantum contextuality for the extended KCBS inequality with n = 7 is 1.106 [19], while the maximum degree of quantum contextuality for our inequality with n = 7 is 1.124, which is a higher degree of quantum contextuality under the same number of measurement settings.

APPENDIX G: EXPERIMENTAL DETAILS

Equally weighted superpositions of LG modes with different OAMs can be written as

$$LG_{l_1, l_2}(r, \varphi) = \frac{1}{\sqrt{2}} [LG_{l_1}(r, \varphi) + e^{i\xi} LG_{l_2}(r, \varphi)], \quad (G1)$$

where ξ denotes the relative phase between the two modes. In our experiment, the +1st and -1st diffraction orders of the blazed gratings carry the OAMs we needed. A non-optimized initial state we obtained experimentally can be expressed as

$$\begin{split} |\phi\rangle_{\text{non}} &= (A|H,+1\rangle + Be^{i\xi}|H,+3\rangle) + e^{i\Delta}(A|V,-1\rangle \\ &+ Be^{i\xi}|V,-3\rangle), \end{split} \tag{G2}$$

where the two weights (*A* and *B*, here be not normalized) and the phase ξ can be controlled by the holographic grating, and

Table 8. Efficiencies of the Two Cascaded QPs Sandwiched between Two QWPs

Input State	Output State	Efficiency
$ H, +1\rangle$	$ H,0\rangle$	66.9%
$ V, -1\rangle$	$ H,0\rangle$	66.8%
$ H, +3\rangle$	$ H,0\rangle$	66.5%
$ V, -3\rangle$	$ H,0\rangle$	66.7%

the phase Δ is controlled by the RG. After optimizing the initial state, we obtain the state we needed, as described in Eq. (5).

First, we test the efficiencies of two cascaded QPs with topological charges $q_1 = 1/2$ and $q_2 = 1$ in our experiment, as shown in Table 8. We can see that the two cascaded QPs have almost the same efficiency for the OAMs needed in our experiment. This is the guarantee of realizing projection measurements. In the experiment, the detection efficiency of SPAD is about 60% at 810 nm. Therefore, the overall detection efficiency of the OAM system is about 40%. The detection efficiency can be improved by high efficiency detector and coating *q*-plates.

Second, it is necessary to satisfy the no-signaling conditions [29] in any quantum contextuality. As mentioned in Ref. [22], the influences can be written as

$$\delta(_, 0|o_j, o_k) = |P(0|o_k) - P(0, 0|o_j, o_k) - P(1, 0|o_j, o_k)|,$$

$$\delta(_, 1|o_j, o_k) = |P(1|o_k) - P(0, 1|o_j, o_k) - P(1, 1|o_j, o_k)|, \quad (G3)$$

$$\delta(0, |o_j, o_k) = |P(0|o_j) - P(0, 0|o_j, o_k) - P(0, 1|o_j, o_k)|,$$

$$\delta(1, |o_j, o_k) = |P(1|o_j) - P(1, 0|o_j, o_k) - P(1, 1|o_j, o_k)|, \quad (G4)$$

where

$$P(0,0|o_j, o_k) = P(0|o_j) - P(0,1|o_j, o_k),$$

$$P(0,1|o_j, o_k) = P(0|o_j)P(1|o_k),$$

$$P(1,0|o_j, o_k) = P(1|o_j) - P(1,1|o_j, o_k),$$

$$P(1,1|o_j, o_k) = P(1|o_j)P(1|o_k).$$
(G5)

Research Article

Equation (G3) represents the statistical effect of the first measurement on the second measurement, indicating that the result of projecting the state onto o_k is 0 and 1, respectively. Meanwhile, Eq. (G4) indicates the statistical effect of the second measurement on the first measurement, implying that the result of projecting the state onto o_j is 0 and 1, respectively. In order to obtain $P(0,1|o_j, o_k) = P(0|o_j)P(1|o_k)$, we need to prepare the orthogonal state $|v_i^{\perp}\rangle$ based on the Lüder rule [30],

$$|\nu_{j}^{\perp}\rangle = \frac{(I - |\nu_{j}\rangle\langle\nu_{j}|\rangle)|\phi\rangle}{[\langle\phi|(I - |\nu_{j}\rangle\langle\nu_{j}|)|\phi\rangle]^{1/2}},$$
 (G6)

where I is the identity matrix. In our experiment, the orthogonal states are shown as

$$\begin{aligned} |\nu_{1}^{\perp}\rangle &= \frac{1}{\sqrt{3}}(0,1,1,1)^{T}, |\nu_{2}^{\perp}\rangle = \frac{1}{\sqrt{3}}(1,0,1,1)^{T}, \\ |\nu_{3}^{\perp}\rangle &= \frac{1}{\sqrt{3}}(1,1,0,1)^{T}, |\nu_{4}^{\perp}\rangle = \frac{1}{\sqrt{2}}(0,1,1,0)^{T}, \\ |\nu_{5}^{\perp}\rangle &= \frac{1}{\sqrt{3}}(1,0,1,0)^{T}, |\nu_{6}^{\perp}\rangle = \frac{1}{\sqrt{2}}(1,1,0,0)^{T}, \\ |\nu_{7}^{\perp}\rangle &= \frac{\sqrt{3}}{6}(1,1,1,3)^{T}. \end{aligned}$$
(G7)

As stated in the main text, all the measurements should be implemented in a complete set. In our experiment, the statistics of each count is assumed to follow the Poisson distribution. The measured value for any quantity is obtained by averaging the 100 randomly grouped counting sets. The errors are evaluated by the standard deviation. Based on the experimental data, we easily calculate the probabilities. Substituting Eq. (G5) into Eq. (G4), we get $\delta(0, _|o_j, o_k) = \delta(1, _|o_j, o_k) = 0$. The errors described in Eq. (G4) determine the experimental precision. The influence results shown in Fig. 8 mean no-signaling between the first and second measurements.

Funding. Alexander von Humboldt-Stiftung; Nankai Zhide Foundation; Tianjin Research Innovation Project for Postgraduate Students (2019YJSB033); National Key Research and Development Program of China (2017YFA0303700, 2017YFA0303800); National Natural



Fig. 8. Experimental results for $\delta(_, 0|o_j, o_k)$ (red circles) and $\delta(0, _|o_j, o_k)$ (black squares) in (a) and for $\delta(_, 1|o_j, o_k)$ (red circles) and $\delta(1, _|o_i, o_j)$ (black square) in (b). The numbers of 1 to 24 (even numbers are not marked in figures) represent the settings of (o_1, o_2) , (o_1, o_3) , (o_1, o_4) , (o_1, o_6) , (o_2, o_1) , (o_2, o_3) , (o_2, o_4) , (o_2, o_5) , (o_3, o_1) , (o_3, o_2) , (o_3, o_6) , (o_4, o_1) , (o_4, o_2) , (o_4, o_7) , (o_5, o_2) , (o_5, o_3) , (o_5, o_7) , (o_6, o_1) , (o_6, o_3) , (o_6, o_7) , (o_7, o_4) , (o_7, o_5) , and (o_7, o_6) . Error bars are calculated from the counting statistics.

Science Foundation of China (11534006, 116741841, 11774183, 11875167, 11901317, 12075001, 12104135); China Postdoctoral Science Foundation (2018M631726, 2018M640471); Collaborative Innovation Center of Extreme Optics.

Disclosures. The authors declare no conflicts of interest.

Data Availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

[†]These authors contributed equally to this paper.

REFERENCES

- S. Kochen and E. P. Specker, "The problem of hidden variables in quantum mechanics," J. Math. Mech. 17, 59–87 (1967).
- A. Cabello, M. Gu, O. Gühne, and Z. P. Xu, "Optimal classical simulation of state-independent quantum contextuality," Phys. Rev. Lett. 120, 130401 (2018).
- A. Cabello, M. Gu, O. Gühne, J.-Å. Larsson, and K. Wiesner, "Thermodynamical cost of some interpretations of quantum theory," Phys. Rev. A 94, 052127 (2016).
- M. Howard, J. Wallman, V. Veitch, and J. Emerson, "Contextuality supplies the 'magic' for quantum computation," Nature 510, 351– 355 (2014).
- A. Cabello, V. D'Ambrosio, E. Nagali, and F. Sciarrino, "Hybrid ququart-encoded quantum cryptography protected by Kochen-Specker contextuality," Phys. Rev. A 84, 030302 (2011).
- B. Yan, "Quantum correlations are tightly bound by the exclusivity principle," Phys. Rev. Lett. 110, 260406 (2013).
- G. Kirchmair, F. Zähringer, R. Gerritsma, M. Kleinmann, O. Gühne, A. Cabello, R. Blatt, and C. F. Roos, "State-independent experimental test of quantum contextuality," Nature 460, 494–497 (2009).
- R. Lapkiewicz, P. Li, C. Schaeff, N. Langford, S. Ramelow, M. Wieśniak, and A. Zeilinger, "Experimental non-classicality of an indivisible quantum system," Nature 474, 490–493 (2011).
- O. Moussa, C. A. Ryan, D. G. Cory, and R. Laflamme, "Testing contextuality on quantum ensembles with one clean qubit," Phys. Rev. Lett. **104**, 160501 (2010).
- M. Jerger, Y. Reshitnyk, M. Oppliger, A. Potočnik, M. Mondal, A. Wallraff, K. Goodenough, S. Wehner, K. Juliusson, N. K. Langford, and A. Fedorov, "Contextuality without nonlocality in a superconducting quantum system," Nat. Commun. 7, 12930 (2016).
- A. Peres, "Two simple proofs of the Kochen-Specker theorem," J. Phys. A 24, L175 (1991).
- A. Cabello, J. Estebaranz, and G. García-Alcaine, "Bell-Kochen-Specker theorem: a proof with 18 vectors," Phys. Lett. A 212, 183–187 (1996).
- M. Kernaghan and A. Peres, "Bell-Kochen-Specker theorem for 20 vectors," J. Phys. A 27, L829 (1994).
- S. Yu and C. H. Oh, "State-independent proof of Kochen-Specker theorem with 13 rays," Phys. Rev. Lett. 108, 030402 (2012).

- A. Peres, "Incompatible results of quantum measurements," Phys. Lett. A 151, 107–108 (1990).
- N. D. Mermin, "Simple unified form for the major no-hidden-variables theorems," Phys. Rev. Lett. 65, 3373–3376 (1990).
- A. Cabello, "Experimentally testable state-independent quantum contextuality," Phys. Rev. Lett. 101, 210401 (2008).
- E. Amselem, M. Rådmark, M. Bourennane, and A. Cabello, "Stateindependent quantum contextuality with single photons," Phys. Rev. Lett. **103**, 160405 (2009).
- A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, "Simple test for hidden variables in spin-1 systems," Phys. Rev. Lett. **101**, 020403 (2008).
- A. Cabello, S. Severini, and A. Winter, "(Non-)Contextuality of physical theories as an axiom," arXiv:1010.2163 (2010).
- A. Cabello, S. Severini, and A. Winter, "Graph-theoretic approach to quantum correlations," Phys. Rev. Lett. **112**, 040401 (2014).
- G. Cañas, E. Acuña, J. Cariñe, J. F. Barra, E. S. Gómez, G. B. Xavier, G. Lima, and A. Cabello, "Experimental demonstration of the connection between quantum contextuality and graph theory," Phys. Rev. A 94, 012337 (2016).
- I. Bengtsson, K. Blanchfield, and A. Cabello, "A Kochen–Specker inequality from a SIC," Phys. Lett. A 376, 374–376 (2012).
- A. Cabello, P. Badziąg, M. Terra Cunha, and M. Bourennane, "Simple Hardy-like proof of quantum contextuality," Phys. Rev. Lett. 111, 180404 (2013).
- L. Hardy, "Nonlocality for two particles without inequalities for almost all entangled states," Phys. Rev. Lett. 71, 1665–1668 (1993).
- B. Marques, J. Ahrens, M. Nawareg, A. Cabello, and M. Bourennane, "Experimental observation of Hardy-like quantum contextuality," Phys. Rev. Lett. **113**, 250403 (2014).
- B. Amaral, M. Terra Cunha, and A. Cabello, "Quantum theory allows for absolute maximal contextuality," Phys. Rev. A 92, 062125 (2015).
- A. Cabello, "Simple method for experimentally testing any form of quantum contextuality," Phys. Rev. A 93, 032102 (2016).
- Y. Xiao, Z. P. Xu, Q. Li, J. S. Xu, K. Sun, J. M. Cui, Z. Q. Zhou, H. Y. Su, A. Cabello, J. L. Chen, C. F. Li, and G. C. Guo, "Experimental observation of quantum state-independent contextuality under nosignaling conditions," Opt. Express 26, 32–50 (2018).
- G. Molina-Terriza, J. P. Torres, and L. Torner, "Twisted photons," Nat. Phys. 3, 305–310 (2007).
- L. J. Kong, R. Liu, W. R. Qi, Z. X. Wang, S. Y. Huang, Q. Wang, C. Tu, Y. Li, and H. T. Wang, "Manipulation of eight-dimensional Bell-like states," Sci. Adv. 5, eaat9206 (2019).
- L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, "Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes," Phys. Rev. A 45, 8185–8189 (1992).
- X. L. Wang, J. P. Ding, W. J. Ni, C. S. Guo, and H. T. Wang, "Generation of arbitrary vector beams with a spatial light modulator and a common path interferometric arrangement," Opt. Lett. 32, 3549–3551 (2007).
- L. Marrucci, E. Karimi, S. Slussarenko, B. Piccirillo, E. Santamato, E. Nagali, and F. Sciarrino, "Spin-to-orbital conversion of the angular momentum of light and its classical and quantum applications," J. Opt. **13**, 064001 (2011).
- G. Lüders, "Über die zustandsänderung durch den meßprozeß," Ann. Phys. 8, 322–328 (1951).