## PHOTONICS Research

# Iterative freeform lens design for optical field control 

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#### Abstract

It is still very challenging to determine a freeform lens for converting a given input beam into a prescribed output beam where not only the irradiance distribution but also the phase distribution hardly can be expressed analytically. Difficulties arise because the ray mapping from the input beam to the output beam is not only intertwined with the required double freeform surfaces but also intertwined with the output phase distribution, whose gradient represents the directions of the output rays. Direct determination of such a problem is very difficult. Here, we develop a special iterative wavefront tailoring (IWT) method to tackle this problem. In a certain iteration, the current calculation data of the double freeform surfaces and the output phase gradient are used to update the coefficients of a Monge-Ampère equation describing an intermediate wavefront next to the entrance freeform surface. The solution to the wavefront equation could lead to an improved ray mapping to be used to update the corresponding phase gradient data and reconstruct the double freeform surfaces. In a demonstrative example that deviates much from the paraxial or small-angle approximation, the new IWT method can generate a freeform lens that performs much better than that designed by a conventional ray mapping method for producing two irradiance distributions in the forms of numerals " 1 " and " 2 " on two successive targets, respectively. © 2021 Chinese Laser Press


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## 1. INTRODUCTION

A beam transformer converts a given incident beam into a prescribed output beam with certain irradiance and phase (or wavefront) distributions [1]. It has various applications, including lithography, material processing, laser or LED projector, optical communications, and light detection and ranging (lidar). Refractive, reflective, and diffractive optical elements can be used for different configurations of beam transformers [1]. Here, we focus on the commonly used refractive or reflective beam transformers, where ray optics are usually used in the design process. The design problem is mainly governed by three types of equations [2]: the energy conservation within a bundle of rays, the ray-tracing equations dominated by Snell's law in vector form, and the Malus-Dupin theorem that describes the equal optical path lengths (OPLs) between the input and output wavefronts. Moreover, surface continuity should be considered for fabrication issues. Generally, the design problem is very difficult to handle because the output beam to be produced could have arbitrary amplitude and phase distributions without any a priori simplifying assumptions. Direct formulation of the design problem by merging the three types of equations into
only one could be impossible, especially for the case that the output phase cannot be represented analytically.

Traditional beam transformers are commonly used for the cases in which the input and output beam wavefronts are kept planar and the irradiance distributions are rotationally symmetric. In the 1960 s, Frieden [3] and Kreuzer [4] independently introduced a pair of plano-aspherical lens systems for converting a TEM00 Gaussian laser beam into a circular uniform beam while the output wavefront is kept planar. In their methods, the solutions for the aspheric surfaces can be formulated as an integral equation. Rhodes and Shealy [5] derived a set of ordinary differential equations for achieving the same goal. Many later designs can transform the Gaussian laser beam into collimated output beams with Fermi-Dirac, super-Gaussian, super-Lorentzian, and the flattened Lorentzian distributions (see, e.g., Ref. [6]). Doskolovich et al. [7] proposed an axisymmetric refractive optical element design for a more general case in which the input and output beam wavefronts are rotationally nonplanar.

Freeform beam transformer designs without rotational symmetries are mostly restricted to paraxial approximations. In a double-mirror system design, Nemoto et al. [8] first determined
the coordinate relationships, i.e., the ray mapping, between the input Gaussian beam and the rectangular uniform output beam by applying variable separation to the energy conservation, and then obtained the first mirror surface by applying stationary phase method to the Fresnel diffraction integral and acquired the second mirror surface by compensating for the phase distortion. Shealy and Chao [9] designed a two-mirror system for transforming an elliptical Gaussian beam into a rectangular uniform output beam while keeping the beam collimated. After obtaining an analytical expression of the ray mapping that is independent in two orthogonal directions, they integrated the two reflective freeform surfaces explicitly from two firstorder partial differential equations (PDEs). In the design of a plano-freeform lens pair, Feng et al. [10] calculated the var-iable-separable ray mapping numerically and constructed the double freeform surfaces point by point following the ray mapping. Such a method was extended for tackling nonseparable beam shaping by computing the ray mapping with the help of the $L^{2}$ optimal transport (OT) theory [11,12]. However, Bösel et al. demonstrated that the $L^{2}$ OT ray mapping is only accurate for paraxial or small-angle approximation in the design of double freeform surfaces for collimated beam shaping [13].

Most of the existing methods applicable for nonparaxial cases are devised for planar or spherical beam wavefronts. For these cases, the design problem can be formulated as a nonlinear PDE of the Monge-Ampère (MA) type [14,15]. The mathematical derivation starts with expressing the target plane coordinates as functions of the first freeform surface and its gradient, where the second freeform surface can be eliminated by imposing the OPL constancy condition. The resulting raytracing equations are then imported into the energy conservation between the incident and outgoing beam irradiance distributions to obtain the final MA equation, where the surface integrability condition is imposed to ensure a smooth surface. Such a direct determination has the advantage that the design problem can be described by only one equation but at the cost of an extremely complicated and tedious derivation process. Alternatively, collimated beam shaping problems for reflective or refractive designs can be described using OT theories that acquire an integrable ray mapping by finding a solution to a linear programming problem [16-19]. Rubinstein and Wolansky [20] showed that a single lens with two freeform surfaces for shaping arbitrary collimated beams can also be found by solving a variational problem related to the weighted least action. Oliker et al. [21] designed the required pair of planofreeform lenses using the supporting quadric method (SQM) combined with OT. Mingazov et al. [22] proposed a new version of the SQM for collimated beam shaping by translating it into the gradient method of maximizing a concave function. Doskolovich et al. [23] showed that the computation of an integral ray mapping is reduced to an OT problem with a nonquadratic cost function that can be solved by a linear assignment algorithm. Yadav et al. [24] proposed a generalized least squares method for solving either a single lens with double freeform surfaces or two separate plano-freeform lenses in collimated beam shaping. Wei et al. extended [25] the symplectic flow-mapping scheme for irradiance tailoring [26] to the design of double freeform surfaces for transforming an elliptic

Gaussian beam into a convergent beam with a complex irradiance distribution.

There are very few design methods available for complex input and output wavefronts. Feng et al. [27] provided a ray-mapping method in designing double freeform surfaces that can produce two prescribed irradiance distributions on two successive target planes. However, they still employ the restrictive $L^{2}$ OT ray mapping, which is generally not accurate for a nonparaxial case. Bösel et al. [28] showed the ability to generate an astigmatic wavefront while forming a complex irradiance distribution. Instead of formulating the problem into one PDE, they directly solve the three nonlinear PDEs for the first freeform surface and the projected mapping coordinates, respectively, and then calculate the second freeform surface based on the OPL constancy. Such a strategy needs more computational power, and the ability to generate a more general wavefront without an analytical expression is not demonstrated.

We propose a new iterative wavefront tailoring (IWT)-based method to tackle the freeform lens design problem for producing a prescribed output beam with a complex irradiance distribution and phase profile. A previous IWT method is developed for designing a freeform lens that can generate a prescribed irradiance distribution on a planar or curved target through the intermediate construction of an outgoing wavefront immediately behind the exit freeform surface [29,30]. This method can simplify the formula derivation process and flexibly generate a variety of freeform optical structures with high accuracy. The major contribution of this work consists in how to deal with the difficulties arising from the additional generation of a complex output phase profile, which could realize optical field control using freeform optics in a simple, flexible, and accurate way. The method is described in detail in Section 2. We verify the method's effectiveness in Section 3, where we design a double freeform lens for generating an output beam with complex irradiance and phase distributions that can produce a different irradiance distribution at a certain distance behind the first output plane. A comparison with the $L^{2}$ OT ray mapping method is also included in Section 3.

## 2. METHOD

We consider here the design of a freeform lens with two freeform surfaces performing a one-to-one mapping of the input optical field into a prescribed output optical field, as sketched in Fig. 1. The amplitude and phase distributions of the input beam are denoted by $A(u, v)$ and $\phi(u, v)$ respectively, where $(u, v) \in \Omega \subset \mathbb{R}^{2}$. The amplitude and phase distributions of the output beam are denoted by $A^{\prime}(\xi, \eta)$ and $\phi^{\prime}(\xi, \eta)$, respectively, where $(\xi, \eta) \in \Gamma \subset \mathbb{R}^{2}$. An arbitrary light ray that goes through point $\mathbf{S}=(u, v, h)$ at the input plane strikes the first freeform surface at point $\mathbf{P}=\left(x_{1}, y_{1}, z_{1}\right)$ and then strikes the second freeform surface at point $\mathbf{Q}=\left(x_{2}, y_{2}, z_{2}\right)$. The output light ray intersects with the output plane at point $\mathbf{T}=(\xi, \eta, d)$. The input wavefront located at a given distance from the input plane is described by position vector $\tilde{\mathbf{W}}=(\tilde{x}, \tilde{y}, \tilde{z})$, and the output wavefront located at a given distance from the output plane is described by position vector $\tilde{\mathbf{W}}^{\prime}=\left(\tilde{x}^{\prime}, \tilde{y}^{\prime}, \tilde{z}^{\prime}\right)$. The refractive indices of the medium at the left side of the first freeform surface and the right side of the second freeform surface


Fig. 1. Sketch of the freeform beam transformer.
are denoted by $n_{1}$, and the refractive index of the freeform lens is denoted by $n_{2}$.

In the irradiance control IWT method, an outgoing wavefront located behind the exit surface of the freeform lens to be designed is adopted to help solve a ray mapping. For the problem of optical field control, we employ an intermediate wavefront that is next to the entrance freeform surface to be designed. This intermediate wavefront is described by position vector $\mathbf{W}=(x, y, w)$, as illustrated in Fig. 1. In the following, we will first establish the parametrized intermediate wavefront equation, and then describe the IWT procedure specified for optical field control. For the sake of completeness, we also describe the double-freeform surface reconstruction from a ray mapping.

## A. Parametrized Intermediate Wavefront Equation

Since we consider a lossless system, and there is no crossing of rays from the input plane to the target plane, the irradiance distributions on the two planes should satisfy the energy conservation in differential form,

$$
\begin{equation*}
I(u, v)=I^{\prime}(\xi(u, v), \eta(u, v))\left|\xi_{u} \eta_{v}-\xi_{v} \eta_{u}\right| \tag{1}
\end{equation*}
$$

where $I(u, v)=A(u, v)^{2}, I^{\prime}(\xi, \eta)=A^{\prime}(\xi, \eta)^{2}$, and the subscripts here denote partial derivatives.

We now describe the light propagation properties inside the lens with the help of the intermediate wavefront $\mathbf{W}=$ $(x, y, w)$. According to Fermat's principle, the first partial derivatives of $w$ with respect to $x$ and $y$ can be described as

$$
\begin{equation*}
w_{x}=-\left(x_{2}-x\right) /\left(z_{2}-w\right), \quad w_{y}=-\left(y_{2}-y\right) /\left(z_{2}-w\right) \tag{2}
\end{equation*}
$$

Both $x$ and $y$ can be thought as functions of the independent variables $u$ and $v$, and according to chain's rule,

$$
\begin{equation*}
w_{u}=w_{x} x_{u}+w_{y} y_{u}, w_{v}=w_{x} x_{v}+w_{y} y_{v} \tag{3}
\end{equation*}
$$

Combining Eq. (2) and Eq. (3), we have

$$
\left\{\begin{array}{l}
w_{x}=\left(y_{v} w_{u}-y_{u} w_{v}\right) / \gamma=-\left(x_{2}-x\right) /\left(z_{2}-w\right)  \tag{4}\\
w_{y}=\left(x_{u} w_{v}-x_{v} w_{u}\right) / \gamma=-\left(y_{2}-y\right) /\left(z_{2}-w\right)
\end{array}\right.
$$

Herein, $\gamma=x_{u} y_{v}-x_{v} y_{u}$. From Eq. (4), we can express $\left(x_{2}, y_{2}\right)$ as

$$
\left\{\begin{array}{l}
x_{2}=x+\left(z_{2}-w\right)\left(y_{u} w_{v}-y_{v} w_{u}\right) / \gamma  \tag{5}\\
y_{2}=y+\left(z_{2}-w\right)\left(x_{v} w_{u}-x_{u} w_{v}\right) / \gamma
\end{array}\right.
$$

We denote the unit outgoing ray vectors as $\mathbf{O}=$ $\left(O_{1}, O_{2}, O_{3}\right)$. In the framework of geometrical optics, the
phase is equivalent to the rays, and the ray directions are determined by the first partial derivatives of the phase (see, e.g., Refs. [31,32]). Therefore, the relationship between $\mathbf{O}$ and the output phase $\phi^{\prime}(\xi, \eta)$ is

$$
\begin{equation*}
O_{1}=\phi_{\xi}^{\prime}, O_{2}=\phi_{\eta}^{\prime}, O_{3}=\sqrt{1-\phi_{\xi}^{\prime 2}-\phi_{\eta}^{\prime}} \tag{6}
\end{equation*}
$$

$(\xi, \eta)$ and $\left(x_{2}, y_{2}\right)$ can be linked through a collinear relationship,

$$
\left\{\begin{array}{l}
\xi=x_{2}+\left(d-z_{2}\right) O_{1} / O_{3}=x_{2}+s  \tag{7}\\
\eta=y_{2}+\left(d-z_{2}\right) O_{2} / O_{3}=y_{2}+t
\end{array}\right.
$$

Herein, $s$ means the displacement between $\xi$ and $x_{2}$, and $t$ means the displacement between $\eta$ and $y_{2}$. Inserting Eq. (5) into Eq. (7), we obtain

$$
\left\{\begin{array}{l}
\xi=x+s+\left(z_{2}-w\right)\left(y_{u} w_{v}-y_{v} w_{u}\right) / \gamma  \tag{8}\\
\eta=y+t+\left(z_{2}-w\right)\left(x_{v} w_{u}-x_{u} w_{v}\right) / \gamma
\end{array}\right.
$$

Generally, $s, t$, and also $z_{2}$ are dependent on $\xi$ and $\eta$. For a degenerate case in which the output beam wavefront is planar, both $s$ and $t$ are zeros, and the difficulty mainly lies in handling $z_{2}$. Such a problem is similar to the irradiance control problem on a curved target where the $z$-values are not constant [30]. Here, we focus on a more complicated case where the output phase is difficult to be expressed analytically. In such a case, $s$ and $t$, which are directly relevant with the phase gradient, are difficult to be expressed explicitly with $\xi$ and $\eta$.

Our solution for simplifying the complicated optical field control problem is to retain $s$ and $t$ on the right side of Eq. (8) and consider them as functions of the independent variables of $u$ and $v$. Equation (8) and its differentiation with respect to $u$ and $v$ are inserted into Eq. (1) to eliminate the two variables $\xi$ and $\eta$. The resulting equation is a second-order PDE of MA type,

$$
\begin{equation*}
w_{u u} w_{v v}-w_{u v}^{2}+A_{1} w_{u u}+A_{2} w_{u v}+A_{3} w_{v v}+A_{4}=0 \tag{9}
\end{equation*}
$$

where the coefficients $A_{i}(i=1,2,3,4)$ are functions depending on $x, y, w, s, t, z_{2}$, and their first partial derivatives, and the second partial derivatives of $x$ and $y$ (see Appendix A). Rewrite Eq. (8) as a vector-valued function, $(\xi, \eta)=\mathbf{m}(u, v)$, and then a nonlinear boundary condition to Eq. (9) can be specified as

$$
\begin{equation*}
\mathbf{m}(\partial \Omega) \rightarrow \partial \Gamma \tag{10}
\end{equation*}
$$

where $\partial \Omega$ and $\partial \Gamma$ are the boundaries of $\Omega$ and $\Gamma$, respectively. You may notice that Eq. (9) is unable to be solved unless $x, y, s, t$, and $z_{2}$ are given in advance. However, $x, y, s, t$, and $z_{2}$ are coupled with the lens data. To solve this contradiction, we introduce the following iterative procedure.

## B. IWT Procedure for Optical Field Control

The IWT procedure for optical field control is illustrated in Fig. 2, and its detailed description is included as follows.

Step 0: We first define an input ray sequence described by the unit input ray vector $\mathbf{I}(u, v)$ and an input wavefront $\tilde{\mathbf{W}}(u, v)$ from the given input beam information. To start the iterative process, we need to provide an initial estimate of the ray mapping, $(\xi, \eta)=(\xi(u, v), \eta(u, v))$.

Step 1: From the ray mapping, we acquire the corresponding data of the first partial derivatives of the output phase, $\phi_{\xi}^{\prime}(\xi(u, v), \eta(u, v))$ and $\phi_{\eta}^{\prime}(\xi(u, v), \eta(u, v))$. Then, we immediately determine an output ray sequence


Fig. 2. Diagram of the IWT method for optical field control.
$\mathbf{O}(\xi(u, v), \eta(u, v))$ according to Eq. (6). From the output ray sequence, we can reconstruct an output wavefront $\tilde{\mathbf{W}}^{\prime}(\xi(u, v), \eta(u, v))$ based on least squares, where we employ a concise relationship between the wavefront points and the outgoing ray vectors [27],

$$
\left\{\begin{array}{l}
\left(\tilde{\mathbf{W}}_{i+1, j}^{\prime}-\tilde{\mathbf{W}}_{i, j}^{\prime}\right) \cdot\left(\mathbf{O}_{i+1, j}+\mathbf{O}_{i, j}\right)=0  \tag{11}\\
\left(\tilde{\mathbf{W}}_{i, j+1}^{\prime}-\tilde{\mathbf{W}}_{i, j}^{\prime}\right) \cdot\left(\mathbf{O}_{i, j+1}+\mathbf{O}_{i, j}\right)=0
\end{array}\right.
$$

Equation (11) means that the chord linking the adjacent wavefront points is perpendicular to the average of the outgoing vectors through the two points. Equation (11) can be written as a linear system of equations of the distances between $\mathbf{T}$ and $\tilde{\mathbf{W}}^{\prime}$, which can be solved with least squares based on Hermann's method [33] and the MATLAB backslash operator [34].

Step 2: Since we have determined the input ray sequence and wavefront, and the output ray sequence and wavefront, we can compute the double freeform surfaces $\mathbf{P}(u, v)=$ $\left(x_{1}(u, v), y_{1}(u, v), z_{1}(u, v)\right)$ and $\mathbf{Q}(u, v)=\left(x_{2}(u, v), y_{2}(u, v)\right.$, $\left.z_{2}(u, v)\right)$ to realize the desired transformation using least squares based on Snell's law and the condition of the OPL constancy [27]. More details are included in Subsection 2.C.

Step 3: After obtaining the data of the double freeform surfaces, we can determine $s$ and $t$. We can also obtain an intermediate ray sequence $\mathbf{R}=\operatorname{Unit}(\mathbf{Q}-\mathbf{P})=(\mathbf{Q}-\mathbf{P}) /|\mathbf{Q}-\mathbf{P}|$, from which we reconstruct an intermediate wavefront $\mathbf{W}=(x, y, \tilde{w})$ using least squares. Herein, $\tilde{w}$ is used to differentiate from $w$ in Eq. (9).

Step 4: We insert the data of $x, y, s, t$, and $z_{2}$ into Eq. (9) and Eq. (10). Taking $\tilde{w}$ as the initial value, we solve the MA equation numerically. After obtaining a solution $w$, we can acquire a new ray mapping, $(\xi, \eta)=(\xi(u, v), \eta(u, v))$, based on Eq. (8).

Step 5: The new ray mapping is inserted into Step 1 to repeat the loop. The stop criterion can be chosen as the difference
between the current ray mapping and the previous one or a certain number of iterations.

Step 6: After obtaining the satisfied ray mapping, we need to specify its corresponding output ray sequence and wavefront as in Step 1 and reconstruct the final double freeform surfaces as in Step 2.

Such an iterative procedure can make the freeform lens design for complicated optical field control easier to implement. Although a sequence of MA equations needs to be solved, a multiscale strategy [29,30] for the irradiance control IWT methods can be adapted to speed up the computation here.

## C. Double Freeform Surface Reconstruction from the Ray Mapping

Here, we recall the reconstruction of double freeform surfaces following the one-to-one mapping between an input ray sequence and an output ray sequence using a least squares-based iterative procedure [27], which is effective and fast. The basic steps are as follows.

Step 0: We first give an initial approximation of the entrance freeform surface, which can be simply set as a plane.

Step 1: After that, we can obtain a corresponding exit freeform surface based on the constant OPL condition,

$$
\begin{equation*}
n_{1}[\tilde{\mathbf{W}}, \mathbf{P}]+n_{2}[\mathbf{P}, \mathbf{Q}]+n_{1}\left[\mathbf{Q}, \tilde{\mathbf{W}}^{\prime}\right]=\text { Const, } \tag{12}
\end{equation*}
$$

where $[\mathbf{A}, \mathbf{B}]$ refers to the distance between $\mathbf{A}$ and $\mathbf{B}$.
Step 2: The required unit ray sequence $\mathbf{R}$ inside the lens can be acquired as $\mathbf{R}=\operatorname{Unit}(\mathbf{Q}-\mathbf{P})$. After that, the required normal field of the first freeform surface can be acquired based on Snell's law in vector form, $\mathbf{N}=\operatorname{Unit}\left(n_{2} \mathbf{R}-n_{1} \mathbf{I}\right)$.

Step 3: Notice that the normal field $\mathbf{N}$ is not necessarily integrable and there generally does not exist an exact freeform surface that is perpendicular to $\mathbf{N}$ everywhere. Therefore, we also employ a least squares method to compute an approximate freeform surface $\mathbf{P}$, and the basic relationship between $\mathbf{P}$ and $\mathbf{N}$ is [27]

$$
\left\{\begin{array}{l}
\left(\mathbf{P}_{i+1, j}-\mathbf{P}_{i, j}\right) \cdot\left(\mathbf{N}_{i+1, j}+\mathbf{N}_{i, j}\right)=0  \tag{13}\\
\left(\mathbf{P}_{i, j+1}-\mathbf{P}_{i, j}\right) \cdot\left(\mathbf{N}_{i, j+1}+\mathbf{N}_{i, j}\right)=0
\end{array}\right.
$$

Equation (13) can be written as a linear system of equations of the distances between $\tilde{\mathbf{W}}$ and $\mathbf{P}$, which can also be solved using Hermann's method and the MATLAB backslash operator.

Step 4: The updated entrance freeform surface $\mathbf{P}$ is inserted into Step 1 to obtain a new exit freeform surface $\mathbf{Q}$. The above process is repeated until a required stop criterion is satisfied.

## 3. RESULTS

As a demonstrative example, we design a freeform lens with the refractive index of 1.5 for transforming a divergent elliptical Gaussian beam into an specified output beam with a complex irradiance distribution and phase profile (see Fig. 3). The input beam is emitted from a point light source located at the original point, and the $1 / e^{2}$ full width divergence angles for $x$ and $y$ directions are $12^{\circ}$ and $36^{\circ}$, respectively. Figure 3(a) shows the input beam irradiance $I(u, v)$ on the $z=20 \mathrm{~mm}$ plane, where $(u, v) \in \Omega=\{(u, v)| | u|\leq 2.1021 \mathrm{~mm},|v| \leq 6.4984 \mathrm{~mm}\}$. The input beam phase distribution $\phi(u, v)$, which corresponds to a spherical wavefront, is shown in Fig. 3(b). Figure 3(c)


Fig. 3. (a) Irradiance and (b) phase distributions of the input beam on the $z=20 \mathrm{~mm}$ plane; the desired (c) irradiance and (d) phase distributions of the output beam on the $z=45 \mathrm{~mm}$ plane, which is aimed for producing (e) a second output irradiance distribution on the $z=75 \mathrm{~mm}$ plane. The phase distributions are normalized to have zero average values.
shows the desired output beam irradiance distribution $I^{\prime}(\xi, \eta)$ on the plane of $z=45 \mathrm{~mm}$, where $(\xi, \eta) \in \Gamma=$ $\{(\xi, \eta)|\xi| \leq 12 \mathrm{~mm},|\eta| \leq 12 \mathrm{~mm}\}$. Figure 3(d) shows the desired output phase distribution $\phi^{\prime}(\xi, \eta)$, which is rather irregular and can be difficult to describe by an analytical formula. This phase distribution is aimed for converting $I^{\prime}(\xi, \eta)$ into a different irradiance distribution $I^{\prime \prime}\left(\xi^{\prime}, \eta^{\prime}\right)$ on the plane of $z=75 \mathrm{~mm}$ [see Fig. 3(e)], where $\left(\xi^{\prime}, \eta^{\prime}\right) \in \Gamma^{\prime}=$ $\left\{\left(\xi^{\prime}, \eta^{\prime}\right)\left|\left|\xi^{\prime}\right| \leq 16 \mathrm{~mm},\left|\eta^{\prime}\right| \leq 16 \mathrm{~mm}\right\}\right.$. $\phi^{\prime}(\xi, \eta)$ is obtained by numerically solving a phase-retrieval problem from the two output irradiance distributions [35],

$$
\begin{align*}
& \phi_{\xi \xi}^{\prime} \phi_{\eta \eta}^{\prime}-\phi_{\xi \eta}^{\prime 2}+\chi\left(1-\phi_{\eta}^{\prime 2}\right) \phi_{\xi \xi}^{\prime}+2 \chi \phi_{\xi}^{\prime} \phi_{\eta}^{\prime} \phi_{\xi \eta}^{\prime}+\chi\left(1-\phi_{\xi}^{\prime 2}\right) \phi_{\eta \eta}^{\prime} \\
& \quad+\chi^{4} d^{\prime 2}\left\{1-I^{\prime}(\xi, \eta) / I^{\prime \prime}\left[(\xi, \eta)+\nabla \phi^{\prime} / \chi\right]\right\}=0, \tag{14}
\end{align*}
$$

where $d^{\prime}$ denotes the distance between the two output planes, $\chi=\sqrt{1-\left|\nabla \phi^{\prime}\right|^{2}} / d^{\prime}$, and the coordinates $\left(\xi^{\prime}, \eta^{\prime}\right)$ for describing $I^{\prime \prime}\left(\xi^{\prime}, \eta^{\prime}\right)$ have been replaced with $(\xi, \eta)+\nabla \phi^{\prime} / \chi$.

The computation size is desired as $256 \times 256$. To speed up the computation, we employ a multiscale strategy that is similar to that in Refs. [29,30]. The initial computation size is set to $32 \times 32$. In this step, the initial ray mapping is obtained from solving a standard MA equation corresponding to the $L^{2}$ OT (see e.g., Ref. [27]),

$$
\begin{equation*}
\psi_{u u} \psi_{v v}-\psi_{u v}^{2}=I(u, v) / I^{\prime}(\nabla \psi) \tag{15}
\end{equation*}
$$

where $\psi$ is a potential whose gradient $\nabla \psi=(\xi, \eta)$. After implementing the computations on the grid of $32 \times 32$ points for three iterations, the final $32 \times 32$ points ray mapping is linearly interpolated into the one of $64 \times 64$ points, which is used to start the iterative computations on the grid of $64 \times 64$ points. Such a process is repeated until implementing the computations on the grid of $256 \times 256$ points for three iterations. The final ray mapping specifies the final output ray sequence


Fig. 4. Evolution of the $\epsilon$ value as the iteration increases.
$\mathbf{O}_{i, j}$ and wavefront $\tilde{\mathbf{W}}_{i, j}^{\prime}$, both of which are used for obtaining the final entrance and exit freeform surfaces according to Section 2.C. We solve the standard MA equation shown in Eq. (15) and the generalized MA equation shown in Eq. (9) [and also Eq. (14)] following the numerical technique outlined in Ref. [29]. A Newton-Krylov solver, Kelley's nsoli.m is adopted to solve the nonlinear system of equations obtained from discretizing the MA equations [36], where maxit (the maxmium number of nonlinear iterations) is set as 200 , maxitl (the upper bound for linear iteration per nonlinear iteration) is set as 20 , and etamax (the maximum error tolerance for residual in inner iteration) is set as 0.1 . Since $I^{\prime}(\xi, \eta)$ can be difficult to represent analytically, we employ linear interpolation to acquire its values corresponding to the scattered grid points of $(\xi, \eta)$ in each iteration of the Newton-Krylov solver. We also use linear interpolation to obtain $\phi_{\xi}^{\prime}(\xi, \eta)$ and $\phi_{\eta}^{\prime}(\xi, \eta)$ when obtaining a ray mapping in each IWT iteration. The computations are implemented in MATLAB 2020a on an Intel Core i7-6700 k CPU at 4.0 GHz with 48 GB RAM, running Windows 10 ( 64 bit), which takes around $2.43 \mathrm{~s}, 11.68 \mathrm{~s}, 40.11 \mathrm{~s}$, and 93.93 s for the grid points of $32 \times 32,64 \times 64,128 \times 128$, and $256 \times 256$, respectively. The solution to the MA equations accounts for more than $90 \%$ of the total running time.

Let $\epsilon$ denote the mean absolute deviation of the angles between the computed normals to the entrance freeform surface, i.e., $\mathbf{P}_{u} \times \mathbf{P}_{v}$, and the required normals for reconstructing the entrance freeform surface in each IWT iteration. Figure 4 illustrates the effectiveness of the proposed multiscale IWT algorithm in the reduction of the $\epsilon$ value. The $\epsilon$ value for the final entrance freeform surface is $0.0064^{\circ}$. Figures $5(\mathrm{a})$ and 5(b) show the final entrance and exit freeform surfaces, respectively, where you can observe structures that can cause difficulties in analytical expression.

The scattered data points of the double freeform surfaces are inserted into Rhinoceros to create a solid lens model. We implement Monte Carlo ray tracing with $2 \times 10^{7}$ rays in LightTools 9.0 to demonstrate the performance of the designed freeform lens. Figure 6(a) visualizes the ray-tracing results of the designed freeform lens in LightTools. Figures 6(b) and 6(c) show the simulated irradiance distributions on the first and second output planes, respectively. It can be seen that the simulated irradiance distribution on the first output plane agrees well with the desired output irradiance distribution shown in Fig. 3(c). The simulated irradiance distribution on the


Fig. 5. Designed (a) entrance and (b) exit freeform surfaces.
second output plane is also in line with the desired one shown in Fig. 3(e), implying that the output phase on the first output plane is achieved well.

We provide an $L^{2}$ OT ray-mapping design for comparison. The $L^{2}$ OT ray mapping is also calculated in a multiscale way: the computed potential $\psi$ by solving Eq. (15) for a coarser grid is linearly interpolated into an initial value for a finer grid. The double-surface reconstruction following the $L^{2}$ OT ray mapping is the same as that described in Section 2.C. Since the design geometry is far beyond the paraxial and small-angle approximation, the $\epsilon$ value for the $L^{2}$ OT ray-mapping design is as high as $2.3197^{\circ}$, resulting in severe deformations in the simulation results, as shown in Figs. 6(d), 6(e), and 6(f).

We employ the relative root-mean-square deviation (RRMSD) to quantify the deviation of the simulated
irradiance distribution from the prescribed one: RRMSD $=$ $\left\|\mathbf{I}_{\text {pres }}-\mathbf{I}_{\text {simu }}\right\|_{F} /\left\|\mathbf{I}_{t}\right\|_{F}$, where $\mathbf{I}_{\text {pres }}$ and $\mathbf{I}_{\text {simu }}$ denote the matrices of the prescribed and simulated irradiance distributions, respectively, within the given domain, and $\|\cdot\|_{F}$ refers to the Frobenius norm [37]. For the $L^{2}$ OT ray-mapping method, the RRMSD values for the two simulated irradiance distributions can be higher than 0.329 . The RRMSD values can be greatly reduced to be lower than 0.045 when we employ the new proposed method.

Figure 7 (a) provides the final ray mapping for the new proposed method. Figure $7(\mathrm{~b})$ shows the $L^{2}$ OT ray mapping. It seems that the two ray mappings demonstrate similar deformations: those regions corresponding to the edges of the numerals are more strongly deformed. However, sufficient differences exist between the two ray mappings, as visualized in Fig. 7(c). The average distance between the two ray mapping is 0.2775 mm . As can be seen from Fig. 7(c), the ray-mapping differences around the central region have relatively small amplitudes, demonstrating that $L^{2}$ OT ray mapping is more accurate for the paraxial region. However, it does not seem that the central regions on the two simulated irradiance distributions have better performances than the surrounding regions. The holes in the two simulated irradiance distributions of the $L^{2}$ OT raymapping design are caused by a small "bump" around the center of the entrance freeform surface. This small bump is a result of implementing least squares surface reconstruction with a preset value of the center surface point when the normal error is large. We can observe whirls in Fig. 7(c), which may reflect the fact that the $L^{2}$ OT ray mapping cannot lead to curlfree surface integration. Note that larger errors will occur when we reconstruct the double freeform surfaces according to the $L^{2}$ OT ray mapping using a point-by-point integration strategy, as described in Ref. [11].


Fig. 6. Simulated results of the freeform lens designed with the new proposed method: (a) ray-tracing illustration in LightTools, and simulated irradiance distributions on the (b) first and (c) second output planes, respectively. Simulated results of the freeform lens designed with the $L^{2}$ OT ray mapping method: (d) ray-tracing illustration in LightTools, and simulated irradiance distributions on the (e) first and (f) second output planes, respectively. In each design, the mesh sizes for the two simulated irradiance distributions are $288 \times 288$, where the prescribed domains ( $\Gamma$ and $\Gamma^{\prime}$ ) correspond to the central grids of $256 \times 256$ points.


Fig. 7. (a) Final ray mapping of the new proposed method; (b) the $L^{2}$ OT ray mapping; and (c) their vector differences in the form of arrows originated from the grid points shown in (a). For better visualization, each ray mapping with the grid points of $256 \times 256$ is interpolated into that with the grid points of $32 \times 32$.

The discrepancies between the new proposed method and the $L^{2}$ OT ray-mapping method can be narrowed for those designs which satisfy or do not deviate much from paraxial or small-angle approximation. Figure 8 provides a comparison of two freeform lenses designed with the two methods, respectively, for a case that the input beam becomes circular (the $1 / e^{2}$ full width divergence angles for both $x$ and $y$ directions are $36^{\circ}$ ). The RRMSD values for the simulated irradiance distributions of the two freeform lenses become very close. The reason is that the $L^{2}$ OT ray mapping becomes much closer to the ray mapping computed by the new proposed method. The average distance between the two ray mappings is 0.0275 mm . However, the freeform lens designed with the new proposed method still performs better enough in terms of surface accuracy. The $\epsilon$ value for the freeform lens designed with the new proposed method is $0.0071^{\circ}$, while the $\epsilon$ value for the freeform lens designed with the $L^{2}$ OT ray-mapping method is $0.2119^{\circ}$. The normal error difference between the two lenses can result in
larger irradiance discrepancies on receiving planes far away from the lenses.

## 4. CONCLUSION

We have proposed a new IWT-based method in designing double freeform optical surfaces for transforming a given optical field into a prescribed one. This method simplifies the design by gradually tailoring an intermediate wavefront next to the entrance freeform surface and incorporating the data of the output phase gradient and the $z$ coordinate of the second freeform surface associated with the ray mapping into the IWT procedure. The solution to the wavefront equation leads to a ray mapping between the input beam and the output beam. Approximate first and second freeform surfaces are reconstructed following the ray mapping via an iterative reconstruction procedure based on ray-tracing equations and the constancy of OPLs. The new reconstructed double freeform


Fig. 8. Simulated results of the freeform lens designed with the new proposed method: (a) ray-tracing illustration in LightTools, and simulated irradiance distributions on the (b) first and (c) second output planes, respectively. Simulated results of the freeform lens designed with the $L^{2}$ OT ray mapping method: (d) ray-tracing illustration in LightTools, and simulated irradiance distributions on the (e) first and (f) second output planes, respectively.
surfaces can be used to obtain an updated wavefront equation that brings a new ray mapping. Such an iterative process can be implemented in a multiscale way to reduce calculation complexities. Results show that we can obtain a doublefreeform lens generating two different irradiance distributions on two successive target planes with much better performances than those of the $L^{2}$ OT ray mapping method for a case that deviates much from the paraxial or small-angle approximation.

Compared with the previous IWT methods for irradiance control, where a single freeform surface could be enough [29,30], the proposed method can realize additional generation of a prescribed output phase distribution requiring at least double freeform surfaces. This method is also applicable for designing double freeform reflective surfaces or a double planofreeform lenses system. Such a special iteration strategy may also work with other methods, including the direct determination.

Currently, this method is restricted to those cases in which there are no zero values in the output beam irradiance distributions and there are no extreme local curvatures (or singularities) in the lens surfaces and the input and output wavefronts. We also need to mention that the starting geometry of the lens should be chosen to avoid unphysical solutions.

## APPENDIX A

The four coefficients of Eq. (9) are

$$
\begin{aligned}
A_{1}= & {\left[\left(x_{v} y_{v v}-y_{v} x_{v v}\right) w_{u}+\left(y_{v} x_{u v}-x_{v} y_{u v}\right) w_{v}\right] / \gamma } \\
& +\left(\gamma \kappa_{v} w_{v}-X_{v} x_{v}-Y_{v} y_{v}\right) /\left(z_{2}-w\right), \\
A_{2}= & {\left[\left(y_{u} x_{v v}-x_{v} y_{u v}+y_{v} x_{u v}-x_{u} y_{v v}\right) w_{u}\right.} \\
& \left.+\left(x_{v} y_{u u}-y_{u} x_{u v}+x_{u} y_{u v}-y_{v} x_{u u}\right) w_{v}\right] / \gamma \\
& +\left[\left(X_{u} x_{v}+Y_{v} y_{u}+X_{v} x_{u}+Y_{u} y_{v}\right)\right. \\
& \left.-\gamma\left(\kappa_{u} w_{v}+\kappa_{v} w_{u}\right)\right] /\left(z_{2}-w\right), \\
A_{3}= & {\left[\left(x_{u} y_{u v}-y_{u} x_{u v}\right) w_{u}+\left(y_{u} x_{u u}-x_{u} y_{u v}\right) w_{v}\right] / \gamma } \\
& +\left(\gamma \kappa_{u} w_{u}-X_{u} x_{u}-Y_{u} y_{u}\right) /\left(z_{2}-w\right), \\
A_{4}= & {\left[\left(x_{u v} y_{v v}-x_{v v} y_{u v}\right) w_{u}^{2}+\left(x_{v v} y_{u u}-x_{u u} y_{v v}\right) w_{u} w_{v}\right.} \\
& \left.+\left(x_{u u} y_{u v}-x_{u v} y_{u u}\right) w_{v}^{2}\right] / \gamma \\
& +\left[\kappa_{u} /\left(z_{2}-w\right)\left(x_{v} y_{v v}-y_{v} x_{v v}\right) w_{u}^{2}\right. \\
& +\left(x_{u} y_{u v}-y_{u} x_{u v}\right) w_{v}^{2} \\
& \left.+\left(y_{u} x_{v v}+y_{v} x_{u v}-x_{v} y_{u v}-x_{u} y_{v v}\right) w_{u} w_{v}\right] \\
& +\left[\kappa_{v} /\left(z_{2}-w\right)\left(y_{v} x_{u v}-x_{v} y_{u v}\right) w_{u}^{2}+\left(y_{u} x_{u u}-x_{u} y_{u u}\right) w_{v}^{2}\right. \\
& \left.+\left(x_{v} y_{u u}+x_{u} y_{u v}-y_{u} x_{u v}-y_{v} x_{u u v}\right) w_{u} w_{v}\right] \\
& +\left[1 /\left(z_{2}-w\right)\left(X_{u} x_{v v}-Y_{v} y_{u v}-X_{v} x_{u v}+Y_{u} y_{v v}\right) w_{u}\right. \\
& \left.+\left(Y_{v} y_{u u}-X_{u} x_{u v}+X_{v} x_{u u}-Y_{u} y_{u v}\right) w_{v}\right] \\
& +\left[1 / \kappa\left(z_{2}-w\right)\left(X_{u} \kappa_{v}-X_{v} \kappa_{u}\right) \beta\right. \\
& +\left(Y_{v} \kappa_{u}-Y_{u} \kappa_{v}\right) \alpha+\left(X_{u} Y_{v}-X_{v} Y_{u}\right) \\
& \left.-I / I^{\prime}(X+\kappa \alpha, Y+\kappa \beta)\right] .
\end{aligned}
$$

Herein $\quad X=x+s, \quad Y=y+t, \quad \alpha=y_{u} w_{v}-y_{v} w_{u}$, $\beta=x_{v} w_{u}-x_{u} w_{v}, \gamma=x_{u} y_{v}-x_{v} y_{u}$, and $\kappa=\left(z_{2}-w\right) / \gamma$.

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