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Fractal topological band-gap structure induced by singularities in the one-dimensional Thue-Morse system

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The physical origin of the fractal topological band-gap structure in the one-dimensional Thue-Morse system has been revealed, which is characterized by the evolutions of two types of topological singularities with zeroscattering properties and the paths of phase vortex points, which are the mirrored paths of the first-type singularities. The field distribution of the upper and lower gap-edge states will interchange when the traditional gaps are closed and reopened. The topologically protected edge-states are found at both traditional gaps and fractal gaps. Our work broadens the topological properties of quasicrystals or aperiodic systems and provides potential applications in new optoelectronic devices. © 2021 Chinese Laser Press

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1. INTRODUCTION

Topological band-gap (TBG) structures have received extensive attention for electronic and photonic systems [1,2]. Many fantastic topological phenomena are realized in photonic crystals (PhCs) based on the topological photonic band-gap (TPBG) [3,4], such as topological edge states [5-10], Weyl points and nodal lines [11-13], one-way rotating states [14], logic gates [15], and sensors [16]. The TPBGs of one-dimensional (1D) systems are intensively studied, because the systems are simple and can be solved in strict theory. Recently, the connection between the surface impedance of 1D semi-infinite PhC and the geometry phase, also called the Zak phase, was revealed [17]. More importantly, in this original work, a simple method with the self-consistent gauge, based on the reflection coefficient of the semi-infinite 1D PhC models, is established to determine the singularities, which dominate the evolution of TPBG of 1D PhCs. With the introduction of synthetic dimensions or parameter freedoms to the 1D periodic systems, rich topological phenomena, such as the Weyl points and the nodal lines (surfaces) in high-dimensional spaces, are also observed and well studied [11–13,18]. In the reflection phase map, there are phase vortex points (PVPs), but only the PVPs whose eigenvector turns to zero are singularities [11,12]. It is revealed that the different evolutions of singularities between bands correspond to different topological phase transitions [12,13].

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Different from the periodic systems, the quasicrystals (QCs) and aperiodic systems, which lack translational symmetry but possess long-range order, display Bragg diffraction spots, and also have complex band-gap structures. Extremely rich transport properties and complex band-gap structures of QCs also imply the rich topology beyond the periodic systems [19]. The first effort to strictly define the topological properties of bandgap structures is done by Kraus et al. [20,21] for 1D QCs, and the topologically protected edge-states are demonstrated experimentally in photonic systems [20,22,23]. The topology of QCs can also be directly retrieved from diffraction experiments [24]. More excitingly, very recently the phenomenon of fractal TBG (FTBG), which is not observed in the periodic systems, has been found in two-dimensional (2D) Floquet QCs and the Bott index originally to describe the topology of random systems is introduced into QCs to characterize the topological phase transition [25]. Characterized by a none-zero spin Bott index, the quantum spin Hall (QSH) effect is also realized in Penrose-type QC lattices, associated with robust edge-states and guantized conductance [26,27]. However, to date, unlike the periodic systems whose topological phase transitions are from the evolution of singularities, any relation between FTBG of QCs and the singularities has never been suggested nor has ever been explored. Without understanding the relation, we cannot answer certain questions, such as "what is the origin of FTBG and how does it evolve with the structural

parameters of QCs?" or "what are the similarities and differences between TBG of periodic systems and QCs?".

Since there is lack of support from Bloch's theorem, how waves (electrons, photons, or otherwise) are transmitted through QCs or aperiodic systems is not fully resolved to this day [25]. Generally, researchers have to deal with finite structures for the study of QCs, so that the study of finite-length PhCs is more instructive. Also, inspired by the studies of 1D periodic systems [12,13,18] and with the introduction of synthetic dimensions or parameter freedoms to QCs, more complex band-gap structures in higher dimensional space can possibly open a new window for us to understand the FTBG of QCs. The topological properties of the 1D periodic systems are signed by the singularities of the reflection coefficient, which is from the zero-scattering of each cell, resulting in the non-zero Zak phase or Chern number [12,13,17], even for finite periodic systems [28]. Then it is reasonable for us to make the assumption that the topological number of the 1D QCs or aperiodic systems can also be obtained from the singularities characterized by zero-scattering. Precise theory needs to be established based on the Bott index of 1D QCs and aperiodic systems, which has been proved to be equivalent to the Chern number in 2D translationally invariant systems [29]. In previous works [30,31], it is revealed that the band-gaps of QCs and aperiodic systems can be separated into two classes, the traditional gaps (TGs) and the fractal gaps (FGs), due to the scattering from the periodic interfaces and the multi-scale nested structure of interfaces, respectively. It is natural to think that the topological properties of TGs and FGs may be quite different because of their different scattering origin.

In this work, we show that the FTBG features of 1D Thue-Morse (TM) systems with one synthetic dimension are generated by the evolutions of two types of singularities and PVPs. The former are characterized by the zero-scattering of the two basic scatterers that compose the TM systems, while the latter are the ordinary transmission resonance from Bragg scattering. Both TGs and FGs of TM systems are closed and reopened when the evolution paths of the first-type singularities pass through them. The second-type singularities only appear when the optical path ratio between two kinds of layers satisfies the ratio of two odd integers, and is indicated by the closing of TGs. A π -phase-jump of the reflection coefficient when a TG or FG is closed and reopened is the sign of the topological phase transition. Before and after topological phase transition, unlike the 1D periodic systems whose upper and lower gap-edge states will interchange both the spatial inversion symmetry (SIS) and the field distribution, only the latter will appear for TM systems because of the absence of SIS. The topologically protected edge-states are also found for both TGs and FGs. Although this work is done for 1D TM systems, we believe the similar generation mechanism of FTBG is also available for other 1D or higher dimensional QCs and aperiodic systems.

2. STRUCTURE

The 1D TM system can be generated by the recursive relationship $S_n = S_{n-1} + \tilde{S}_{n-1}$, $S_1 = AB$, where the \tilde{S}_{n-1} is obtained by exchanging A and B in the S_{n-1} . For example, an S_3 TM system can be represented as ABBABAAB [30]. The system is

composed of the dielectric layers A and B with widths d_A and d_B , the relative permittivities $\epsilon_A = 4$ and $\epsilon_B = 6.25$, and the relative permeability $\mu_A = \mu_B = 1$, respectively. Without loss of generality, we suppose that the background material outside the finite TM system is the same as the A-kind layer. So all Bkind layers can be thought as scatterers, and they are submerged inside the background material. The electric field in the *i*th layer can be written as $E_i(z) = A_i^+ e^{ik_i(z-z_i)} + A_i^- e^{-ik_i(z-z_i)}$. The transfer matrix between layers can be expressed as $M_i(A_i^+, A_i^-)^T = (A_{i+1}^+, A_{i+1}^-)^T \cdot A_i^+$ and A_i^- are the coefficients of the forward and backward electric field in the *i*th layer. k_i is the wave vector in the *i*th layer. M_i is the transfer matrix between layers. The coefficients of transmission and reflection and the field distribution can be obtained by the transfer matrix of a finite TM system. The central wavelength of the mth TG is $\frac{2}{m}(l_A + l_B)$, where $l_A = \sqrt{\epsilon_A} d_A$ and $l_B = \sqrt{\epsilon_B} d_B$ are optical paths of layer A and B, respectively [31]. In this work, $l_A + l_B$ is set as 600 nm, while the structural parameter is $\sigma = \frac{l_B}{l_A + l_B}$. Our study is focused on the frequency regions below the third TG.

It should be noted that the band-gap structure becomes more and more detailed with the increasing order of TM systems. However, when the TM order *m* is large enough, e.g., m > 4, the main band-gap structure of TM systems becomes stable, and the physics studied in this work is not influenced by the order of TM systems. So when we discuss the TGs and FGs, we generally choose the order *m* (even or odd) of the system larger than four, and it is irrelevant with the topology of band-gap structure of TM systems. The main difference between even-order and odd-order systems is the location of inversion center, which is only important when we use a certain order TM as a super-cell with a periodic boundary condition and study the topology of the band-gap of such periodic systems (see Appendix A).

3. RESULTS AND DISCUSSION

To have a comprehensive understanding about the singularities, we first observe the evolution of the PBG structure with different σ for a finite TM system. Analog to the periodic systems, from the observation of PBG of an S7 TM system, we find that there "seemingly" are two types of singularities with the judgement of perfect transmission [28] and other criterions, such as the π -phase-jump of reflection coefficient, which will be discussed later. Figures 1(a)-1(c) are the transmission spectra of an S_7 TM system with $\sigma = \frac{5}{12}, \frac{1}{2}, \frac{7}{12}$, and the two types of singularities are indicated by the red and blue dots, respectively. The first-type singularities (red dots) satisfy the frequency condition $sin(k_B d_B) = 0$, which is also the condition for the singularities in 1D PhCs [17]. We find that when the first-type singularities pass through a TG or FG, both of them will be closed and reopened. The second-type singularities (blue dots) only appear when the TGs are closed and reopened. The condition for the existence of the second-type singularities will be given in the following section.

To understand the field distribution characteristic of singularities, the electric field |E(z)| (black lines) of states at singularities and the refractive index distributions (blue lines)



Fig. 1. (a)–(c) Transmission spectra of the S_7 TM system with the structural parameter $\sigma = \frac{5}{12}, \frac{1}{2}, \frac{7}{12}$, respectively. The central positions of the first three orders of TGs are labeled by dotted lines. The red dots and blue dots indicate two types of singularities. (d) The refractive index distribution (blue line) and |E(z)| (black line) of the PhC at singularity with frequency $f = 5 \times 10^{14}$ Hz and the same parameters as in (b). (e) and (f) Refractive index distribution (blue lines) of the TM system at two types of singularities with frequencies $f_1 = 5 \times 10^{14}$ Hz and $f_2 = 2.5 \times 10^{14}$ Hz, respectively.

are shown in Figs. 1(d)-1(f) for a 1D PhC and a TM system, respectively. The 1D PhC shown in Fig. 1(d) is composed with the same two kinds of layers as the TM system in Fig. 1(b). It is known that if we take B-kind layers as the scatterers in the background material of A-kind, the zero-scattering occurs at the singularities of 1D PhC [28]. Because of the zero-scattering of B-kind layers, the |E(z)| of a finite 1D PhC in Fig. 1(d) shows two special properties, (i) the flat field profile through the whole system, which means the amplitude of total field is unchanged, (ii) the flat field amplitude in A-kind layers, which means the field in A-kind layers only contains the forward propagating component. Figures 1(e) and 1(f) also show the similar zero-scattering properties at two types of singularities of TM systems. Because of the same frequency condition of the singularities, the |E(z)| shown in Fig. 1(e) has the same properties as those in Fig. 1(d). For the second-type singularities, the |E(z)| is more complex. First, the field profile is still flat through the whole system, which implies the zeroscattering property of the whole TM system. Second, from these A-kind layers with flat field amplitude, we can find that the scattering from S_2 = ABBA and $\tilde{S_2}$ = BAAB is zero for the state of the second-type singularities. Because any TM system whose order is greater than two can be decomposed into $S_2 = ABBA$ and $S_2 = BAAB$, the scattering of the whole TM system should be zero for all second-type singularities.

The physical origin and the field distribution of the firsttype singularities can be explained by the transfer matrix method. Regarding the A-kind layers as the background material, we can find two kinds of scatterers, (A)B(A) and (A)BB(A) in the TM system. The elements of the transfer matrix for scatterer (A)B(A) are $M_{11} = M_{22}^* = \cos(k_B d_B) + \frac{i}{2} \left(\frac{k_A}{k_B} + \frac{k_B}{k_A}\right) \sin(k_B d_B)$, $M_{12} = M_{21}^* = -\frac{i}{2} \left(\frac{k_A}{k_B} - \frac{k_B}{k_A}\right) \sin(k_B d_B)$. When the condition of the first-type singularities $\sin(k_B d_B) = 0$ is satisfied, the transfer matrix becomes *I* or *-I*, where *I* is the unitary matrix. For scatterer (A)BB(A), we can insert an infinite thin A-kind layer between two B-kind layers, and then the transfer matrix of (A)BB(A) becomes *I*. Obviously, at such a TM system, the incident field can not sense the scattering of B-kind layers at all.

The physical origin of the second-type singularities, which appear only when the *m*th TG is closed, can also been analyzed by similar method. For the central wavelength of the *m*th TG, the second-order trace map [32-34] is

$$x_{2} = \left[1 - \frac{1}{2} \left(\frac{\sqrt{\epsilon_{A}}}{\sqrt{\epsilon_{B}}} + \frac{\sqrt{\epsilon_{B}}}{\sqrt{\epsilon_{A}}}\right)\right] \cos(4m\sigma\pi) + 1 + \frac{1}{2} \left(\frac{\sqrt{\epsilon_{A}}}{\sqrt{\epsilon_{B}}} + \frac{\sqrt{\epsilon_{B}}}{\sqrt{\epsilon_{A}}}\right).$$
(1)

The sufficient and necessary condition of the *m*th TG being closed [31] is $x_2 = 2$, and then we have $\sigma = \frac{n}{2m}$, $n \in N^*$. We emphasize that this condition is different from the *m*th-gap-closed condition of 1D PhC, which is $\sigma = \frac{n}{m}$ and also the condition of nodal line (surface) in the higher-dimensional space with synthetic dimensions [12]. The new condition causes several special properties of the TM system. Firstly, for the *m*th TG, there are 2m - 1 closing points with σ changing in the TM system, while there are m - 1 closing points in 1D PhC. Secondly, for the *m*th gap of the TM system, besides the m - 1 closing points which satisfy $\sin(k_B d_B) = 0$, there are *m* second-type singularities satisfying the condition

$$\sigma = \frac{2n-1}{2m}, \quad n \in N^*.$$

The physical explanation of the condition for second-type singularities is the zero-scattering of $S_2 = ABBA$ and $\tilde{S}_2 = BAAB$, which could be thought of as second-order units of the TM system. From the trace map theory [31], all traces of higher order units also satisfy $x_n = x_{n-2}^2(x_{n-1}-2) + 2 = 2$, which means the TGs are closed for all ordered TM systems when the condition $x_2 = 2$ is satisfied. It is noted that the first TG can be closed for TM systems while it is always opened for 1D PhCs.

We note that two types of topological singularities in this work have not been strictly demonstrated so far. The gapclosing-reopening is a phenomenon of topological transition, but could not be used as the judgement of topological transition or the existing of topological singularities. Zero-scattering is a much stronger support of the existing of topological singularities, but still not enough. The theoretical demonstration of two types of singularities could be done by counting the number change of resonant modes in upper and lower bands before and after the gap-closing-reopening, which can be regarded as the strict characteristic of topological phase transition for 1D aperiodic systems [35].

Next we will study the maps of transmission and reflection phases to show the origins and characteristics of FTBG of QCs and aperiodic systems. First, we introduce a phase space $\{\sigma, f\}$ with the synthetic dimension of structural parameter σ and the frequency f dimension. The transmission map of the S_5 TM



Fig. 2. (a) Transmission map of the S_5 TM system in the parameter space { σ , f}. The blue (red) area indicates a gap (band). (b) The reflection phase map of the S_5 TM system. The first-type singularities are located at the singular lines (white lines), which are characterized by the π -phase-jump. The black lines are the mirrored lines. Both of them form the basic structure of the FTBG feature. (c), (d) and (e), (f) Local maps of transmission and reflection phase around the first-type and second-type singularities, respectively. A series of PVPs are indicated by the white circles in (d). (g) Transmission distribution error graph of the S_5 TM system after 500 times is calculated when the disorder intensity is 0.1.

system is shown as Fig. 2(a). The blue (red) area indicates the low (high) transmission region, corresponding to the gap (band) area. At first glimpse, we can see the significant self-similarity feature as the rhombus region signed by the white dashed lines. The self-similarity is from the red (high-transmission) lines cutting the whole map into different regions and the repeated pattern of TG and FG existing in each region. The deeper physics is more clearly shown by the reflection phase map in Fig. 2(b). Theoretically, the absolute phase of the reflection coefficient has no physical meaning because it can be eliminated by redefining the starting point. However, with the strictly defined common starting point, the phase difference of reflection from different systems has physical meaning, which can be used to determine the different topology of the systems. The sign of the reflection phase and the corresponding surface impedance can be used to characterize the topology of photonic crystals [17,36-40], acoustic systems [41,42], and quasicrystals [10,43]. Further more, it is mathematically demonstrated that the topology of the systems, which could be periodic, quasi-periodic or random, could be characterized by the reflection matrix [44,45]. In Fig. 2(b), the red (high-transmission) lines in Fig. 2(a) can be separated into two classes according to their different properties. We find that the white lines characterized by the π -phase-jump satisfy the condition of the first-type singularities $sin(k_B d_B) = 0$, and we call them singular lines (SLs). Obviously, the SLs are evolution paths of the first-type singularities, which will close all TGs and FGs on their paths and cause the topological phase transitions for both TGs and FGs. The black lines are the mirrored lines (MLs) which satisfy the condition $sin(k_A d_A) = 0$, and we will show that these lines connect PVPs. In addition to the two classes of high-transmission lines, the self-similarity in Fig. 2(a) is also characterized by the closing of TGs, which are signed by white circles for the first and second TGs, respectively. As discussed above, the closing points of TGs are corresponding to the second-type singularities which cause the topological phase transition only at TGs. Then we can obtain the origin of self-similar FTBGs in the TM system, which are constructed by the evolution paths of the first-type singularities, their mirrored lines, and the second-type singularities together.

For more details of the FTBG, we carefully studied two special cases of the transmission and reflection phase maps around the two types of singularities, which are shown in Figs. 2(c), 2(d) and Figs. 2(e), 2(f), respectively. In Figs. 2(c) and 2(d), an SL and an ML intersect at the central frequency of the second TG (signed by two white dashed lines). The second TG is closed since the SL cuts through it, and there is a topological transition with a π -phase-jump at two sides of SL. Along the ML represented by the black solid line in Fig. 2(d), we can find a series of PVPs indicated by the white circles which are corresponding to the ordinary resonant transmission for different systems. In the vicinity of the SL and ML intersection, a high-transmission in a wide frequency range is indicated by the white solid line in Fig. 2(c). When $\sigma = \frac{1}{2}$, the relative bandwidth of the transmittance higher than 0.9 can reach 30%. As illustrated as Fig. 2(g), the high-transmission region is also robust against 10% disorder strength introduced into d_A , when the optical length $l_A + l_B$ is kept as constant. Such a robust high transmission case in wide frequency range is quite rare in low-dimension systems, which can be used to design specific optoelectronic devices. In Figs. 2(e) and 2(f), there is a second-type singularity at the central frequency of the first TG when $\sigma = \frac{1}{2}$. Focusing on the frequency range of the first TG (signed by the white dashed lines) and increasing σ continuously near $\sigma = \frac{1}{2}$, a typical topological transition appears such that the first TG becomes thinner, totally closed, and then reopened. Exactly at the point with $f = 2.5 \times 10^{14}$ Hz and $\sigma = \frac{1}{2}$, a π -phase-jump occurs which makes it possible to realize the topological edge-state in the first TG.

An important sign of topological phase transition in 1D binary PhC is the spatial symmetry switching (symmetry to antisymmetry or antisymmetry to symmetry) guaranteed by the SIS for the upper and lower gap-edge states when a gap is closed and reopened [17]. Although the TM systems lack the SIS, we can still find a similar sign for the topological phase transition of such aperiodic systems. For a 1D binary PhC, besides the symmetry switching, there is a field-distributioninterchange (FDI) phenomenon occurring for the upper and lower gap-edge states at the topological phase transition. For example, when the singularity is located at the lower band, the field of the upper gap-edge state is concentrated within the layers, while that of the lower gap-edge state is concentrated at the interfaces between the layers. Actually, such an FDI for gap-edge states widely exists at the topological phase transition in 2D PhCs where the gap is closed as a Dirac point and reopened, and the chiral properties are also interchanged. For a TM system, an FDI clearly exists when the first-type singularity passes through a TG, which is shown in Figs. 3(a)-3(c). Comparing Figs. 3(b) and 3(c), the field distribution almost perfectly interchanges for the upper and lower gap-edge states after the topological phase transition. Such an FDI also exists



Fig. 3. (a) Local transmission map near the second TG of the S_7 TM system in the parameter space. The blue (red) area indicates a gap (band). A (B) is the lower (upper) gap-edge state marked by a yellow dot (triangle) when $\sigma = 0.4$. C (D) is the lower (upper) gap-edge state marked by a green dot (triangle) when $\sigma = 0.6$. (b) The |E(z)| of state A (B) is indicated by the blue (red) lines. (c) The |E(z)| of state C (D) is indicated by the blue (red) lines. The black lines show the refractive index distribution, *s* and *L* means the total length of the system.

for the topological phase transition when the TG is closed by the second-type singularities. To better demonstrate the topologically nontrivial properties of the TM system, we use a certain order TM structure as a super-cell and set periodic boundary conditions to the super-cell, which is the method to study the topology of quasi-crystal systems in other works [10,43]. Then, we obtain all the gaps of TM systems and see the gap-closing-reopening process with different σ . Surprisingly, we find the change of the spacial inversion symmetry of the upper and lower gap-edge states, which is the typical sign of topological phase transitions of periodic systems (see Appendix A).

The topologically protected edge-states are widely studied for topologically nontrivial systems. It is well known that the condition for the existence of an interface state can be represented as $Z_{SR} + Z_{SL} = 0$, where Z_{SR} and Z_{SL} are the surface impedance on the right-hand side and left-hand side of the boundary. The relation between the impedance Z_S and the reflection coefficient r is $Z_S = \frac{1+r}{1-r}Z_0$, where Z_0 is the vacuum impedance [17]. In Fig. 2 and Appendix A, we have shown that, before and after the gap is closed, the reflection phase of the gap changes from 0 to π or $-\pi$ to 0, and the sign of the corresponding surface impedance changes from positive



Fig. 4. (a) Structure composed of two S_7 TM systems which are spliced together. The structural parameters of the left and right TM systems satisfy $\sigma_l = \frac{3}{5}$ and $\sigma_r = \frac{2}{5}$, respectively. The black dashed line indicates the interface, and the blue (yellow) areas indicate A-kind (B-kind) layers. (b) The transmission spectrum of the structure shown in (a). (c) The |E(z)| of the topologically protected edge-state with a frequency satisfying $f = 2.5 \times 10^{14}$ Hz. (d) Edge-states in the space of the reflection phase ϕ and frequency $f {\phi, f}$ when the S_5 TM system with $\sigma = 0.4$ is connected to a reflector with an adjustable phase.

infinity to 0 or 0 to negative infinity. So there must be a topological edge-state when we connect two TM systems with reflection phases $\theta_r \in [0, \pi]$ and $\theta_l \in [-\pi, 0]$, since the phase requirement of a bound state can always be satisfied for a frequency in the gap range. For FTBG, physically, the π -phasejump beside the two types of singularities in Fig. 2 guarantees the existence of the topological edge-states in both TGs and FGs when we connect two TM systems with different σ at two sides of a singularity. The specific structure to realize the topological edge-state at central frequency $f = 2.5 \times 10^{14}$ Hz of the first TG, which is closed at $\sigma = \frac{1}{2}$ by the second-type singularity, is shown as Fig. 4(a). Figure 4(b) shows the transmission spectrum of the structure shown in Fig. 4(a) and a distinct transmission peak at the center of the first TG is marked by a black arrow. Figure 4(c) shows the |E(z)| of the topologically protected edge-state, which exponentially decays at both TM systems from the interface between them.

In fact, to show the existing of a topologically protected edge-state, we choose any value of σ , as long as $Z_{\text{SR}} + Z_{\text{SL}} = 0$ is satisfied in the gap. To better show that the edge-state is topological, we propose to connect the S_5 TM system with $\sigma = 0.4$ which is topologically non-trivial and an artificial reflector with adjustable reflection phase ϕ tuned from 0 to π . As illustrated as Fig. 4(d), the edge-states are represented by the blue circles for different ϕ , and the gray areas indicate the band. Obviously, an edge-state crosses the entire gap, because of the topology difference.

When FGs are closed by the first-type singularities, the topologically protected edge-state can also be realized. A series of FGs of an S_8 TM system are shown in Figs. 5(a) and 5(b), where the white line in the reflection phase map is the SL with π -phase-jump. Nevertheless, the physical origin of FGs is much more complicated [31]. Unlike TGs, the FGs are obviously asymmetric in the parameter space at two sides of the SL shown



Fig. 5. (a) and (b) Maps of transmission and the reflection phase at the initial position of the S_8 TM system in the parameter space. (c) Transmission spectrum near the edge-state corresponding to the structure composed of two S_8 TM systems with $\sigma_l = 0.5458$ and $\sigma_r = 0.6333$. (d) The |E(z)| of the topologically protected edge-state with frequency $f = 4.156907 \times 10^{14}$ Hz.

in Figs. 5(a) and 5(b), which makes it much difficult to choose appropriate structural parameters to realize the topologically protected edge-state between two TM systems [46]. Here we give an example to show the edge-state in the FG. In Fig. 5(c), a distinct transmission peak with the frequency $f = 4.156907 \times 10^{14}$ Hz can be found in the original gap area. The |E(z)| of the topologically protected edge-state is shown in Fig. 5(d) with the exponential decay from the interface between two TM systems.

Further theoretical and experimental research can be carried out from the following aspects. Firstly, the relationship between the topological number of QCs (the change of Bott index) and the evolution of singularities can be explored in detail. Secondly, the robust edge-states with high-Q can be experimentally demonstrated, and the broadband transparent feature caused by the topological properties of QCs can be widely used on optical or electromagnetic device design. Thirdly, the mechanism based on singularities could be extended to the high-dimensional QCs, such as the study of more complicated singularities and the characteristics of their evolution paths.

4. CONCLUSION

In conclusion, we study the FTBG feature of a 1D TM system in the structural parameter-frequency space, which is generated by the evolving paths of two types of topological singularities and the PVPs. The first-type singularities can evolve along π phase-jump lines which satisfy the condition $sin(k_B d_B) = 0$, while the paths of PVPs, which satisfy the condition $sin(k_A d_A) = 0$, are the mirrored lines of the evolving paths of the first-type singularities. The second-type singularities appear in the mth TG when the structural parameter satisfies $\sigma = \frac{2n-1}{2m}, n \in N^*$. The self-similarity of the FTBG feature could be understood by the reappearing satisfaction of these conditions. Different from 1D PhCs, there is no symmetry-interchange at the topological transition in a TM system, while the FDI of the upper and lower gap-edge states will take place in a TM system before and after the TG closing, which can be regarded as a special feature of the topological phase transition in the QCs. The topologically protected edge-states with the exponential decay from the interface between two TM systems are realized at both TGs and FGs. These works will open new windows to understand the fractal topological band-gap structures.

APPENDIX A: THE INVERSION CENTER AND SPACIAL INVERSION SYMMETRY FOR TM SUPER-CELLS WITH PERIODIC BOUNDARY CONDITIONS

Generally, for a finite TM system, there is no inversion center due to the lack of translational symmetry. But, if we use a certain-order-TM structure as a super-cell and set a periodic condition on the super-cell, we can find two inversion centers for both odd- and even-ordered TM super-cells. We call such systems super-cell-TM systems. For an odd-order $S_{2m+1} (m \in N^*)$ TM super-cell, the two inversion centers are the middle positions of the S_{2m} and \tilde{S}_{2m} TM super-cells, while these, for even-order or S_{2m} TM super-cells are the start

point and the middle position, respectively. For example, the S_4 TM system with periodic boundary conditions can be represented as ... |ABBABAAB|BAABABBA|..., and the symbol | means one inversion center. Meanwhile, $\epsilon_{z+z_{\rm inv}}=\epsilon_{\scriptscriptstyle -z+z_{\rm inv}}$ and $E_{z+z_{inv}} = E_{-z+z_{inv}}$, where z_{inv} means the coordinate of the inversion center, are automatically satisfied. The supercell-TM systems have almost the same main gaps as the TM systems, and also show the gap-closing-reopening processes as TM systems with different σ . It has been demonstrated that the reflection phase in the gap range at one inversion center is from 0 to π or $-\pi$ to 0 in the binary 1D PhCs [17]. Similarly, under periodic boundary conditions, the reflection phase of a gap at the inversion center in the super-cell-TM system is also from 0 to π or $-\pi$ to 0. So the gaps can be divided into two types according to the different range of the reflection phase with different topological properties.

It has been demonstrated that the changes of the spatial inversion symmetry of the upper and lower gap-edge states, can be regarded as a strong evidence of topologically nontrivial phase transition [12,13,17]. For the super-cell-TM systems, the reflection phase of the gap-edge can only be 0 or π . The total electric field at the inversion center can be represented as $(1 + e^{i\theta_r})E_i$, where θ_r and E_i are the reflection phase and incident wave, respectively. Choosing a particular inversion center, the gap-edge state is symmetric (S-state) with $\theta_r = 0$, while that is antisymmetric (A-state) with $\theta_r = 0$.

Here we choose an S_5 super-cell-TM system as an example where $\sigma = 0.125$, 0.4, 0.6, 0.875, corresponding to four



Fig. 6. (a) and (b) Maps of transmission and the reflection phase at the initial position of the S_5 TM system under the periodic boundary conditions in the parameter space. A–H represent the lower and upper gap-edge states for $\sigma = 0.125$, 0.4, 0.6, 0.875, respectively. (c)–(j) The |E(z)| of the gap-edge states marked in (a). *L* is the total length of the S_5 TM super-cell.

areas in the second TG of the TM system. The |E(z)| of the upper and lower gap-edge states are shown in Fig. 6. For the certain inversion center (red dash line), the symmetry of |E| of the upper and lower gap-edge states is exchanged before and after the gap is closed. For example, the upper (lower) gap-edge state is A-state (S-state) with $\sigma = 0.125$, while the upper (lower) gap-edge state is S-state (A-state) with $\sigma = 0.4$. In fact, in all processes of gap closing and reopening of TM systems caused by two types of singularities, whether it is TG or FG, this change of spacial inversion symmetry can be observed in the super-cell-TM systems as long as we select the appropriate inversion center. From the similar studies of quasi-crystals with super-cells [10,43], we can conclude that the gap-closing-reopening caused by two kinds of singularities of TM systems is really the topological phase transition.

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- 46. On the transmission map, the width, central frequency, and structure of the FGs are different at two sides of the SL, which is the track of the first-type singularity. If we choose two σ very near the SL, which correspond to the two TM systems, the reflection phase of the left (ϕ_l) and right (ϕ_r) TM systems can match well, which satisfies $\phi_l + \phi_r = 0$. But, due to the slow decaying of the field magnitude, it is difficult to observe the topological edge-state between the two TM systems. However, if we choose σ a little farther away from the SL, due to the gap structure difference, it is difficult to satisfy $\phi_l + \phi_r = 0$ at a certain frequency inside the two gaps.