PHOTONICS Research

Universal single-mode lasing in fully chaotic two-dimensional microcavity lasers under continuous-wave operation with large pumping power [Invited]

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For a fully chaotic two-dimensional (2D) microcavity laser, we present a theory that guarantees both the existence of a stable single-mode lasing state and the nonexistence of a stable multimode lasing state, under the assumptions that the cavity size is much larger than the wavelength and the external pumping power is sufficiently large. It is theoretically shown that these universal spectral characteristics arise from the synergistic effect of two different kinds of nonlinearities: deformation of the cavity shape and mode interaction due to a lasing medium. Our theory is based on the linear stability analysis of stationary states for the Maxwell–Bloch equations and accounts for single-mode lasing phenomena observed in real and numerical experiments of fully chaotic 2D microcavity lasers.

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1. INTRODUCTION

Universality is a key concept in quantum chaos study. We can find a common feature in a quantum system if the corresponding classical system exhibits fully chaotic dynamics [1–3]. The most representative example is the Bohigas–Giannoni–Schmit conjecture concerning universal spectral fluctuations of quantized fully chaotic systems [4–11]. Universality in quantum chaos has also been observed for electron transport in mesoscopic devices [12,13].

Because of an analogy between the classical-quantum and the ray-wave correspondence, quantum chaos theory can be directly applied to "wave-chaotic" systems, which exhibit chaotic dynamics in the ray-optic limit [14–25]. A representative example is a two-dimensional (2D) microcavity laser, whose resonant modes can be viewed as those of quantum billiards [26,27]. By a quantum-chaos approach, the emission patterns of 2D microcavity lasers with various cavity shapes have been successfully explained and predicted [14–27]. However, there is another important aspect of 2D microcavity lasers that cannot be elucidated only by quantum chaos theory, that is, nonlinear interactions among resonant modes due to a laser gain medium. From the viewpoint of universality, it is of interest to uncover how this additional nonlinear effect manifests itself depending on the chaoticity or integrability of underlying ray dynamics inside a cavity.

A recent experimental study of semiconductor 2D microcavity lasers has demonstrated that single-mode lasing is achieved with a stadium-shaped (i.e., fully chaotic) cavity, while multimode lasing with an elliptic (i.e., nonchaotic) cavity [28,29]. This drastic difference was attributed to the difference of spatial modal patterns between the stadium and elliptic cavities. It was numerically shown that for the stadium cavity, an arbitrary low-loss modal pair has a significant spatial overlap, while for the elliptic cavity, there exist low-loss modal pairs whose spatial overlaps are small [29]. This means that any lasing mode of the stadium cavity tends to strongly interact with the other modes, while multiple lasing modes can coexist for the elliptic cavity because of small interactions among them.

It is important that all of the various fully chaotic stadium-type cavities of different aspect ratios and sizes studied in the experiments of Ref. [29] have shown single-mode lasing while all of various integrable elliptic cavities have multimode lasing.

Therefore, it was conjectured that single-mode lasing is universal for fully chaotic cavity lasers [29]. In order to ascertain this universality, it is important to further examine it experimentally and numerically for various 2D cavities. Such studies are important because they would add evidence of the universality. However, these pieces of evidence cannot directly reveal the insight of the universality. In past studies of universalities not only in quantum chaos but also in second-order phase transitions and critical phenomena, establishment of a theory that verifies a universality was the most difficult, challenging, and important task [7–11,30].

For a theoretical verification of the conjecture of universal single-mode lasing, it is necessary to evaluate the stability of a stationary-state solution of a full nonlinear model such as the Maxwell–Bloch equations [31–34]. Stability analysis for one-dimensional lasers in a low pumping regime just above the lasing threshold has been established by Lamb [35–37]. It also has been applied to 2D microcavity lasers and explained the spontaneous symmetry breaking of a lasing pattern [32,38]. However, Lamb's perturbation theory becomes invalid for a high pumping regime where the universal single-mode lasing can be observed because his perturbation theory expresses the population inversion by a power series expansion of lasing modes.

In this paper, we introduce a different expansion method for the population inversion in the Maxwell–Bloch equations that is applicable to a high pumping regime. Furthermore, we explicitly derive the stability matrix for stationary-state solutions, which describes the interactions among a huge number of lasing modes. The matrix elements turn out to be greatly simplified under the assumptions that the cavity size is much larger than the wavelength and the external pumping power is sufficiently large. Moreover, the eigenvalues of this matrix can be analytically evaluated by applying a theorem in linear algebra. This enables us to theoretically show that for a fully chaotic 2D microcavity laser, at least one single-mode lasing state is stable, while all multimode lasing states are unstable. This result provides a theoretical ground for universal single-mode lasing in fully chaotic 2D microcavity lasers.

2. FUNDAMENTAL EQUATIONS

For modeling the microcavity, we assume that it is wide in the xy-directions and thin in the z-direction. This allows us to separate the electromagnetic fields into transverse-magnetic (TM) and transverse-electric (TE) modes. Here we focus only on TM modes, where the electric field vector is expressed as $\boldsymbol{E} = (0, 0, E_z)$. We also assume that the atoms in the lasing medium have spherical symmetry and two energy levels. The relaxation due to the interaction with the reservoir can be described phenomenologically with decay constants $\gamma_\perp$ for the microscopic polarization ρ and $\gamma_\parallel$ for the population inversion W. We also need to include phenomenologically the effect of the external energy injected into the lasing medium by the pumping power W_∞.

By applying the slowly varying envelope approximation, one can reduce the Maxwell equation as follows:

$$\frac{\partial}{\partial t}\tilde{E} = \frac{i}{2}(\nabla_{xy}^2 + 1)\tilde{E} - \alpha_L\tilde{E} + \frac{2\pi N\kappa\hbar}{\varepsilon}\tilde{\rho}, \tag{1}$$

where $\tilde{E}$ and $\tilde{\rho}$ are respectively the slowly varying envelopes of the z-component of the electric field and the microscopic polarization, N is the number density of the atoms, κ is the coupling strength, ε is the permittivity, and α_L represents the losses describing absorption inside the cavity. In the above, space and time are made dimensionless by the scale transformation $((n_{\rm in}\omega_s/c)x,\ (n_{\rm in}\omega_s/c)y) \to (x, y)$, $t\omega_s \to t$, respectively, where $n_{\rm in}$ denotes the effective refractive index inside the cavity and ω_s is the oscillation frequency of the fast oscillation part of the electric field. In the same way, we have the equation for the electric field outside the cavity,

$$\frac{n_{\rm out}^2}{n_{\rm in}^2}\frac{\partial}{\partial t}\tilde{E} = \frac{i}{2}\left(\nabla_{xy}^2 + \frac{n_{\rm out}^2}{n_{\rm in}^2}\right)\tilde{E}, \tag{2}$$

where $n_{\rm out}$ denotes the refractive index outside the cavity. For the boundary condition at infinity, we adopt the outgoing wave condition.

The optical Bloch equations are also transformed to the following form:

$$\frac{\partial}{\partial t}\tilde{\rho} = -\tilde{\gamma}_\perp\tilde{\rho} - i\Delta_0\tilde{\rho} + \tilde{\kappa}W\tilde{E}, \tag{3}$$

$$\frac{\partial}{\partial t}W = -\tilde{\gamma}_\parallel(W - W_\infty) - 2\tilde{\kappa}(\tilde{E}\tilde{\rho}^* + \tilde{E}^*\tilde{\rho}), \tag{4}$$

where the dimensionless parameters are defined as follows: $\tilde{\gamma}_\perp \equiv \gamma_\perp/\omega_s$, $\tilde{\gamma}_\parallel \equiv \gamma_\parallel/\omega_s$, $\Delta_0 \equiv [\omega_0 - \omega_s]/\omega_s$, and $\tilde{\kappa} \equiv \kappa/\omega_s$ has the dimension of the inverse of the electric field and ω_0 is the transition frequency of the two-level atoms.

A theoretical method to obtain stationary-state solutions of Eqs. (1)–(4) has been developed as "steady-state *ab initio* laser theory (SALT)" [33,34,39–45]. However, the existence of a stationary-state solution does not always mean its experimental observability. That is, a stationary-state solution must be stable so that it can be experimentally observed, especially when the experiment is performed with continuous-wave operation, where the long-term dynamical effect is expected to be important. Such a dynamical stability is not explicitly incorporated in the SALT approach.

In the following, we carry out the stability analysis of a stationary-state solution for the Maxwell–Bloch equations. Although the equations for the stability analysis are very complicated in general, they turn out to be greatly simplified thanks to full chaoticity and the short wavelength limit. By applying the analysis to a 2D microcavity laser where the ray dynamics are fully chaotic, we show that at least one single-mode lasing state is stable and all of the multimode lasing states are unstable when the size of the cavity is much larger than the wavelength and the pumping power is sufficiently large.

3. DYNAMICS OF ALMOST STATIONARY LASING STATES

We assume that near a stationary state the light field and polarization can be expressed as follows:

$$\tilde{E} = \sum_i E_i(t)e^{-i\Delta_i t}U_i(x, y), \tag{5}$$

$$\tilde{\rho} = \sum_i \rho_i(t) e^{-i\Delta_i t} V_i(x, y), \quad (6)$$

where Δ_i represents the lasing oscillation frequency. Note that the lasing mode i depends on the pumping power and can be a fusion of several modes that coalesce by frequency-locking and separate into individual lasing modes with different frequencies as the pumping power decreases [28,38]. U_i is supposed to be normalized.

Then, from Eq. (1), we obtain

$$\begin{aligned}
&\frac{dE_i(t)}{dt} + \sum_{j\neq i} \frac{dE_j(t)}{dt} e^{-i\Delta_{ij}t} U_{ij} \\
&= \left[i\left(\Delta_i + \frac{1}{2}\right) - (\alpha_L + \tilde{\gamma}_{ii}) \right] E_i(t) \\
&\quad + \sum_{j\neq i} \left\{ \left[i\left(\Delta_j + \frac{1}{2}\right) - \alpha_L \right] U_{ij} - \tilde{\gamma}_{ij} \right\} E_j(t) e^{-i\Delta_{ij}t} \\
&\quad + \frac{2\pi N\kappa\hbar}{\epsilon} \sum_j e^{-i\Delta_{ij}t} \rho_j(t) \int_D U_i^*(x, y) V_j(x, y) dxdy, \quad (7)
\end{aligned}$$

where Δ_{ij} denotes the frequency difference between modes j and i, i.e., $\Delta_{ij} \equiv \Delta_j - \Delta_i$ and U_{ij} is defined by the inner product $U_{ij} \equiv \int_D U_i^*(x, y) U_j(x, y) dxdy$, and it will be shown later that $\tilde{\gamma}_{ii}$ is related to the flux of the light field intensity from inside to outside the cavity through the cavity edge and $\tilde{\gamma}_{ij}$ is defined as follows:

$$\tilde{\gamma}_{ij} \equiv -\frac{i}{2} \int_D dxdy U_i^*(x, y) \nabla^2 U_j(x, y). \quad (8)$$

In the above, D denotes the area inside the cavity.

From Eq. (3), we have

$$\rho_j(t) V_j(x, y) = \frac{\tilde{\kappa} W}{\tilde{\gamma}_\perp - i\Delta_{0j}} E_j(t) U_j(x, y). \quad (9)$$

Therefore, when the light field is almost stationary, from Eqs. (4) and (9), one can express the population inversion W by the light field amplitudes E_i and the spatial patterns U_i, i.e.,

$$W = W_\infty \Big/ \left\{ 1 + \left[\sum_i \sum_j \frac{2\tilde{\kappa}^2 E_i E_j^* U_i U_j^* e^{-i\Delta_{ji}t}}{(\tilde{\gamma}_\perp + i\Delta_{0j})(\tilde{\gamma}_\parallel - i\Delta_{ji})} + \text{c.c.} \right] \right\}. \quad (10)$$

The conventional approach to treat the nonlinear terms in W is to perturbatively expand the right-hand side of Eq. (10) for small light field amplitudes [35,36]. This method is only applicable to just above the lasing threshold and cannot correctly describe the case when the external pumping power W_∞ is very large. In the following, we present a different approach applicable to the high-pumping cases. Note that our method can be applied to semiconductor lasers in the same way as the conventional approach [37].

We introduce the dimensionless quantities L and C related to the total intensity and mode interference, respectively, as follows:

$$L(x, y) \equiv 1 + \sum_m a_m |U_m|^2, \quad (11)$$

where a_m denotes the dimensionless light field intensity of the mode m weighted by the Lorentzian gain $g(\Delta_m) \equiv \tilde{\gamma}_\perp / (\tilde{\gamma}_\perp^2 + \Delta_{0m}^2)$, i.e., $a_m \equiv (4\tilde{\kappa}^2/\tilde{\gamma}_\parallel) g(\Delta_m) |E_m|^2$, and

$$C(x, y) \equiv \sum_{\substack{l,j \\ l\neq j}} \frac{2\tilde{\kappa}^2 E_l E_j^* U_l U_j^* e^{-i\Delta_{jl}t}}{(\tilde{\gamma}_\perp + i\Delta_{0j})(\tilde{\gamma}_\parallel - i\Delta_{jl})} + \text{c.c.} \quad (12)$$

Then the denominator of the term in the right-hand side in Eq. (10) is expressed as $L + C = L(1 + C/L)$. The basic idea of our approach is to expand it in the power series of C/L under the condition $|C|/|L| < 1$ almost everywhere in the cavity. From Eqs. (7), (9), and (10), we obtain

$$\begin{aligned}
&\frac{dE_i(t)}{dt} + \sum_{j\neq i} \frac{dE_j(t)}{dt} e^{-i\Delta_{ij}t} U_{ij} \\
&= \left[i\left(\Delta_i + \frac{1}{2}\right) - (\alpha_L + \tilde{\gamma}_{ii}) \right] E_i(t) \\
&\quad \times \sum_{j\neq i} \left\{ \left[i\left(\Delta_j + \frac{1}{2}\right) - \alpha_L \right] U_{ij} - \tilde{\gamma}_{ij} \right\} E_j(t) e^{-i\Delta_{ij}t} \\
&\quad + \xi W_\infty \sum_k \frac{e^{-i\Delta_{ik}t} E_k}{\tilde{\gamma}_\perp - i\Delta_{0k}} \\
&\quad \times \int_D dxdy \frac{U_i^* U_k}{L(x, y)} \left\{ 1 - \frac{C(x, y)}{L(x, y)} + \left[\frac{C(x, y)}{L(x, y)} \right]^2 - \ldots \right\}, \quad (13)
\end{aligned}$$

where $\xi \equiv 2\pi N\kappa\tilde{\kappa}\hbar/\varepsilon$. Since we are focusing on the vicinity of the stationary state of the slowly varying envelope, $dE_i(t)/dt$ is very small. Therefore, we can assume $E_i(t) \sim e^{\epsilon_i t}$ where $\epsilon_i \ll |\Delta_{ij}|$ for all i and j ($j \neq i$). When Eq. (13) is integrated over t, the second terms on both sides have the coefficients of $1/(\epsilon_j - i\Delta_{ij})$ while the first terms are of $1/\epsilon_i$. Consequently, the contributions of the terms concerning fast oscillations like the second terms are much smaller than those of the first terms. Accordingly, one can ignore the terms oscillating faster than $e^{-i\Delta_{ik}t}$.

By ignoring the terms oscillating faster than $e^{-i\Delta_{ik}t}$, Eq. (13) is reduced to

$$\begin{aligned}
\frac{dE_i}{dt} \simeq &\left[i\left(\Delta_i + \frac{1}{2}\right) - (\alpha_L + \tilde{\gamma}_{ii}) \right] E_i \\
&+ \frac{\xi W_\infty E_i}{\tilde{\gamma}_\perp - i\Delta_{0i}} \int_D dxdy \frac{|U_i|^2}{L(x, y)} \\
&- \xi W_\infty E_i \int_D dxdy \frac{|U_i|^2}{[L(x, y)]^2} \sum_{\substack{k \\ k\neq i}} \frac{2\tilde{\kappa}^2 |E_k|^2 |U_k|^2}{(\tilde{\gamma}_\perp - i\Delta_{0k})(\tilde{\gamma}_\parallel - i\Delta_{ki})} \\
&\times \left(\frac{1}{\tilde{\gamma}_\perp + i\Delta_{0k}} + \frac{1}{\tilde{\gamma}_\perp - i\Delta_{0i}} \right). \quad (14)
\end{aligned}$$

Therefore, we obtain

$$\frac{\mathrm{d}|E_i|^2}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(E_i E_i^*) = E_i^* \frac{\mathrm{d}E_i}{\mathrm{d}t} + E_i \frac{\mathrm{d}E_i^*}{\mathrm{d}t}$$
$$= (-2\alpha_L + \tilde{\gamma}_{ii} + \tilde{\gamma}_{ii}^*)|E_i|^2$$
$$+ 2\xi W_\infty g(\Delta_i) \int_D \mathrm{d}x\mathrm{d}y[L(x,y)]^{-2}$$
$$\times |E_i|^2|U_i|^2\{L(x,y) - \sum_{k,k\neq i} 2\tilde{\kappa}^2 g(\Delta_k) g_\parallel(\Delta_i - \Delta_k)$$
$$\times [2\tilde{\gamma}_\perp + (\Delta_i - \Delta_0)(\Delta_i - \Delta_k)/\tilde{\gamma}_\perp$$
$$+ (\Delta_i - \Delta_k)(\Delta_i + \Delta_k - 2\Delta_0)/\tilde{\gamma}_\parallel]|E_k|^2|U_k|^2\}, \quad (15)$$

where $g_\parallel(\Delta_i - \Delta_k)$ is a Lorentzian defined as $g_\parallel(\Delta_i - \Delta_k) \equiv \tilde{\gamma}_\parallel/[\tilde{\gamma}_\parallel^2 + (\Delta_i - \Delta_k)^2]$. If Δ_i is far from Δ_0, the second term of the second line does not contribute because $g(\Delta_i)$ almost vanishes. Therefore, we assume $|\Delta_i - \Delta_0| \ll \tilde{\gamma}_\parallel$. Since the terms concerning Δ_k contribute in $L(x,y)$ if Δ_k is as close to Δ_0 as Δ_i due to the Lorentzian $g(\Delta_k)$, we obtain

$$L(x,y) \simeq 1 + \sum_{k,|\Delta_k - \Delta_i| \ll \tilde{\gamma}_\parallel} \frac{4\tilde{\kappa}^2}{\tilde{\gamma}_\parallel} g(\Delta_k)|E_k|^2|U_k|^2. \quad (16)$$

Because of the Lorentzian $g_\parallel(\Delta_i - \Delta_k)$, only the terms whose Δ_k values are close to Δ_i such that $|\Delta_i - \Delta_k| \ll \tilde{\gamma}_\parallel$ contribute to the sum over k in Eq. (15). Accordingly, we have $g_\parallel(\Delta_i - \Delta_k) \simeq 1/\tilde{\gamma}_\parallel$ and

$$2\tilde{\gamma}_\perp \gg (\Delta_i - \Delta_0)(\Delta_i - \Delta_k)/\tilde{\gamma}_\perp$$
$$+ (\Delta_i - \Delta_k)(\Delta_i + \Delta_k - 2\Delta_0)/\tilde{\gamma}_\parallel. \quad (17)$$

Consequently, we obtain

$$\frac{\mathrm{d}|E_i|^2}{\mathrm{d}t} = (-2\alpha_L + \tilde{\gamma}_{ii} + \tilde{\gamma}_{ii}^*)|E_i|^2$$
$$+2\xi W_\infty g(\Delta_i) \int_D \mathrm{d}x\mathrm{d}y[L(x,y)]^{-2}|E_i|^2|U_i|^2$$
$$\times \left[1 + \sum_{k,|\Delta_k - \Delta_i| \ll \tilde{\gamma}_\parallel} \frac{4\tilde{\kappa}^2}{\tilde{\gamma}_\parallel} g(\Delta_k)|E_k|^2|U_k|^2 \right.$$
$$\left. - \sum_{k,k\neq i,|\Delta_k - \Delta_i| \ll \tilde{\gamma}_\parallel} \frac{4\tilde{\kappa}^2}{\tilde{\gamma}_\parallel} g(\Delta_k)|E_k|^2|U_k|^2 \right]$$
$$= (-2\alpha_L + \tilde{\gamma}_{ii} + \tilde{\gamma}_{ii}^*)|E_i|^2$$
$$+2\xi W_\infty g(\Delta_i) \int_D \mathrm{d}x\mathrm{d}y[L(x,y)]^{-2}|E_i|^2|U_i|^2$$
$$\times \left[1 + \frac{4\tilde{\kappa}^2}{\tilde{\gamma}_\parallel} g(\Delta_i)|E_i|^2|U_i|^2 \right]. \quad (18)$$

Therefore, we finally obtain the equation for the time evolution of the light field intensity $I_i \equiv |E_i|^2$ of the lasing mode i,

$$\frac{\mathrm{d}I_i}{\mathrm{d}t} \simeq S_i I_i, \quad (19)$$

where S_i denotes the balance of the loss, gain, and saturation of the mode i, and is defined as

$$S_i \equiv -2(\alpha_L + \gamma_i) + 2\xi W_\infty g(\Delta_i) \int_D \mathrm{d}x\mathrm{d}y \frac{|U_i|^2 L_i(x,y)}{[L(x,y)]^2}, \quad (20)$$

and $L_i(x,y)$ is related to the dimensionless light field intensity of the mode i, i.e., $L_i(x,y) \equiv 1 + a_i|U_i|^2$. γ_i is derived by applying Green's theorem to $(\tilde{\gamma}_{ii} + \tilde{\gamma}_{ii}^*)$ and represents the rate of the flux of the light field intensity going outside the cavity through the cavity edge for the lasing mode i:

$$\gamma_i \equiv -\frac{i}{4} \oint_{\partial D} \mathrm{d}s \left(U_i^* \frac{\partial U_i}{\partial n} - U_i \frac{\partial U_i^*}{\partial n} \right), \quad (21)$$

where $\partial/\partial n$ is a normal derivative on the cavity edge. It is important to note that Eq. (19) is derived without any assumption of small intensities, and hence it can be applied to the strongly pumped regimes.

4. STABILITY ANALYSIS OF STATIONARY LASING STATES

A. Stability Matrix

The light field intensities that make the right-hand side of Eq. (19) vanish correspond to stationary-state solutions. The stability of a stationary-state solution is evaluated by the time evolution of the small displacements δI_i from the intensities $I_{s,i}$ for the stationary-state subject to the differential equation $\mathrm{d}\boldsymbol{\delta I}/\mathrm{d}t = \tilde{M}\boldsymbol{\delta I}$. Here the displacement vector is defined by $\boldsymbol{\delta I} \equiv (\delta I_1 \; \delta I_2 \; \cdots \; \delta I_n)^T$ and the matrix $\tilde{M}$ is given by $\tilde{M} \equiv PMP^{-1}$, and

$$M_{ij} \equiv \left\{ S_j + 2\xi W_\infty b_j^2 \int_D \mathrm{d}x\mathrm{d}y \frac{|U_j|^4}{[L(x,y)]^2} \right\} \delta_{ij}$$
$$-4\xi W_\infty b_i b_j \int_D \mathrm{d}x\mathrm{d}y \frac{|U_i|^2|U_j|^2 L_i(x,y)}{[L(x,y)]^3}, \quad (22)$$

where $P \equiv \mathrm{diag}(|E_1|, \ldots, |E_N|)$ and $b_i \equiv [a_i g(\Delta_i)]^{1/2} = 2\tilde{\kappa} g(\Delta_i)|E_i|/\tilde{\gamma}_\parallel^{1/2}$. For simplicity, we introduced the matrix P and assumed that its inverse matrix P^{-1} exists. However, $|E_i|^{-1}$ in P^{-1} always cancels out $|E_i|$ in P. Therefore, $\tilde{M}$ is always well-defined and the following discussion is valid irrespective of the existence of P^{-1}.

B. Single-Mode Lasing States

From Eq. (19), one can see that the fixed point $(0, \ldots, 0, I_{s,j}, 0, \ldots, 0)$, which satisfies $S_j = 0$, corresponds to the single-mode lasing state of the mode j whose light field intensity is equal to $I_{s,j} = |E_{s,j}|^2$. Then, the matrix $\tilde{M}$ for this fixed point becomes a diagonal matrix, i.e.,

$$\tilde{M}_{jj} = -2\xi W_\infty b_j^2 \int_D \mathrm{d}x\mathrm{d}y \frac{|U_j|^4}{L_{s,j}^2} < 0, \quad (23)$$

and for $i \neq j$,

$$\tilde{M}_{ii} = 2\xi W_\infty g(\Delta_i) \int_D \mathrm{d}x\mathrm{d}y |U_i|^2 \frac{L_{s,i} - L_{s,j}^2}{L_{s,j}^2 L_{s,i}}, \quad (24)$$

where $L_{s,i}$ is related to the light intensity of the single-mode lasing corresponding to the fixed point $(0, \ldots, 0, I_{s,i}, 0, \ldots, 0)$, i.e., $L_{s,i} \equiv 1 + a_{s,i}|U_i|^2$ and $a_{s,i} \equiv (4\tilde{\kappa}^2/\tilde{\gamma}_\parallel) g(\Delta_i) I_{s,i}$.

From Eq. (23), one can see that the single-mode lasing state of the mode j is stable in the direction $(0 \cdots 0\ \delta I_j\ 0 \cdots 0)^T$.

It is important to note that $L_{s,i(j)}$ contains the spatial pattern of the mode $i(j)$. As shown in the Section 5.B, if the spatial pattern $|U_i|^2$ overlaps with $|U_j|^2$ inside the cavity, $\tilde{M}_{ii}$ can be negative. Then, the single-mode lasing state of the mode j can be stable in the direction $(0 \cdots 0\ \delta I_i\ 0 \cdots 0)^T$.

C. Multimode Lasing States

From Eq. (19), one can see that a multimode lasing state corresponds to the solutions $I_{s,i}$ of the simultaneous equations $S_i = 0$ $(i = 1, 2, \ldots, N)$. The number N of the lasing modes that have nonzero light field intensities is an arbitrary natural number more than 1 because $I_i = 0$ always satisfies the stationary-state condition for Eq. (19). Then, from Eq. (22), we have

$$M_{ii} = 2\xi W_\infty b_i^2 \int_D \mathrm{d}x\mathrm{d}y \frac{|U_i|^4 L_i(x,y)}{[L(x,y)]^3}\left[\frac{L(x,y)}{L_i(x,y)} - 2\right], \tag{25}$$

$$M_{ij} = -4\xi W_\infty b_i b_j \int_D \mathrm{d}x\mathrm{d}y \frac{|U_i|^2|U_j|^2 L_i(x,y)}{[L(x,y)]^3}. \tag{26}$$

Consequently, if and only if all of the eigenvalues of the $N \times N$ matrix $\tilde{M}(\equiv PMP^{-1})$ are negative, this multimode lasing state is stable.

5. FULLY CHAOTIC 2D MICROCAVITY LASERS

A. Spatial Patterns of Stationary Lasing Modes

For bounded chaotic systems, theories on quantum ergodicity have shown that the probability density of finding a quantum particle in a small area whose state is described by the eigenfunction of a quantized fully chaotic system approaches a uniform measure as the energy of the particle increases and the wavelength becomes shorter [46–51].

For open chaotic mapping systems, it has been shown that the long-lived eigenstates tend to be localized on the forward trapped set of the corresponding classical dynamics as the wavelength decreases [52–56]. This tendency can be considered as a manifestation of quantum ergodicity in open systems.

A similar tendency has also been numerically observed for chaotic 2D microcavities [29], where resonance wave functions of low-loss modes are supported by the forward trapped set of the corresponding ray dynamics with Fresnel's law [57,58] in the short wavelength limit. Because of this property, the overlap of wave functions between an arbitrary pair of low-loss modes takes a large value [29]. Therefore, we assume that the spatial patterns of the wave functions for low-loss modes are similar to each other and expressed as $|U_i(x,y)|^2 = [1+\epsilon_i(x,y)]\,|U_0(x,y)|^2$, which implies

$$\int_D \mathrm{d}x\mathrm{d}y \epsilon_i(x,y)|U_0(x,y)|^2 = 0, \tag{27}$$

because of the normalization for all of the spatial patterns of the lasing modes. We assume that the fluctuation $\epsilon_i(x,y)$ is so small almost everywhere and random that $|\epsilon_i(x,y)|U_0(x,y)|^2| \ll 1$ and $\int_D \mathrm{d}x\mathrm{d}y\epsilon_i(x,y) \simeq 0$. We also assume $\int_D \mathrm{d}x\mathrm{d}y\epsilon_i(x,y)|U_0(x,y)|^{-2} \simeq 0$.

B. Stability of Single-Mode Lasing States

In the case of a fully chaotic 2D microcavity, one can apply the wave function property in Eq. (27) to evaluate $\tilde{M}_{ii}$ in Eq. (24) whose sign determines the stability of the single-mode lasing state of the mode j in the direction $(0 \cdots 0\ \delta I_i\ 0 \cdots 0)^T$. Indeed, $\tilde{M}_{ii}$ is reduced for the first order of $\epsilon_i(x,y)|U_0(x,y)|^2$ as follows:

$$\begin{aligned}\tilde{M}_{ii} &= 2\xi W_\infty g(\Delta_i)\int_S \mathrm{d}x\mathrm{d}y(1+\epsilon_i)|U_0|^2 \\ &\quad\times\frac{a_{s,i}(1+\epsilon_i) - a_{s,j}^2(1+\epsilon_j)^2|U_0|^2}{a_{s,j}^2(1+\epsilon_j)^2(1+\epsilon_i)a_{s,i}|U_0|^6}|U_0|^2 \\ &= \frac{2\xi W_\infty g(\Delta_i)}{a_{s,j}^2 a_{s,i}}\int_S \mathrm{d}x\mathrm{d}y\{[1+O(\epsilon_i(x,y)|U_0(x,y)|^2)]\frac{a_{s,i}}{|U_0|^2} \\ &\quad - [1+O(\epsilon_j(x,y)|U_0(x,y)|^2)]a_{s,j}^2\} \\ &\simeq \frac{2\xi W_\infty g(\Delta_i)}{a_{s,j}^2 a_{s,i}}\left(a_{s,i}\int_S \mathrm{d}x\mathrm{d}y\frac{1}{|U_0|^2} - Aa_{s,j}^2\right),\end{aligned} \tag{28}$$

where S denotes the support of $|U_0(x,y)|^2$ and A is its area.

If and only if $\tilde{M}_{ii}$ is negative, the single-mode lasing state of the mode j is stable in the direction $(0 \cdots 0\ \delta I_i\ 0 \cdots 0)^T$. Therefore, from Eq. (28), we obtain the stability condition for the single-mode lasing of the mode j,

$$\left(\frac{a_{s,j}}{A}\right)^2 > \frac{a_{s,i}}{A}\int_S \mathrm{d}x\mathrm{d}y\frac{1}{A^2|U_0|^2}. \tag{29}$$

Accordingly, the single-mode lasing state of the mode j is stable when its intensity is large enough to satisfy Eq. (29) for all of the other mode i values. Note that the integral in Eq. (29) is estimated to be approximately unity because $|U_0|^{-2}$ can be approximated to be A. Consequently, the mode that has the largest single-mode intensity is stable, and hence there always exists one stable single-mode lasing state at least.

C. Instability of Multimode Lasing States

Next, we show that all of the multimode lasing states are unstable under the assumptions that all of the spatial patterns U_i of the single-mode lasing states are similar to each other, and the pumping power W_∞ is very high, which implies that the light field intensities are very large.

According to the assumption in Section 5.A for the spatial patterns of the lasing modes in a fully chaotic microcavity, we obtain

$$L_i(x,y) \simeq (1+\epsilon_i)a_i|U_0|^2, \tag{30}$$

$$L(x,y) \simeq \sum_{m=1}^{N}(1+\epsilon_m)a_m|U_0|^2 \tag{31}$$

for the support S of $|U_0|^2$ and we assumed $a_i \gg 1$. Then, the matrix elements in Eqs. (25) and (26) can be expressed by using the fluctuations $\epsilon_i(x,y)$. Therefore, from the properties assumed for $\epsilon_i(x,y)$, we obtain for the first order of $\epsilon_i(x,y)|U_0(x,y)|^2$,

$$M_{ii} \simeq -4\xi A W_\infty b_i^2 \frac{a_i}{a_{\mathrm{tot}}^3}\left(1 - \frac{a_{\mathrm{tot}}}{2a_i}\right), \tag{32}$$

$$M_{ij} \simeq -4\xi A W_\infty b_i b_j \frac{a_i}{a_{\rm tot}^3}, \tag{33}$$

where $a_{\rm tot} \equiv \sum_{m=1}^N a_m$.

It is important to note that $\tilde{M}(N)$ can be factorized as $\tilde{M}(N) = -4\xi A W_\infty/(a_{\rm tot}^3) PQBRBP^{-1}$, where $Q \equiv {\rm diag}(a_1, \ldots, a_N)$, $B \equiv {\rm diag}(b_1, \ldots, b_N)$, and the diagonal elements of R are given by $R_{ii} \equiv 1 - a_{\rm tot}/(2a_i)$, while the off-diagonal elements $R_{ij} = 1$. Then, as is explained in Appendix A, all of the eigenvalues of $\tilde{M}(N)$ are equal to those of $M'(N)$ defined as

$$M'(N) \equiv -\frac{4\xi A W_\infty}{a_{\rm tot}^3} {\rm diag}(\sqrt{a_1}, \ldots, \sqrt{a_N}) \times BRB\, {\rm diag}(\sqrt{a_1}, \ldots, \sqrt{a_N}). \tag{34}$$

All of the eigenvalues of $M'(N)$ are real because it is a real symmetric matrix. According to the theorem of linear algebra, the number of the negative eigenvalues of $M'(N)$ is equal to that of the sign changes of the sequence $\{1, |M'(1)|, |M'(2)|, \ldots, |M'(N)|\}$, where the minor determinants of $M'(N)$ are given as explained in Appendix A,

$$|M'(k)| = \left(\frac{2\xi A W_\infty}{a_{tot}^2}\right)^k \left(1 - \sum_{i=1}^k \frac{2a_i}{a_{\rm tot}}\right) \prod_{i=1}^k b_i^2. \tag{35}$$

Since the term $(1 - \sum_{i=1}^k 2a_i/a_{\rm tot})$ decreases monotonically for k and equals -1 when $k = N$, the above sequence of the minor determinants of $M'(N)$ changes the sign once. Accordingly, $\tilde{M}$ has one negative eigenvalue and $(N-1)$ positive eigenvalues, which means the fixed point corresponding to the multimode lasing state is an unstable saddle point.

6. SUMMARY AND DISCUSSION

By introducing an expansion method different from conventional theories [35–37] for the population inversion in the Maxwell–Bloch equations and evaluating the eigenvalues of a stability matrix describing the interactions among a huge number of lasing modes, we theoretically showed that in a fully chaotic 2D microcavity laser, at least one single-mode lasing state is stable, while all multimode lasing states are unstable, when the external pumping power is sufficiently large and the cavity size is much larger than the wavelength to the extent that $\tilde{\gamma}_\parallel \gg |\Delta_{ij}|$, where Δ_{ij} is the difference between the adjacent lasing frequencies. This result provides a theoretical ground for recent experimental observations of universal single-mode lasing in fully chaotic 2D microcavity lasers [28,29].

It is important to note that the theory presented in this paper should be applied to explain the observation of single-mode lasing in the experiments of continuous-wave pumping cases. Generally, the lifetime of an unstable multimode lasing state can be much longer than a pulse width. Thus, for a pulsed operation, it is likely that the collapse of an unstable multimode lasing state cannot be achieved within a pulse width, even if the size of a fully chaotic 2D microcavity is sufficiently large, and multimode lasing is observed and universal single-mode lasing seems to disappear [59,60]. In this case, as the pulse width is increased, the number of lasing modes decreases [28]. Multimode lasing in a fully chaotic 2D microcavity can also be observed when the condition $\tilde{\gamma}_\parallel \gg |\Delta_{ij}|$ is not satisfied [61,62]. This condition for multimode lasing coincides with that derived for one-dimensional lasers [63].

The theory presented in this paper cannot give the threshold pumping power for single-mode lasing, but it is useful for understanding the single-mode lasing mechanism. In addition, it does not take into consideration these phenomena that might affect lasing characteristics such as the thermal effect. However, according to previous studies [28,29], we can at least say that the threshold for single-mode lasing is achievable in real experiments. It is of interest to further demonstrate the predicted single-mode lasing experimentally for various fully chaotic cavities. It is also important to elucidate experimentally, numerically, and theoretically how multimode lasing states in a low pumping regime and/or in a small cavity change into a single-mode lasing state as the pumping power and/or the size of the cavity are increased.

APPENDIX A

Let us suppose that $\boldsymbol{x}$ is the eigenvector corresponding to the eigenvalue λ of the matrix $M \equiv {\rm diag}(B_1, \ldots, B_N) C\, {\rm diag}(A_1, \ldots, A_N)$, where the diagonal element of the matrix C is defined as $C_{ii} \equiv C_i$ and every off-diagonal element is equal to 1, that is,

$$M\boldsymbol{x} = {\rm diag}(B_1, \ldots, B_N) C\, {\rm diag}(A_1, \ldots, A_N)\boldsymbol{x} = \lambda \boldsymbol{x}. \tag{A1}$$

The left-hand side of Eq. (A1) is rewritten as follows:

$$\begin{aligned} &{\rm diag}(B_1, \ldots, B_N) C\, {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2}) \\ &\quad \times {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2}) {\rm diag}(B_1^{1/2}, \ldots, B_N^{1/2}) \\ &\quad \times {\rm diag}(B_1^{-1/2}, \ldots, B_N^{-1/2})\boldsymbol{x} \\ &= {\rm diag}(B_1, \ldots, B_N) C\, {\rm diag}(B_1^{1/2}, \ldots, B_N^{1/2}) \\ &\quad \times {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2}) {\rm diag}(B_1^{-1/2}, \ldots, B_N^{-1/2}) \\ &\quad \times {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2})\boldsymbol{x}. \end{aligned} \tag{A2}$$

On the other hand, the right-hand side of Eq. (A1) is rewritten as follows:

$$\begin{aligned} &\lambda\, {\rm diag}(A_1^{-1/2}, \ldots, A_N^{-1/2}) {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2}) \\ &\quad \times {\rm diag}(B_1^{1/2}, \ldots, B_N^{1/2}) {\rm diag}(B_1^{-1/2}, \ldots, B_N^{-1/2})\boldsymbol{x} \\ &= \lambda\, {\rm diag}(A_1^{-1/2}, \ldots, A_N^{-1/2}) {\rm diag}(B_1^{1/2}, \ldots, B_N^{1/2}) \\ &\quad \times {\rm diag}(B_1^{-1/2}, \ldots, B_N^{-1/2}) {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2})\boldsymbol{x}. \end{aligned} \tag{A3}$$

Therefore, we have

$$\begin{aligned} &{\rm diag}(B_1, \ldots, B_N) C\, {\rm diag}(B_1^{1/2}, \ldots, B_N^{1/2}) \\ &\quad \times {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2})\boldsymbol{x}' \\ &= \lambda\, {\rm diag}(A_1^{-1/2}, \ldots, A_N^{-1/2}) {\rm diag}(B_1^{1/2}, \ldots, B_N^{1/2})\boldsymbol{x}', \end{aligned} \tag{A4}$$

where $\boldsymbol{x}' \equiv {\rm diag}(B_1^{-1/2}, \ldots, B_N^{-1/2}) {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2})\boldsymbol{x}$. Operating ${\rm diag}(B_1^{-1/2}, \ldots, B_N^{-1/2}) {\rm diag}(A_1^{1/2}, \ldots, A_N^{1/2})$ to both sides of Eq. (A4) yields

$$\mathrm{diag}(A_1^{1/2},\ldots,A_N^{1/2})\mathrm{diag}(B_1^{1/2},\ldots,B_N^{1/2})C \times \mathrm{diag}(B_1^{1/2},\ldots,B_N^{1/2})\mathrm{diag}(A_1^{1/2},\ldots,A_N^{1/2})\boldsymbol{x}' = \lambda\boldsymbol{x}'. \quad \text{(A5)}$$

From Eq. (A5), one can see that the eigenvalue of the matrix M' defined as

$$M' \equiv \mathrm{diag}(A_1^{1/2},\ldots,A_N^{1/2})\mathrm{diag}(B_1^{1/2},\ldots,B_N^{1/2})C \times \mathrm{diag}(B_1^{1/2},\ldots,B_N^{1/2})\mathrm{diag}(A_1^{1/2},\ldots,A_N^{1/2}) \quad \text{(A6)}$$

is equal to λ. Since $M'_{ij} = C_{ij}(A_iA_jB_iB_j)^{1/2}$ and C is a real symmetric matrix, M' is a real symmetric matrix. The determinant of M' is given as follows:

$$|M'| = |\mathrm{diag}(A_1^{1/2},\ldots,A_N^{1/2})||\mathrm{diag}(B_1^{1/2},\ldots,B_N^{1/2})||C| \times |\mathrm{diag}(B_1^{1/2},\ldots,B_N^{1/2})||\mathrm{diag}(A_1^{1/2},\ldots,A_N^{1/2})| = \left(\prod_{i=1}^{N} A_iB_i\right)\left(\prod_{i=1}^{N}(C_i-1)\right)\left(1+\sum_{i=1}^{N}(C_i-1)^{-1}\right). \quad \text{(A7)}$$

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