Analytic solutions of Bloch equations with phase term for two-pulse photon echo by the phase modulation pulses*

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In this paper, a method is proposed to obtain the analytical solutions of Bloch equations with phase term by using the elementary matrix transform. The effect of the initial phase on the components of Bloch vector is discussed. The result shows that by considering the initial phase of the input pulse, the amplitudes of the in-phase component and the in-quadrature component can be dynamically controlled, meanwhile the population inversion is almost immune to the initial phase. Additionally, the signal electric field expressions of the phase effect of two-pulse photon echo (2PE) is derived, which is also modulated by the initial phase.
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Photon echo (PE) technology has important applications in quantum information storage and processing[1]. 2PE is generally applied to quantum communication protocols[2]. From gas to rare earth-doped ions crystal (REIC), PE technology can be used in different systems and has remarkable performance in efficiency[3], which can be used as a new source of excitation for quantum protocols. In the traditional PE sequences, the rephase π-pulse can be controlled by the additional parameters such as the variable electric[4], magnetic field[5], the AC-Stark shift or light-shift effect[6] and a fast chirp pulse[7]. These schemes have been applied in different fields and achieved good results, but so far there has been no report on the modulation of the initial phase of the PE. The main reason is that the analytical solution of Bloch equation with phase term is quite complex, and the phase term is generally ignored, including some numerical iterations[8].

In the previous work, the signal phase is sometimes mentioned in Bloch equations, such as in Ref.[9], spatial phase was introduced in Bloch equations to investigate effect of the angled beams on PE, and linear frequency chirped as phase function was introduced in Bloch equation to program the spatial-spectral grating in Ref.[10].

In this paper, the main contents are as follows. Firstly, based on elementary matrix transform, a method to solve the analytical solutions of Bloch equations with phase term is presented, then we analyze the effect of the initial phase on each component of Bloch vector. The expression of 2PE with different initial phases is derived and the effects of the phases of each pulse on the 2PE is analyzed.

We start with the Bloch equations with phase term, the Bloch equations in rotating frame can be written as[11]

$$\begin{pmatrix} \partial u(\Delta,t)/\partial t \\ \partial v(\Delta,t)/\partial t \\ \partial w(\Delta,t)/\partial t \end{pmatrix} = \begin{pmatrix} 0 & -\Delta & -\Omega\sin(\varphi) \\ \Delta & 0 & \Omega\cos(\varphi) \\ \Omega\sin(\varphi) & -\Omega\cos(\varphi) & 0 \end{pmatrix} \times \begin{pmatrix} u(\Delta,t) \\ v(\Delta,t) \\ w(\Delta,t) \end{pmatrix}, \quad (1)$$

where $u(\Delta, t)$, $v(\Delta, t)$, $w(\Delta, t)$ are the components of Bloch vector $\boldsymbol{B}(\Delta, t)$. $u(\Delta, t)$ and $v(\Delta, t)$, respectively, which are called in-phase components and in-quadrature components of Bloch vector. These components are related to the diploe moment, while $w(\Delta, t)$ is population inversion. Δ is the detuning, and φ is the initial phase of the input pulse. Ω is known as Rabi frequency.

Eq. (1) can be expressed in the form of partial differential equation $\partial\boldsymbol{B}(\Delta, t)/\partial t=\boldsymbol{AB}(\Delta, t)$, where $\boldsymbol{A}$ is the coefficient matrix of $\boldsymbol{B}(t)$. In order to solve $e^{t\boldsymbol{A}}$, we first write the expression of $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{pmatrix} 0 & -\Delta & -\Omega\sin\varphi \\ \Delta & 0 & \Omega\cos\varphi \\ \Omega\sin\varphi & -\Omega\cos\varphi & 0 \end{pmatrix}. \quad (2)$$

Then the solution to a system of linear ordinary dif-

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ferential equations can always be written as:

$$\boldsymbol{B}(\Delta,\ t)=e^{t\boldsymbol{A}}\boldsymbol{B}(\Delta,\ 0), \quad (3)$$

where $\boldsymbol{B}(\Delta, 0)=(u(\Delta, 0), v(\Delta, 0), w(\Delta, 0),)$ denotes the initial value of Bloch vector.

The elementary matrix e^{tA} is the key point to solve Eq.(2). The expression of e^{tA} is written as

$$e^{t\boldsymbol{A}}=a_1(t)\boldsymbol{I}+a_2(t)\boldsymbol{A}+a_3(t)\boldsymbol{A}^2. \quad (4)$$

The coefficient $a_i(t)$ (i=1, 2, 3) is determined by functions $f(t\lambda_i)=e^{t\lambda_i}$ and $T(t\lambda_i)=a_1(t)+a_2(t)\lambda_i+a_3(t)\lambda_i^2$, where λ_i (i=1, 2, 3) is eigenvalue of coefficient matrix $\boldsymbol{A}$. From Eq.(3), two functions are satisfied as

$$f(t\lambda_i)=T(t\lambda_i). \quad (5)$$

By putting every eigenvalue of $\boldsymbol{A}$ into this equation, then we can get it $a_i(t)$.

By letting $|\lambda\boldsymbol{I}-\boldsymbol{A}|=0$,we get three eigenvalues of the eigenvalue matrix. From Eq.(2) we can get a set of eigenvalues $\lambda_1=0$, $\lambda_2=\mathrm{i}\sqrt{\Omega^2+\Delta^2}$ and $\lambda_3=-\mathrm{i}\sqrt{\Omega^2+\Delta^2}$. We define that $\Omega'=\sqrt{\Omega^2+\Delta^2}$,where Ω' represents the generalized Rabi frequency. Then the eigenvalues can be rewritten in the form of set as

$$\lambda_i=\{0,\ \mathrm{i}\Omega',\ -\mathrm{i}\Omega'\},\ i=1,\ 2,\ 3. \quad (6)$$

By substituting Eq.(6) of eigenvalues value into Eq.(5), we can get the coefficient a_i about time t as

$$a_0(t)=1, a_1(t)=\frac{\sin\Omega' t}{\Omega'}, a_2(t)=\frac{1-\cos\Omega' t}{\Omega'^2}. \quad (7)$$

By substituting Eq.(7) into Eq.(3), the elementary solution matrix e^{tA} is expressed as:

$$e^{tA}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{\sin\Omega' t}{\Omega'}\boldsymbol{A}+\frac{1-\cos\Omega' t}{\Omega'^2}\boldsymbol{A}^2. \quad (8)$$

Finally, we substitute Eq.(8) into Eq.(3), the expansion form of the analytical solution of Bloch equation with phase term is expressed as

$$u(\Delta,t)=\left[1-\frac{1-\cos\Omega' t}{\Omega'^2}\left(\Delta^2+\Omega^2\sin^2\varphi\right)\right]u(\Delta,0)+\left(-\Delta\frac{\sin\Omega' t}{\Omega'}+\frac{1}{2}\frac{1-\cos\Omega' t}{\Omega'^2}\Omega^2\sin 2\varphi\right)v(\Delta,0)+\left(-\Omega\sin(\varphi)\frac{\sin\Omega' t}{\Omega'}-\frac{1-\cos\Omega' t}{\Omega'^2}\Delta\Omega\cos\varphi\right)w(\Delta,0)\ ,(9)$$

$$v(\Delta,t)=\left(\Delta\frac{\sin\Omega' t}{\Omega'}+\frac{1}{2}\frac{1-\cos\Omega' t}{\Omega'^2}\Omega^2\sin 2\varphi\right)u(\Delta,0)+\left[1-\frac{1-\cos\Omega' t}{\Omega'^2}\left(\Delta^2+\Omega^2\cos^2\varphi\right)\right]v(\Delta,0)+\left(\frac{\sin\Omega' t}{\Omega'}\Omega\cos(\varphi)-\frac{1-\cos\Omega' t}{\Omega'^2}\Delta\Omega\sin\varphi\right)w(\Delta,0)\ ,(10)$$

$$w(\Delta,t)=\left(\frac{\sin\Omega' t}{\Omega'}\Omega\sin\varphi-\frac{1-\cos\Omega' t}{\Omega'^2}\Delta\Omega\cos\varphi\right)\times u(\Delta,0)+\left(-\frac{\sin\Omega' t}{\Omega'}\Omega\cos\varphi-\frac{1-\cos\Omega' t}{\Omega'^2}\Delta\Omega\sin\varphi\right)\times v(\Delta,0)+\left(1-\frac{1-\cos\Omega' t}{\Omega'^2}\Omega^2\right)w(\Delta,0)\ .(11)$$

Eqs.(9)—(11) is the analytic solutions of Bloch equation with phase term. The effect of phase on each component of Bloch vector at different detuning will be analyzed in the following.

If the initial state of the system starts at ground state, this state corresponds to the initial value of the Bloch vector expressed as $\boldsymbol{B}(\Delta, 0)=(0, 0, -1)$, by substituting such initial value conditions into Eqs.(9)—(11). In Fig.1, each component of Bloch vector are plotted versus the different input pulse's phases for various line shape, in which the phases φ are equal to 0, π/6 and π/2, the time of input pulse $t=T$=200 ns, Rabi frequency $\Omega=\pi/2T$, detuning Δ=0.

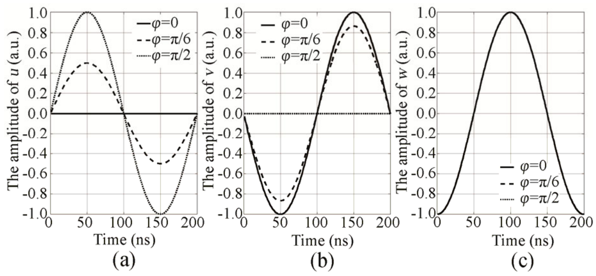

Fig.1 Profiles of (a) in-phase *u*, (b) in-quadrature *v* and (c) population inversion *w* with different phase values when *Δ*=0

From Fig.1(a) and (b), the amplitudes of the in-phase component $u(\Delta=0,\ t)$ and the in-quadrature component $v(\Delta=0,\ t)$ are respectively $\sin\varphi$ and $\cos\varphi$ when the detuning is 0. If phase $\varphi=0$, then $u(\Delta=0,\ t)\equiv 0$,this situation widely discussed in the previous paper[12]. Meanwhile, as shown in Fig.1(c), the population inversion is not affected by the phase, which is in line with the existing theory.

When $\Delta\neq 0$, the simulation conditions are the same as Fig.1. We simulate the components of Bloch vector, as shown in Fig.2.

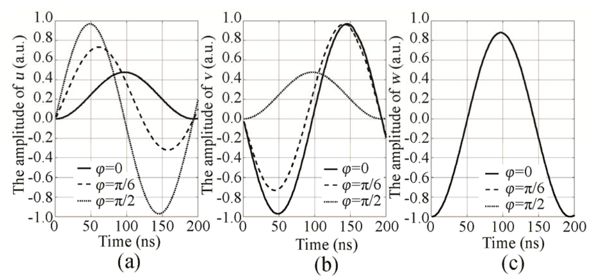

Fig.2 Profiles of (a) in-phase *u*, (b) in-quadrature *v* and (c) population inversion *w* with different phase values when *Δ*≠0

From Fig.2(a) and (b), the phase can also affect the amplitude of in-phase component $u(\Delta,\ t)$ and the in-quadrature component $v(\Delta,\ t)$ when $\Delta\neq 0$, but their amplitudes are not satisfied with $\sin\varphi$ and $\cos\varphi$. As shown in Fig.2(c), the population inversion $w(\Delta,\ t)$ is not affected as well.

Photon echoes occur in the inhomogeneous broadening of the rephase phenomenon[13], when the motion of Bloch vector is reversed after the initial dephasing of the atomic dipole $\langle\wp(t)\rangle$.If the transition dipole moment $\wp_{12}$ is same for all atoms ,then the total mean dipole moment $\langle\wp(t)\rangle$ produced by all N atoms is[13]

$$\langle\wp(t)\rangle=\frac{\wp_{12}N}{4\pi}\int_{-\infty}^{\infty}\left(u(\Delta,t)-\mathrm{i}v(\Delta,t)\right)\mathrm{e}^{-i\omega_0 t}g(\Delta)d\Delta+ c.c., \qquad (12)$$

where N is considered as a group of inhomogeneously broadened two-level atoms distributed over a region of space that is small compared with the wavelength. By an inhomogeneously broadened system we mean one in which different atoms have different detuning Δ with some distribution $g(\Delta)$ centered on Δ=0 is expressed as Gaussian distribution[14]

$$g(\Delta)=\sqrt{2\pi}T_2\exp\left[-\frac{1}{2}\Delta^2\left(T_2\right)^2\right], \qquad (13)$$

where T_2 is inhomogeneous life. In a solid, the inhomogeneous broadening is generally the result of variations in the field of crystal lattice, which affects different atoms differently. In this paper, we consider that the inhomogeneous lifetime T_2=1 μs.

The signal electric field $E_s(t)$ of medium is directly proportional to the total dipole moment $\langle\wp(t)\rangle$ of the medium[16]. The photon echo is generated when the amplitude of the dipole moment $E_s(t)$ reaches its maximum. The signal electric field equation derived from Maxwell-Bloch equation is[15]:

$$E_s(t)=\frac{KL\wp_{12}N}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\left(\mathrm{i}u(\Delta,t)+v(\Delta,t)\right)\mathrm{e}^{-i\omega_0 t}g(\Delta)d\Delta+ c.c., \qquad (14)$$

where k is wave number, L is the length of medium, ε_0 is dielectric constant of vacuum. By combining equations Eqs.(12) and Eq.(14), signal electric field satisfy the relationship that $E_s(t)\propto \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\langle\wp(t)\rangle$, it is shown that the phase delay of the electric field is π/2 over the phase delay of the dipole moment.

To simplify the discussion, we assume that the shapes of the pulses are rectangular. The duration of the two pulses is equal, and the amplitude of the second pulse is twice that of the first pulse. Two pulses are π pulse and π/2 pulse, respectively and satisfy the equations $\int_{t_1}^{t_1'}\Omega_1\mathrm{d}\tau=\frac{\pi}{2}$ and $\int_{t_2}^{t_2'}\Omega_2\mathrm{d}\tau=\pi$. The initial phases of the two pulses are φ_1 and φ_2. The sequence and phase of pulses are shown in Fig.3.

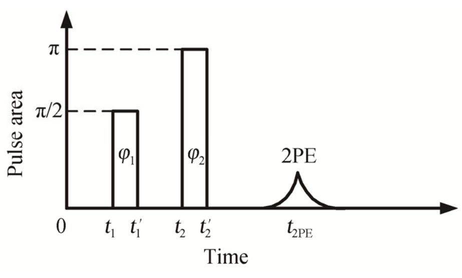

Fig.3 Schematic diagram of 2PE

Under the condition of ultra-short wave condition, we assume that Ω>>Δ, so we can get the following approximation $\Omega'\approx\Omega$ and the terms of order Δ/Ω can be neglected. In this paper, we assume that the width of all input pulses is t=50 ns.

If the atoms starts in the ground state at time $t=t_1$ with its initial value of Bloch vector $\boldsymbol{B}(\Delta, 0)$=(0, 0, −1), after the first pulse, from Eqs.(9)—(11), we can get that the components of Bloch vector are

$$u\left(\Delta,t=t_1'\right)\approx\sin(\varphi_1), \qquad (15)$$

$$v\left(\Delta,t=t_1'\right)\approx-\cos(\varphi_1), \qquad (16)$$

$$w\left(\Delta,t=t_1'\right)\approx 0. \qquad (17)$$

When $t>t_1'$, the external field is zero, so that Ω=0, the Bloch vector will rotate freely without an external field, and its rotation frequency is Δ, that is:

$$u(\Delta,t)=u(\Delta,0)\cos\Delta t-v(\Delta,0)\sin\Delta t, \qquad (18)$$

$$v(\Delta,t)=u(\Delta,0)\sin\Delta t+v(\Delta,0)\cos\Delta t, \qquad (19)$$

$$w(\Delta,t)=w(\Delta,0). \qquad (20)$$

Assuming the initial time t_1=0, we now substitute from Eqs. (18)—(20) into Eq.(14) and obtain the expression at $t_2>t>t_1'$

$$E_s(t)=\frac{kL\wp_{12}N}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\left(\mathrm{i}\sin\varphi_1-\cos\varphi_1\right)\mathrm{e}^{-\mathrm{i}\omega_0 t}\mathrm{e}^{-\mathrm{i}\Delta(t-t_1')}g(\Delta)+\left(-\mathrm{i}\sin\varphi_1-\cos\varphi_1\right)\mathrm{e}^{\mathrm{i}\omega_0 t}\mathrm{e}^{\mathrm{i}\Delta(t-t_1')}g(\Delta)\mathrm{d}\Delta=-\frac{kL\wp_{12}N}{\varepsilon_0}\cos\left(\omega_0 t+\varphi_1\right)G\left(t-t_1'\right). \qquad (21)$$

The Fourier transform of the spectral distribution $g(\Delta)$ is certain autocorrelation function, that we denote by $G(t)$, its expression is

$$G(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(\Delta)\mathrm{e}^{-\mathrm{i}\Delta t}\mathrm{d}\Delta=\exp\left(-t^2/2T_2^2\right). \qquad (22)$$

This shows that the macroscopic dipole moment of the collective atomic system oscillates at the center frequency ω_0, and its initial phase receives the phase modulation of the first pulse. When $t>t_1'$, $G(t-t_1')$ become very small and the photon echo doesn't occur.

We simulate the process of free induction decay, the duration of first pulse $t=t_1'-t_1$=50 ns ,as shown in the Fig.4, the different initial phases φ_1 of the first pulse are equal to 0, π/4 and π/2.

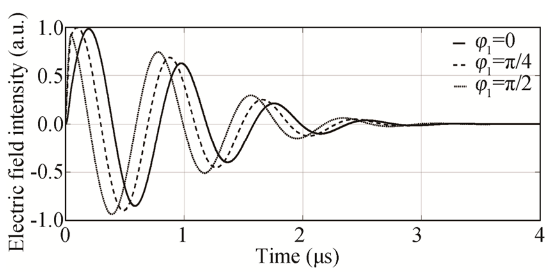

Fig.4 Free induction decay with different phases

As can be seen from Fig.4, as the phase increases, the waveform of the signal electric field shifts to the left compared with the waveform at φ_1=0, which is consistent with Eq.(21).

Eqs.(15)—(17) as the initial value of Eqs.(18)—(20), we get the results:

$$u(\Delta, t=t_2) \approx \sin(\varphi_1 + \Delta(t_2 - t_1')) , \quad (23)$$

$$v(\Delta, t=t_2) \approx -\cos(\varphi_1 + \Delta(t_2 - t_1')) , \quad (24)$$

$$w(\Delta, t=t_2) \approx 0 . \quad (25)$$

At the time t_2, the system receives the second pulse, which phase is φ_2. By substituting Eqs.(23)—(25) into Eqs.(9)—(11), we get the value of each component of Bloch vector at the end of the pulse at t_2':

$$u(\Delta, t=t_2') \approx \sin(\varphi_1 - 2\varphi_2 + \Delta(t_2 - t_1')) , \quad (26)$$

$$v(\Delta, t=t_2') \approx \cos(\varphi_1 - 2\varphi_2 + \Delta(t_2 - t_1')) , \quad (27)$$

$$w(\Delta, t=t_2') \approx 0 . \quad (28)$$

By substituting Eqs.(26) and (27) into Eqs.(15)—(17), we can get the time-domain expression of $t>t_2'$ moment and signal electric field as

$$E_s(t) = \frac{kL\wp_{12}N}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} e^{i(\varphi_1-2\varphi_2+\Delta(t_2-t_1'))} e^{-i\omega_0 t} e^{-i\Delta(t-\tau)} g(\Delta)\,d\Delta +$$

$$c.c = \frac{kL\wp_{12}N}{4\pi\varepsilon_0} e^{i(\varphi_1-2\varphi_2)} e^{-i\omega_0 t}\int_{-\infty}^{\infty} e^{-i\Delta(t-2(t_2-t_1'))} g(\Delta)\,d\Delta +$$

$$c.c. = \frac{kL\wp_{12}N}{\varepsilon_0}\cos(\omega_0 t - \varphi_1 + 2\varphi_2) G(t - 2(t_2 - t_1')) . \quad (29)$$

So, we get the expression of electric field. When $t=2(t_2-t_1')$, the function $G(t-2(t_2-t_1'))$ get the maximum value And decay quickly after $t>2(t_2-t_1')$.The initial phase of 2PE is modulated by the first pulse and the second pulse, which is expressed as $-\varphi_1+2\varphi_2$.

We simulated the 2PE. The simulation of the amplitude in time domain is shown in Fig.5. Under the simulation conditions, we set the first decay duration $t=2(t_2-t_1')=10$ μs and we can obtain the time that when the amplitude of the 2PE gets its maximum is $t_{echo}=2(t_2-t_1')=20$ μs. We give the simulation diagram of $E_s(t)$ from the time $t=t_1=0$.

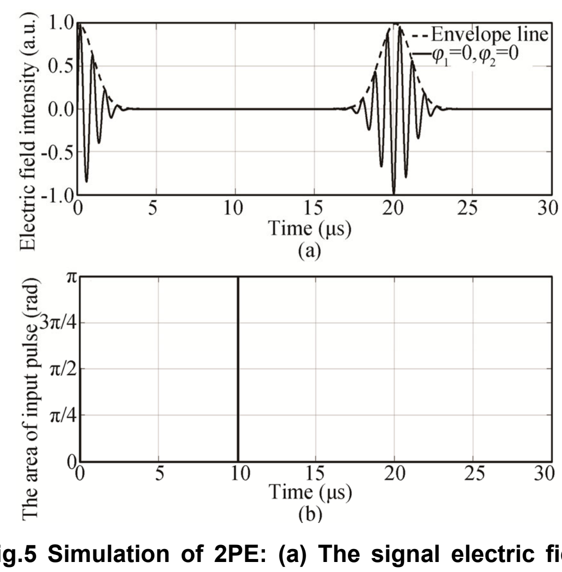

Fig.5 Simulation of 2PE: (a) The signal electric field waveform of 2PE, where the maximum amplitude of 2PE signal is generated at *t*≈20 μs; (b) Time sequence and area of input pulses of 2PE, where the first pulse is input at *t*₁=0, the second pulse is input at *t*₂=10.05 μs, and the pulse width of both pulses is 50 ns

By changing the phases of the input pulses, we can simulate different waveforms of the signal electric field. The influence of different initial phases on 2PE is shown in the Fig.6.

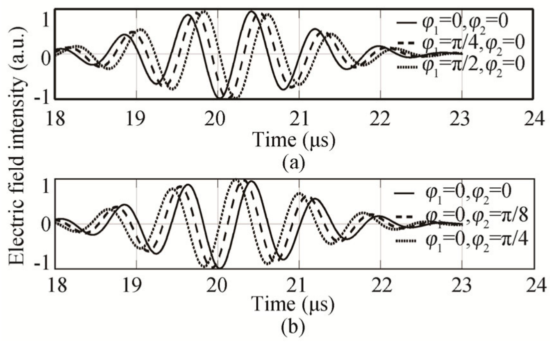

Fig.6 Profiles of 2PE with (a) φ_1=0, π/4, π/2 and φ_2=0 and (b) φ_1=0 and φ_2=0, π/8, π/4, where the amount of phase change of the first pulse is twice of that of the second one, i.e. $\varphi_1=2\varphi_2$

As shown in Fig.6(a), we assume that the initial phase of the second pulse is $\varphi_2=0$, and the initial phase of the first pulse is $\varphi_1=0$, π/4, π/2. As for Fig.6(c), on the contrary, we assume that the initial phase of the first pulse is $\varphi_1=0$, and the initial phase of the second pulse is $\varphi_2=0$, π/8, π/4. For Fig.6(b), as phase φ_1 increases, the waveform shifts to the right. For Fig.6(c), as phase φ_2 increases, the waveform shifts to the left, and they have the same pulse interval, which is consistent with Eq.(29).

In this paper, a method is proposed to obtain the analytical solutions of Bloch equations with phase term by using the elementary matrix e^{tA} transform. The effect of the initial phase on the components of Bloch vector is discussed. The result shows that by considering the initial phase of the input pulse, the amplitudes of in-phase component and in-quadrature component can be dynamically controlled. When the detuning is 0, their amplitudes are of sinφ and cosφ oscillations, while the population inversion is almost immune to the initial phase. Additionally, the signal electric field expressions including the phase effect of 2PE is derived, which is modulated by the initial phases. For 2PE, its initial phases are equal to $-\varphi_1+2\varphi_2$ .The work shows that the initial phases of 2PE can be fully controlled by the phases of input pulses.

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