Tripartite entanglement properties in the system of atoms interacting with three coupled cavities*

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It is considered that three identical two-level atoms are separately trapped in three coupled single-mode optical cavities, and each atom resonantly interacts with cavity via a one-photon transition. The tripartite entanglement dynamics among atoms is studied. The influence of cavity-cavity coupling constant on the tripartite entanglement among atoms is discussed. The results obtained using the numerical method show that the tripartite entanglement among atoms has a nonlinear relation with the cavity-cavity coupling coefficient. On the other hand, the three-body entanglement is the result of the coherent superposition of the two-body entanglements.
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Entangled state of two or more particles is not only a key ingredient for the quantum nonlocality, but also a basic resource to achieve quantum communication and quantum computation. It plays an important role in quantum information processing and quantum computation[1-3]. Quantum entanglement dynamics for two atoms trapped in two coupled cavities was investigated in Ref.[4]. Hao et al[5] studied the dynamics of quantum entanglement in reservoir with memory effects. Liao[6] researched the sudden death of the entanglement between an isolated atom and an atom inside a bimodal cavity. Dynamics properties of the system consisting of a Λ-type atom and a V-type atom trapped individually in two separated cavities were studied by the author of this paper[7]. But, up to now there has not been an appropriate measure to describe the entanglement of three and more subsystems quantitatively due to the high complexity of entanglement in multi-particle system. In recent years, there have also been considerable efforts to characterize tripartite entanglement properties qualitatively and quantitatively. Laskowski Wieslaw et al[8] presented the correlation-tensor criteria for genuine multiqubit entanglement. Hwang Mi-Ra et al[9] investigated tripartite entanglement in a noninertial frame.

On the other hand, it has been shown that the coupled-cavity system is very useful for distributed quantum computation. Therefore, in the last two decades, the coupled-cavity system has attracted much interest. For example, Xiao et al[10] presented a treatment of the entanglement transfer between atoms in two distant cavities coupled by an optical fiber. Zheng et al[11] proposed a generation of two-mode squeezed states for two separated atomic ensembles via coupled cavities. The author of this paper[12] discussed the entanglement properties in the system of a Λ-type atom and a V-type atom trapped in two distant cavities connected by an optical fiber, and so on[13-19]. However, until now, the tripartite entanglement in the system of atoms interacting with coupled cavities has not been discussed. In this paper, we consider the situation that three identical two-level atoms are separately trapped in three single-mode coupled cavities, and the atoms interact resonantly with the cavity fields. By making use of the entanglement tensor approach which was proposed in Ref.[20], we investigate the tripartite entanglement dynamics in the system of three atoms interacting with three coupled cavities. The effects of cavity-cavity coupling constant on the tripartite entanglement are discussed.

We consider a system of three coupled single-mode optical cavities, each containing a single two-level atom, which resonantly interacts with the cavity field by one-photon transitions, as shown in Fig.1. The cavities are coupled to each other with one-photon exchange. The interaction Hamiltonian of the system is written by

$$H_I = f_1(a_A s_1^+ + a_A^+ s_1^-) + f_2(a_B s_2^+ + a_B^+ s_2^-) + f_3(a_C s_3^+ + a_C^+ s_3^-) + J_1(a_A a_B^+ + a_A^+ a_B) + J_2(a_B a_C^+ + a_B^+ a_C), \qquad (1)$$

where the rotating-wave approximation is adopted, a_L^+ (L=A, B, C) and a_L are the creation operator and the annihilation operator of cavity L, respectively, s^+ and s^- are the atomic raising operator and the atomic lowering operator, respectively, f_l (l=1, 2, 3) is the atom-cavity coupling coefficient, and J_l (l=1, 2) is the cav-

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ity-cavity coupling coefficient. For the sake of simplicity, we assume that $f_1=f_2=f_3=f$ and $J_1=J_2$.

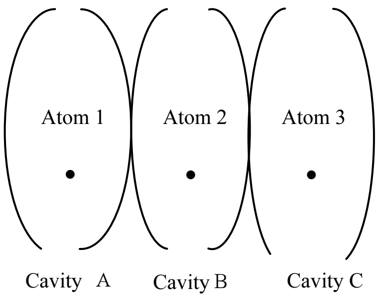


Fig.1 Schematic diagram of the system with three coupled single-mode optical cavities

We define the total excitation number operator as $\hat{N}=\sum_{l=1}^{3}|e\rangle_l\langle e|+a_{\mathrm{A}}^{+}a_{\mathrm{A}}+a_{\mathrm{B}}^{+}a_{\mathrm{B}}+a_{\mathrm{C}}^{+}a_{\mathrm{C}}$, where $|e_l\rangle$ ($l=1,2,3$) denotes excited state of the l atom, and $|g_l\rangle$ denotes ground state of the l atom. It is easily shown that the excitation number operator commutes with the Hamiltonian as Eq.(1). Therefore, the total excitation number is conserved during the dynamic evolution of the Hamiltonian H. The subspace with one excitation number can be spanned by the basis state vectors:

$$
\begin{aligned}
|\varphi_1\rangle&=|egg\rangle_{123}|000\rangle_{\mathrm{ABC}} ,\\
|\varphi_2\rangle&=|geg\rangle_{123}|000\rangle_{\mathrm{ABC}} ,\\
|\varphi_3\rangle&=|gge\rangle_{123}|000\rangle_{\mathrm{ABC}} ,\\
|\varphi_4\rangle&=|ggg\rangle_{123}|100\rangle_{\mathrm{ABC}} ,\\
|\varphi_5\rangle&=|ggg\rangle_{123}|010\rangle_{\mathrm{ABC}} ,\\
|\varphi_6\rangle&=|ggg\rangle_{123}|001\rangle_{\mathrm{ABC}} ,
\end{aligned}
\tag{2}
$$

where the subscripts 1, 2 and 3 label the first, second and third atoms, and A, B and C label cavity A, cavity B and cavity C, respectively.

At any time, the state of the whole system can be written as

$$
|\varphi(t)\rangle=A|\varphi_1\rangle+B|\varphi_2\rangle+C|\varphi_3\rangle+ D|\varphi_4\rangle+E|\varphi_5\rangle+F|\varphi_6\rangle. \tag{3}
$$

The time evolution of the total system is governed by the Schrödinger equation ($\hbar=1$)

$$
\mathrm{i}\hbar\frac{\partial|\varphi(t)\rangle}{\partial t}=H_I|\varphi(t)\rangle . \tag{4}
$$

For a given initial state $|\varphi_1\rangle$, substituting Eq.(3) into Eq.(4), we can obtain the solution of Eq.(4) as

$$
\begin{aligned}
A&=a\cos(\alpha t)+b\cos(\beta t)+\frac{1}{2}\cos(ft) ,\\
B&=-\frac{\mathrm{i}}{J}[\frac{\alpha^2-f^2}{\alpha}a\sin(\alpha t)+\frac{\beta^2-f^2}{\beta}b\sin(\beta t)] ,\\
C&=a\cos(\alpha t)+b\cos(\beta t)-\frac{1}{2}\cos(ft) ,\\
D&=-\frac{\mathrm{i}}{f}[a\alpha\sin(\alpha t)+b\beta\sin(\beta t)+\frac{f}{2}\sin(ft)] ,\\
E&=\frac{1}{fJ}[(\alpha^2-f^2)a\cos(\alpha t)+(\beta^2-f^2)b\cos(\beta t)] ,\\
F&=-\frac{\mathrm{i}}{f}[a\alpha\sin(\alpha t)+b\beta\sin(\beta t)-\frac{f}{2}\sin(ft)] ,
\end{aligned}
\tag{5}
$$

where $\alpha=\sqrt{f^2+J^2-P}$, $\beta=\sqrt{f^2+J^2+P}$, $P=J\times\sqrt{2f^2+J^2}$, $a=\frac{P+J^2}{4P}$, and $b=\frac{P-J^2}{4P}$.

For investigating the tripartite entanglements among atoms, we adopt the entanglement tensor approach which was introduced in Ref.[20] to quantify the degree of entanglement. Entanglement measures of a three-qubit system involve both an inter-three-qubit entanglement measure denoted by E_3 and an inter-two-qubit entanglement measure denoted by E_2. The entanglement measures of E_3 and E_2 can be defined as follows:

$$
\begin{aligned}
E_3&=\frac{1}{4}\sum_{i,j,k=1}^{3}M_{ijk}(1,\ 2,\ 3)M_{ijk}(1,\ 2,\ 3) ,\\
E_2(m,n)&=\frac{1}{3}\sum_{i,j=1}^{3}M_{ij}(m,\ n)M_{ij}(m,\ n) .
\end{aligned}
\tag{6}
$$

Here coherence vectors and the entanglement tensors are defined by

$$
\begin{aligned}
&\lambda_i(1)=\mathrm{Tr}(\hat{\rho}\bullet\hat{\sigma}_i\otimes\hat{I}\otimes\hat{I}) ,\\
&\lambda_j(2)=\mathrm{Tr}(\hat{\rho}\bullet\hat{I}\otimes\hat{\sigma}_j\otimes\hat{I}) ,\\
&\lambda_k(3)=\mathrm{Tr}(\hat{\rho}\bullet\hat{I}\otimes\hat{I}\otimes\hat{\sigma}_k) ,\\
&M_{ijk}(1,\ 2,\ 3)=K_{ijk}(1,\ 2,\ 3)-\lambda_i(1)M_{jk}(2,\ 3)-\\
&\qquad\lambda_j(2)M_{ik}(1,\ 3)-\lambda_k(3)M_{ij}(1,\ 2)-\\
&\qquad\lambda_i(1)\lambda_j(2)\lambda_k(3) ,\\
&M_{ij}(m,\ n)=K_{ij}(m,\ n)-\lambda_i(m)\lambda_j(n) ,\\
&k_{ij}(1,2)=\mathrm{Tr}(\hat{\rho}\bullet\hat{\sigma}_i\otimes\hat{\sigma}_j\otimes\hat{I}) ,\\
&k_{ik}(1,3)=\mathrm{Tr}(\hat{\rho}\bullet\hat{\sigma}_i\otimes\hat{I}\otimes\hat{\sigma}_k) ,\\
&k_{jk}(2,3)=\mathrm{Tr}(\hat{\rho}\bullet\hat{I}\otimes\hat{\sigma}_j\otimes\hat{\sigma}_k) ,\\
&k_{ijk}(1,2,3)=\mathrm{Tr}(\hat{\rho}\bullet\hat{\sigma}_i\otimes\hat{\sigma}_j\otimes\hat{\sigma}_k) ,
\end{aligned}
\tag{7}
$$

where $\hat{\rho}$ is the density operator of the whole system, $\hat{\sigma}_i$ (i=1, 2, 3) is the Pauli matrix, and $\hat{I}$ is unit matrix.

Using Eq.(3) and tracing over the variables of the field, the density matrix of atom 1, atom 2 and atom 3 is written in the Hilbert space spanned by { $|eee\rangle_{123}$, $|eeg\rangle_{123}$, $|ege\rangle_{123}$, $|egg\rangle_{123}$, $|gee\rangle_{123}$, $|geg\rangle_{123}$, $|gge\rangle_{123}$

and $|ggg\rangle_{123}\}$ as

$$\boldsymbol{\rho}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & |A|^2 & 0 & AB^* & AC^* & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & BA^* & 0 & |B|^2 & BC^* & 0\\ 0 & 0 & 0 & CA^* & 0 & CB^* & |C|^2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & |DEF|^2 \end{bmatrix}, \qquad (8)$$

where $|DEF|^2=|D|^2+|E|^2+|F|^2$. Using Eqs.(7) and (8), we can obtain

$$\lambda_x(l)=0 \quad (l=1,\ 2,\ 3),$$
$$\lambda_y(l)=0 \quad (l=1,\ 2,\ 3),$$
$$\lambda_z(1)=-1+2|A|^2 \ ,$$
$$\lambda_z(2)=-1+2|B|^2 \ ,$$
$$\lambda_z(3)=-1+2|C|^2 \ . \qquad (9)$$

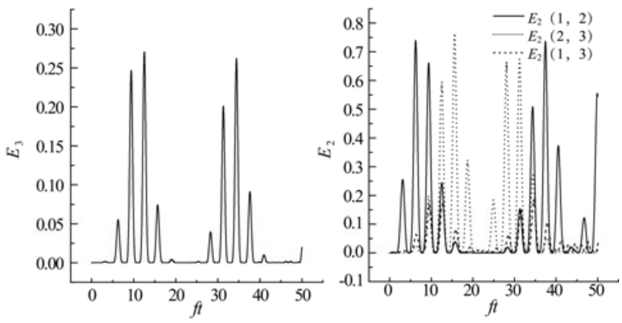
We can also get the nonzero entanglement tensors as

$$M_{xx}(1,\ 2)=M_{yy}(1,\ 2)=AB^*+BA^* \ ,$$
$$M_{xy}(1,\ 2)=-M_{yx}(1,\ 2)=-\mathrm{i}(AB^*-BA^*) \ ,$$
$$M_{zz}(1,\ 2)=-4|A|^2|B|^2 \ ,$$
$$M_{xx}(1,\ 3)=M_{yy}(1,\ 3)=AC^*+CA^* \ ,$$
$$M_{xy}(1,\ 3)=-M_{yx}(1,\ 3)=-\mathrm{i}(AC^*-CA^*) \ ,$$
$$M_{zz}(1,\ 3)=-4|A|^2|C|^2 \ ,$$
$$M_{xx}(2,\ 3)=M_{yy}(2,\ 3)=BC^*+CB^* \ ,$$
$$M_{xy}(2,\ 3)=-M_{yx}(2,\ 3)=-\mathrm{i}(BC^*-CB^*) \ ,$$
$$M_{zz}(2,\ 3)=-4|B|^2|C|^2 \ ,$$
$$M_{xxz}=M_{yyz}=-2|C|^2(AB^*+BA^*) \ ,$$
$$M_{xyz}=-M_{yxz}=\mathrm{i}2|C|^2(AB^*-BA^*) \ ,$$
$$M_{xzx}=M_{yzy}=-2|B|^2(AC^*+CA^*) \ ,$$
$$M_{xzy}=-M_{yzx}=\mathrm{i}2|B|^2(AC^*-CA^*) \ ,$$
$$M_{zxx}=M_{zyy}=-2|A|^2(BC^*+CB^*) \ ,$$
$$M_{zxy}=-M_{zyx}=\mathrm{i}2|A|^2(BC^*-CB^*) \ ,$$
$$M_{zzz}=16|A|^2|B|^2|C|^2 \ . \qquad (10)$$

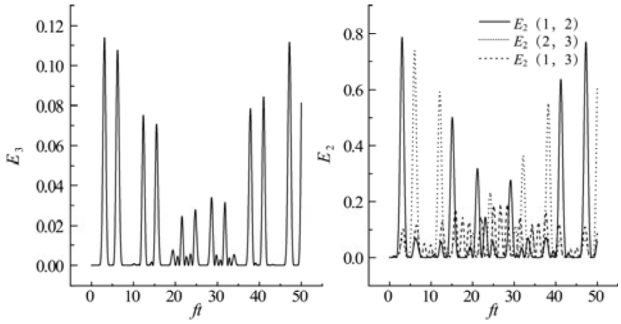
All other entanglement tensors equal zero. Substituting Eqs.(9) and (10) into Eq.(6), we can obtain entanglement measures of E_3 and E_2 as

$$E_3=8|A|^2|B|^2|C|^2(|A|^2+|B|^2+|C|^2+8|A|^2|B|^2|C|^2) \ ,$$
$$E_2(1,\ 2)=\frac{8}{3}|A|^2|B|^2(1+2|A|^2|B|^2) \ ,$$
$$E_2(1,\ 3)=\frac{8}{3}|A|^2|C|^2(1+2|A|^2|C|^2) \ ,$$
$$E_2(2,\ 3)=\frac{8}{3}|B|^2|C|^2(1+2|B|^2|C|^2) \ . \qquad (11)$$

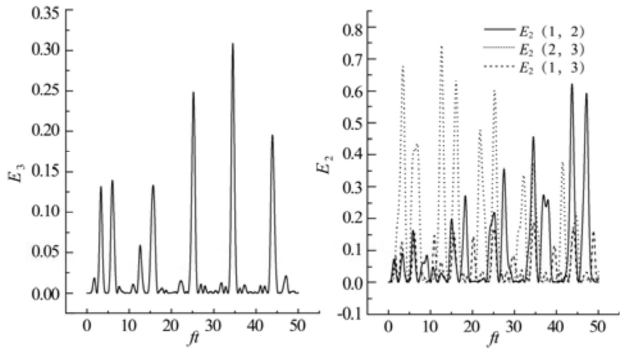
In Fig.2, we plot the time evolutions of entanglement measures of E_3 and E_2 ($E_2(1, 2)$, $E_2(2, 3)$ and $E_2(1, 3)$) as the function of the scale time *ft* with J=0.2f, J=0.5f, J=2f, and J=8f. From Fig.2, we can easily get three results as follows. Firstly, the time evolution of E_3 displays the collapse-revival phenomenon, and collapse-revival period first decreases and then increases as J increases. The mean of E_3 changes irregularly as J increases. For example, when J equals 0.2f, $\overline{E}_3$ equals 0.00210; when J equals 0.5f, $\overline{E}_3$ equals 0.00127; when J equals 2f, $\overline{E}_3$ equals 0.00244; and when J equals 8f, $\overline{E}_3$ equals 0.00271. These results show that the tripartite entanglement measure of E_3 has a nonlinear relation with the increase of cavity-cavity coupling coefficient. Secondly, the amount of $E_2(1, 2)$ entanglement is larger than that of $E_2(1, 3)$ entanglement, and that of $E_2(2, 3)$ entanglement is also larger than that of $E_2(1, 3)$ entanglement. Thirdly, the peak of the entanglement measure E_3 occurs when $E_2(1, 2)$, $E_2(2, 3)$ and $E_2(1, 3)$ all reach or are close to the peaks of themselves. For example, in Fig.2(d) when *ft* equals 15.76, 18.80 or 21.96, E_3 reaches the peak, and at the same time $E_2(1, 2)$, $E_2(2, 3)$ and $E_2(1, 3)$ all reach or are close to their peaks. It means that the three-body entanglement is the result of the coherent superposition of the two-body entanglements.



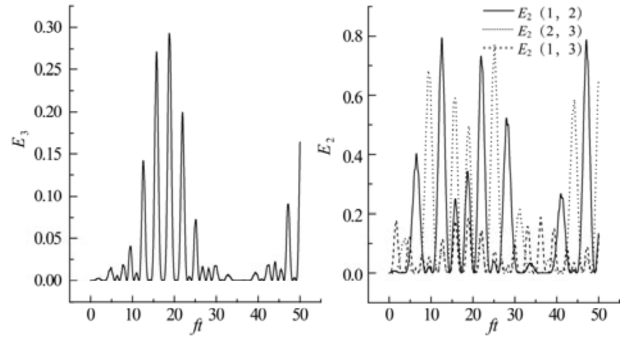
(a) J=0.2f



(b) J =0.5f



(c) J =2.0f



(d) J =8.0f

Fig.2 Time evolutions of entanglement measures E_3, E_2(1, 2), E_2(2, 3) and E_2(1, 3) as a function of the scale time

We introduce the atom-cavity system composed of three identical two-level atoms and three single-mode optical cavities. Each cavity contains an atom, which resonantly interacts with the cavity field by one-photon transition. The cavities are coupled to each other with one-photon exchange. We consider the situation that the total excitation number equals one. The tripartite entanglements among three atoms are studied by using the entanglement tensor approach. The influence of cavity-cavity coupling coefficient on the tripartite entanglement is discussed. The results show that the tripartite entanglement measure of E_3 has a nonlinear relation with the increase of cavity-cavity coupling coefficient. On the other hand, the three-body entanglement is the result of the coherent superposition of the two-body entanglements.

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