Dynamics of quantum discord for a two-qubit system*

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We study the dynamics of quantum discord of a two-qubit system coupled to a common structured reservoir at zero temperature. The conditions to maximize reservoir-induced quantum discord for the two-qubit system wiht an initially factorized state are derived. In particular, when the two qubits are placed in a lossy cavity, high values of quantum discord can be obtained in the dispersive regime, even in the bad-cavity limit. Finally, we show that under certain conditions, the quantum discord dynamics exhibits quantum beats.

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For a given bipartite quantum state, it is important to know whether it is entangled, separable, classically correlated or quantum correlated. Much effort has been invested in subdividing quantum states into separable and entangled states[1,2]. It is well known that the entanglement makes some tasks in quantum information possible which are impossible without it[3,4]. However, entanglement is not the only type of correlation which is useful for quantum technology. Recently, it was found that there are some quantum correlations other than entanglement which also offer some advantages, for example, quantum nonlocality without entanglement[5,6]. In addition, it was shown theoretically and later experimentally that some separable states may also speed up certain tasks over their classical counterparts[7,8]. Therefore, it is desirable to investigate, characterize and quantify quantum correlations more broadly.

On the other hand, we know that all realistic quantum systems interact inevitably with their surrounding environments, which introduces quantum noise into the systems. As a result, the quantum systems can lose their energy (dissipation) and/or coherence (dephasing). Thus, it is of fundamental importance to know the influence of the environment on quantum correlation. In several recent papers, quantum correlation dynamics in open quantum systems has been studied[9-15]. It was shown that the quantum correlation measured by quantum discord is more resistant against the environment than quantum entanglement[8].

In this paper, we study the dynamic action of both quantum and classical correlations by quantum discord for two-qubit systems in noisy environment. We present our physical model and study its time evolution. The dynamics of quantum correlation is studied analytically by means of quantum discord for the given initial states. We discuss the effect on quantum discord in the Markovian and non-Markovian cases, respectively, and provide a summary of the results together with some concluding remarks.

Let us consider an open quantum system consisting of two qubits coupled to a common zero-temperature bosonic reservoir. The Hamiltonian of the total system is

$$H = H_S + H_R + H_{int} , \tag{1}$$

where H_R is the Hamiltonian of the reservoir, and H_S is the Hamiltonian of the two qubits which are coupled to the common reservoer via the interaction H_{int}.

In the dipole and the rotating-wave approximations, and assuming $\hbar$=1, the Hamiltonian for the total system can be obtained as[15]

$$H_S = \omega_1\sigma_+^1\sigma_-^1 + \omega_2\sigma_+^2\sigma_-^2 , \tag{2}$$

$$H_R = \sum_k \omega_k b_k^+ b_k , \tag{3}$$

$$H_{int} = (\alpha_1\sigma_+^{(1)} + \alpha_2\sigma_+^{(2)})\sum_k g_k b_k + h\cdot c , \tag{4}$$

where b_k^+, and b_k are the creation and annihilation operators of quanta of the reservoir, $\sigma_\pm^{(j)}$ and ω_j are the inversion operator and transition frequency of the jth qubit (j = 1,2), respectively, ω_k is the frequency of the k mode of reservoir,

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and $\alpha_j g_k$ describes the coupling strength between the jth qubit and the k mode of the reservoir. α_j is dimensionless real coupling constant measuring the interaction strength of each single qubit with the reservoir. We denote the collective coupling constant by $\alpha_T = (\alpha_1^2+\alpha_2^2)^{1/2}$ and the relative interaction strength by $r_j = \alpha_j/\alpha_T$.

We restrict ourselves to the case in which only one excitation is present in the system and the reservoir is in the vacuum. Initially, the two-qubit system is assumed to be disentangled from its reservoir, and the initial state for the whole system is written as

$$\left|\Psi(0)\right\rangle = [c_{01}\left|1\right\rangle_1\left|0\right\rangle_2 + c_{02}\left|0\right\rangle_1\left|1\right\rangle_2]\bigotimes_k\left|0_k\right\rangle_R, \qquad (5)$$

where $c_{01}=\sqrt{(1-s)/2}$ and $c_{02}=\sqrt{(1+s)/2}\,e^{i\phi}$ are complex numbers defining the initial state for the qubit system, $-1\le s\le 1$, $\left|0\right\rangle_j$ and $\left|1\right\rangle_j$ (j=1,2) are the ground and excited states of the jth qubit, respectively, and $\left|0_k\right\rangle_R$ is the state of the reservoir with zero excitations in the k mode.

As a consequence of the time evolution generated by Eq.(1), the excitation can be shared by the qubits and the reservoir, so that the time evolution of the total system is given by

$$\left|\Psi(0)\right\rangle = c_1(t)\left|1\right\rangle_1\left|0\right\rangle_2\left|0\right\rangle_R + c_2(t)\left|0\right\rangle_1\left|1\right\rangle_2\left|0\right\rangle_R + \sum_k c_k(t)\left|0\right\rangle_1\left|0\right\rangle_2\left|1_k\right\rangle_R, \qquad (6)$$

where $\left|0_k\right\rangle_R$ is the state of the reservoir with only one excitation in the kth mode and $\left|0\right\rangle_R = \otimes_k\left|0_k\right\rangle$.

The reduced density matrix describing the two-qubit systems is obtained from the density operator $\left|\Psi(t)\right\rangle\left\langle\Psi(t)\right|$ after tracing over the reservoir degrees of freedom, which takes the form as

$$\rho(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |c_1(t)|^2 & c_1(t)c_2^*(t) & 0 \\ 0 & c_1^*(t)c_2(t) & |c_2(t)|^2 & 0 \\ 0 & 0 & 0 & 1-|c_1|^2-|c_2|^2 \end{pmatrix}. \qquad (7)$$

Therefore, the two-qubit dynamics is completely characterized by the amplitudes of $c_{1,2(t)}$.

By solving the Schrödinger equation, we can obtain the motion equations for $c_{1,2}(t)$ as

$$\dot{c}_1(t) = -\int_0^t dt_1[\alpha_1^2 c_1(t_1) + \alpha_1\alpha_2 c_2(t_1)e^{-i\delta_{21}t_1}]f(t-t_1)e^{i\delta_1(t-t_1)}, \qquad (8)$$

$$\dot{c}_2(t) = -\int_0^t dt_1[\alpha_1\alpha_2 c_1(t_1)e^{i\delta_{21}t_1} + \alpha_2^2 c_2(t_1)]f(t-t_1)e^{i\delta_2(t-t_1)}, \qquad (9)$$

where $\delta_j = \omega_j - \omega_c$, $\delta_{21} = \delta_2 - \delta_1$, and the correlation function $f(t-t_1)$ is related to the spectral density $J(\omega)$ of the reservoir by

$$f(t-t_1) = \int d\omega J(\omega)e^{i(\omega-\omega_c)(t-t_1)}; \qquad (10)$$

It is obvious that the solution of $c_{1,2}(t)$ is determined by the explicit form of $J(\omega)$. We consider the spectral distribution of the Lorentzian form[15] as

$$J(\omega) = \frac{W^2}{\pi}\cdot\frac{\lambda}{(\omega-\omega_c)^2+\lambda^2}; \qquad (11)$$

where ω_c is the fundamental frequency of the cavity, the weight W is proportional to the vacuum Rabi frequency, and λ is the width of the distribution and therefore describes the cavity losses (photon escape rate).

Substituting Eq.(11) into Eq. (10), one can obtain the analytical form of $f(t-t_1)$, and then the analytical solution for the amplitudes $c_1(t)$ and $c_2(t)$ can be easily solved as

$$c_1(t) = [r_2^2 + r_1^2\varepsilon(t)]c_1(0) - r_1r_2[1-\varepsilon(t)]c_2(0); \qquad (12)$$

$$c_2(t) = -r_1r_2[1-\varepsilon(t)]c_1(0) + [r_1^2 + r_2^2\varepsilon(t)]c_2(0); \qquad (13)$$

with

$$\varepsilon(t) = e^{-(\lambda-i\delta)t/2}[\cosh(\Omega t/2) + \frac{\lambda - i\delta}{\Omega}\sinh(\Omega t/2)], \qquad (14)$$

where $\delta_1 = \delta_2 = \delta$ and $\Omega = \sqrt{\lambda^2 - \Omega_R^2 - 2i\delta\lambda}$ with $\Omega_R = \sqrt{4W^2\alpha_T^2 + \delta^2}$ as the generalized Rabi frequency and $R=W\alpha_T$ as the vacuum Rabi frequency.

It is well known that the total correlations between two subsystems A and B described by a bipartite quantum state as $\rho_{AB} = \rho_S$ is generally measured by quantum mutual information[16] as

$$\tau(\rho_S) = S(\rho_A) + S(\rho_B) - S(\rho_S), \qquad (15)$$

where $S(\rho) = -\mathrm{Tr}(\rho\log_2\rho)$ is the Von Neumann entropy of density matrix ρ, and $\rho_A(\rho_B)$ is the reduced density operator for subsystem A(B).

Quantum mutual information is classified as quantum correlation D and classical correlation C[17]. And the quantum correlation D is quantified by the so-called quantum discord[18-20]. Then the quantum nature of correlation between two quantum systems is different between quantum mutual information I and classical correlation,

$$D(\rho_S) = \tau(\rho_S) - C(\rho_S), \qquad (16)$$

which means that for obtaining the amount of quantum correlation, one has to find its classical part. The classical correlation $C(\rho_S)$ is defined as the maximum information about one subsystem ρ_i, which depends on the type of measurement performed on the other subsystem. For a local projective measurement Π_k performed on the subsystem B with a given outcome k, we denote the prcbability as

$$P_k = \mathrm{Tr}[(I_A\otimes\Pi_k)\rho_S(I_A\otimes\Pi_k)], \qquad (17)$$

where I_A is the identity operator for the subsystem A. Then the classical correlation is

$$C(\rho_S)=\max_{\Pi_k}[S(\rho_A)-\sum_k P_k S(\rho_A^{(k)})]=$$
$$S(\rho_A)-\min_{\Pi_k}[\sum_k P_k S(\rho_A^{(k)})]\,, \tag{18}$$

where the maximum is taken over the complete set of orthogonal projectors Π_k, and $\rho_A^{(k)}=\frac{1}{P_k}\mathrm{Tr}[(I_A\otimes\Pi_k)\rho(I_A\otimes\Pi_k)]$ is the state of the subsystem A on condition that the measurement of the outcome is labelled by k.

For the system considered in this paper, the optimization problem in the definition of the classical correlations can be solved exactly, and a simple analytical expression for this quantity can be derived. Indeed, calculate the action of the one-qubit projectors,

$$\Pi_l^{(2)}=I\otimes|k\rangle\langle k|\text{, with } k=a,b\,, \tag{19}$$

and

$$|a\rangle=\cos\theta|\uparrow\rangle+\mathrm{e}^{\mathrm{i}\phi}\sin\theta|\downarrow\rangle; \tag{20}$$
$$|b\rangle=\sin\theta|\uparrow\rangle-\mathrm{e}^{\mathrm{i}\phi}\cos\theta|\downarrow\rangle. \tag{21}$$

On the general two-qubit state given by Eq.(7) and using Eq.(16), it is straightforward to prove that the classical correlations do not explicitly depend on ö, and are maximized for è = (2n+1)ð/2 with $n\in Z$. The analytic expression for $C(\rho_S)$ is given by

$$C(\rho_S)=[1-|c_1(t)|^2-|c_2(t)|^2]\log_2[1-|c_1(t)|^2-|c_2(t)|^2]-$$
$$\sum_{j=1,2}[1-|c_j(t)|^2]\log_2[1-|c_j(t)|^2]. \tag{22}$$

The analytic expression of the discord is

$$D(\rho_S)=|c_1(t)|^2\log_2(1+\frac{|c_2(t)|^2}{|c_1(t)|^2})+|c_2(t)|^2\log_2(1+\frac{|c_1(t)|^2}{|c_2(t)|^2}). \tag{23}$$

We first consider the bad-cavity limit case. Time evolutions of the quantum discord in bad-cavity limit and resonant limit are shown in Fig.1. In Fig.1, the time is measured in units of λ with $\tau=\lambda t$. In the bad-cavity case, e.g., for R=0.1λ and for small values of the detuning $\delta<R$, the behavior of the quantum discord does not change appreciably compared with that of the resonant case. For values of the detuning $\delta\approx R$, i.e., when approaching the dispersive regime, the dynamics for an initially factorized state (s=1) shows a monotonic increase to the stationary value of the quantum discord in the resonant case as shown in Fig.1(a). However, a significant change occurs in the bad-cavity limit when the system is prepared in an initial entangled state. Indeed, one can prove that in this regime, contrary to situation in the resonant case, the finite time $\bar{t}$ which satisfies $D(\bar{t})$=0 does not exist anymore as shown in Fig.1(b).

We now focus on the dispersive regime $\delta\gg\lambda\gg R$. Fig.2 shows the evolution of quantum discord as a function of scaled

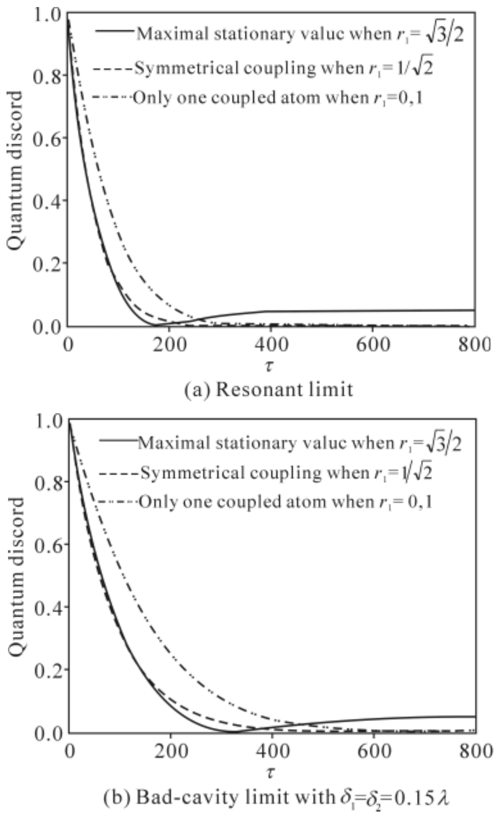


Fig.1 Time evolutions of the quantum discord in the bad-cavity limit (R=0.1) with s=0 and φ=0 and resonant limit

time $\tau=\lambda t$ in bad-cavity limit with $\delta_1=\delta_2=10\lambda$. In this case, the quantum correlation shows oscillations as a function of time for all of the initial atomic states for which a finite stationary quantum discord is obtained, $D_S\neq 0$. Due to the presence of these oscillations and for an initially factorized state, the quantum discord is greater than the stationary value D_S even in the bad-cavity limit, as shown in Fig.2.

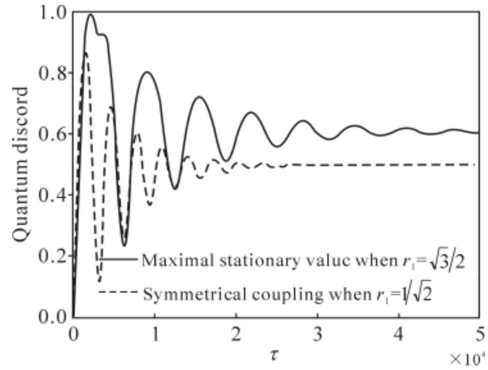


Fig.2 Evolution of quantum discord as a function of the scaled time $\tau=\lambda t$ in dispersive regime with $\delta_1=\delta_2=10\lambda$ under the bad-cavity limit (R=0.1) with s=1

When it is in good-cavity limit, i.e., in the strong-coupling case, quantum discord oscillations are present for any initial atomic state. Moreover, for $\delta\approx\lambda\ll R$, when both atoms are effectively coupled to the cavity field, i.e., $r_1\neq 0$

and 1, the dynamics of quantum correlation is characterized by the quantum discord of quantum beats, as shown in Fig.3. For initially entangled states, this phenomenon is more evident for $\varphi = \pi$, because the value of stationary entanglement in this case is higher, and the behavior of the quantum is more regular.

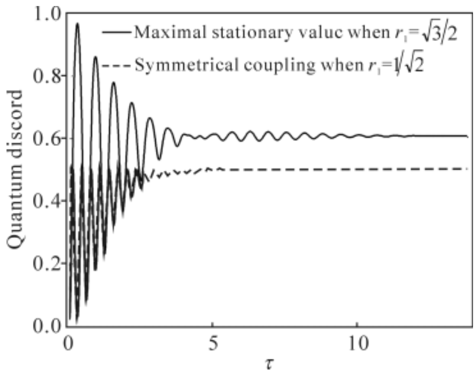
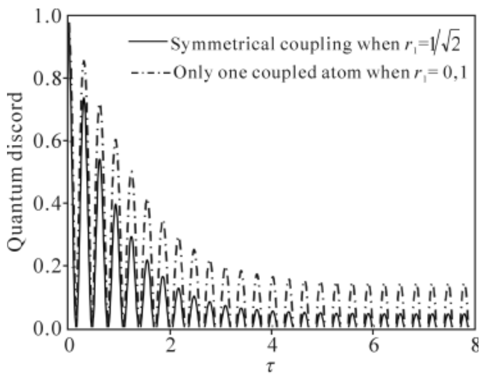


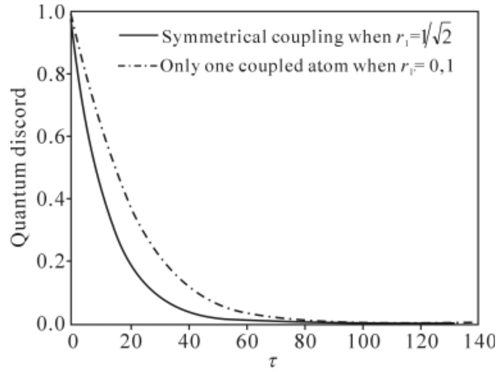
Fig.3 Quantum discord as a function of $\tau=\lambda t$ outside the dispersive region with $\delta_1=\delta_2=0.7\lambda$ in the good-cavity limit (R=10) with s=1

When only one of the two qubits is effectively coupled to the cavity field, i.e., for r_1=0 and 1, for maximally entangled initial states s = 0 in the resonant regime δ = 0, the system performs damped oscillations between the states $|\psi_+\rangle$ and $|\psi_-\rangle$ which are equally populated at the beginning. Hence, quantum discord revivals with maximum amplitude are present in the dynamics, as shown in Fig.4(a). Increasing the detuning, the amplitude of the oscillations decreases, and the revivals disappear, while the frequency does not change appreciably as shown in Fig.4(a). While for greater values of the detuning, the oscillations completely disappear, and the quantum discord decays exponentially as shown in Fig.4(b).

Finally, we note that similar to the behavior discussed in the bad-cavity limit, when the qubits are initially in a factorized state, the presence of the detuning enhances the generation of quantum discord in a short period compared with that in the resonant coupling case. In general, in the strongly dispersive regime, the qubits do not exchange energy with cavity, which is only virtually excited. Thus a high-degree reservoir-induced quantum correlation can be generated in both the good- and bad-cavity limits.



(a) $\delta_1=\delta_2=0.7\lambda$



(b) $\delta_1=\delta_2=50\lambda$

Fig.4 Time evolution of quantum discord with $\tau=\lambda t$ in the good-cavity limit (R=10) with s=0 and φ = 0

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