Propagation properties of controllable dark-hollow beams through fractional Fourier transform systems*

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Based on the definition of fractional Fourier transform (FrFT) in the cylindrical coordinate system, the propagation properties of a controllable dark-hollow beam (CDHB) are investigated in detail. An analytical formula is derived for the FrFT of a CDHB. By using the derived formula, the properties of a CDHB in the FrFT plane are illustrated numerically. The results show that the properties of the intensity of the beam in the FrFT are closely related to not only the fractional order but also initial beam parameter, beam order and the lens focal length of the optical system for performing FrFT. The derived formula provides an effective and convenient way for analyzing and calculating the FrFT of a CDHB.

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In recent years, optical beams with zero central intensity along the beam axis, which are called dark-hollow beams (DHBs), have attracted much attention because of their increasing applications in modern and atom optics. DHBs have been widely studied in both experimental and theoretical aspects. Several theoretical models have been presented to describe DHBs. The typical example is a TEM^*_{01} beam. A co-propagating TEM^*_{01} beam was first considered as an atom-focusing lens[1]. Subsequently, a doughnut beam was used as the optical trap for an atom[2]. Another model to describe DHBs is the high-order Bessel-Gaussian beams, which can be produced by use of an axicon[3]. In addition, some other models can also be used to describe DHBs, such as hollow Gaussian beams[4]. More recently, Mei et al presented a new theoretical model called controllable dark-hollow beams (CDHBs)[5] to describe DBHs with circular symmetry. The CDHB is shown to be an ideal and convenient model to describe dark-hollow beams, and can be used for guiding, focusing and trapping ultra-cold atoms, even Bose–Einstein condensates. Propagation properties of a CDHB through various optical systems have been studied extensively within the framework of the paraxial or nonparaxial approximation[5-7].The fractional Fourier transform (FrFT) is a generalization of the ordinary Fourier. It was firstly proposed as a new mathematical tool, and subsequently its potential applications in optics were explored in 1993 by Lohmann et al[8]. Since then, it has become a research subject in optics, and much work has been done on its properties, optical implementation and applications. For instance, the FrFT has been used as a new method in signal processing, beam shaping and image encryption. Recently, people have researched the FrFT of various beams used frequently in modern optics[9-16]. However, to the best of our knowledge, research on the FrFT of CDHBs has not been reported elsewhere. Therefore, for the properties and the wide application of these beams, the study on the behavior of CDHBs propagating through FrFT optical system would be of practical interest.

In this paper, therefore, the FrFT is applied to treat the propagation and transformation properties of CDHBs. Based on the definition of FrFT, analytical propagation expressions of CDHBs are derived in FrFT plane. The properties of the beams in the fractional Fourier plane and its dependence on fractional order, initial beam parameters and the lens focal length of the optical system for performing FrFT are studied in detail by using the derived formulas. Some typical numerical examples are given to illustrate the transformation characteristics of the beams in the FrFT plane. Finally, a simple conclusion is given.

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The optical systems for performing FrFT are depicted in Fig.1. Assume a stationary quasi-monochromatic source field is expressed by $\boldsymbol{E}(x_1, y_1)$, and the FrFT of $\boldsymbol{E}(x_1, y_1)$ achieved by the optical systems as shown in Fig.1 is given by[9]:

$$\boldsymbol{E}_p(x_2,y_2)=\frac{1}{\mathrm{i}\,f\lambda\sin\phi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{E}(x_1,y_1)\times \exp\left[-\frac{\mathrm{i}\,\pi(x_1^2+y_1^2+x_2^2+y_2^2)}{\lambda f\tan\phi}\right]\times \exp\left[\frac{2\,\pi\mathrm{i}(x_1x_2+y_1y_2)}{\lambda f\sin\phi}\right]\mathrm{d}\,x_1\,\mathrm{d}\,y_1\ , \tag{1}$$

where f is focal length of the lens, λ is the optical wavelength, x_1, y_1 and x_2, y_2 are the rectangular coordinates in the input plane and the fractional Fourier plane, respectively. ϕ is given by

$$\phi=\frac{p\pi}{2}\ , \tag{2}$$

where p is called the fractional order of the Fourier transform, and it may take any arbitrary real number. When p takes a value of $4k+1$, where k can be any integer, the FrFT reverts to a conventional Fourier transform. In Eq.(1), the factor $\frac{1}{\mathrm{i}\lambda f\sin\phi}$ in front of the integral ensures energy conservation after the FrFT. In fact, the two optical systems for performing the FrFT are equivalent[9].

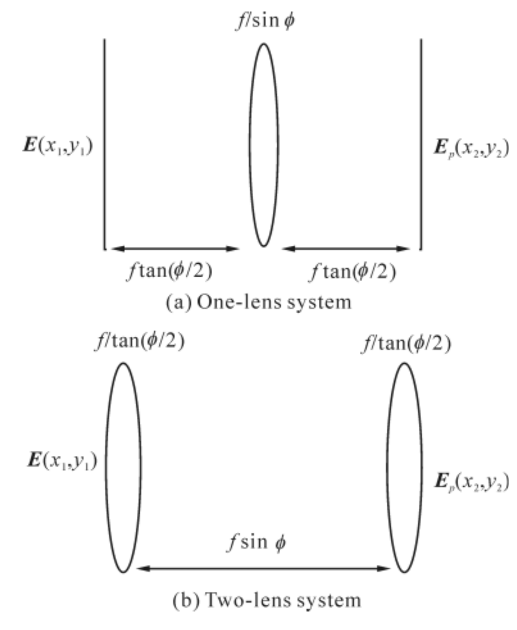

Fig.1 Optical systems for performing the FrFT

By setting $x_1=r_1\cos\theta_1$, $y_1=r_1\sin\theta_1$, $x_2=r_2\cos\theta_2$ and $y_2=r_2\sin\theta_2$ in Eq.(1), we can get the expression of FrFT in the cylindrical coordinate system as follows

$$\boldsymbol{E}_p(r_2,\theta_2)=\frac{1}{\mathrm{i}\lambda f\sin\phi}\int_0^{\infty}\int_0^{2\pi}\boldsymbol{E}(r_1,\theta_1)\exp\left[-\frac{\mathrm{i}\pi(r_1^2+r_2^2)}{\lambda f\tan\phi}\right]\times \exp\left[\frac{2\pi\mathrm{i}r_1r_2}{\lambda f\sin\phi}\cos(\theta_1-\theta_2)\right]r_1\mathrm{d}r_1\mathrm{d}\theta_1\ , \tag{3}$$

where r_1, θ_1 and r_2, θ_2 are the radial and the azimuth angle coordinates in the input and FrFT planes , respectively.

The analytical expression of the CDHB at z=0 in cylindrical coordinate system is as follows[5]:

$$\boldsymbol{E}(r_1,\theta_1)=\sum_{n=1}^{N}\boldsymbol{a}_n\left[\exp\left(-\frac{nr_1^2}{w_0^2}\right)-\exp\left(-\frac{nr_1^2}{v_0^2}\right)\right],\quad N=1,2,3,\cdots, \tag{4}$$

where the amplitude parameter $\boldsymbol{a}_n=\frac{(-1)^{n-1}}{N}\binom{n}{N}$ for the component Gaussian modes is adjustable, which gives us adequate free parameters to control the area of the dark region. Here $\binom{N}{n}$ denotes a binomial coefficient, N is the beam order of the CDHB, w_0 determines mainly the beam waist width, and $v_0=\varepsilon w_0$, where $\varepsilon<1$. The central dark size of the beam can also be controlled by the parameters ε and w_0. Fig.2 shows the contour graphs of the normalized intensity distribution of a CDHB at the plane $z=0$ for different beam parameters N and ε . It is clear from Fig.2 that the central dark size increases with the increasing of parameters N and ε . It is because a CDHB is not a pure mode but a superposition of a

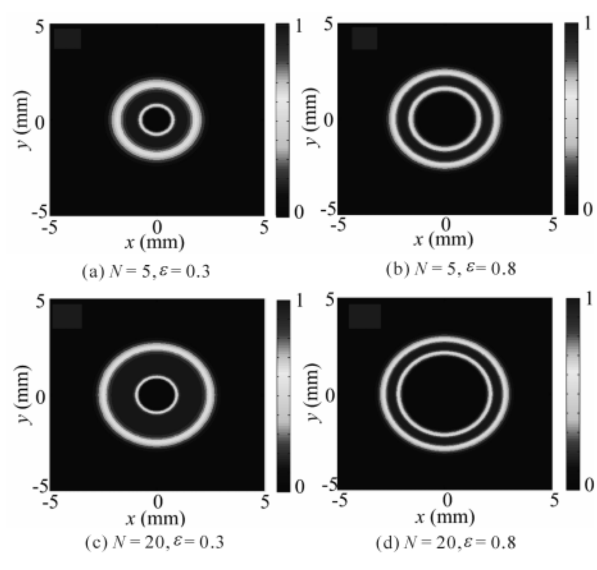

Fig.2 Contour graphs of the normalized intensity distribution of the CDHB with w_0=1.5 mm for differet beam parameters

series of Gaussian beams, which can be divided into a positive and a negative part.

Substituting Eq.(4) into Eq.(3), and using the following integral formula:

$$J_0(x)=\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\mathrm{i}x\cos(\theta_1-\theta_2)\right]\mathrm{d}\theta_1 , \qquad (5)$$

we can transform Eq.(3) into

$$\boldsymbol{E}_r(r_2,\theta_2)=\frac{1}{\mathrm{i}\lambda f\sin\phi}\sum_{n=1}^{N}\boldsymbol{a}_n\int_0^{\infty}\exp\left[-\left(\frac{n}{w_0^2}+\frac{\mathrm{i}\pi}{\lambda f\tan\phi}\right)r_1^2\right]\times J_0\left(\frac{2\pi r_1 r_2}{\lambda f\sin\phi}\right)r_1\,\mathrm{d}r_1-\frac{1}{\mathrm{i}\lambda f\sin\phi}\sum_{n=1}^{N}\boldsymbol{a}_n\times\int_0^{\infty}\exp\left[-\left(\frac{n}{v_0^2}+\frac{\mathrm{i}\pi}{\lambda f\tan\phi}\right)r_1^2\right]J_0\left(\frac{2\pi r_1 r_2}{\lambda f\sin\phi}\right)r_1\,\mathrm{d}r_1 . \qquad (6)$$

After tedious but straightforward integration, we can obtain

$$\boldsymbol{E}_p(r_2,\theta_2)=\frac{2\pi}{\mathrm{i}\lambda f\sin\phi}\exp\left(-\frac{\mathrm{i}\pi r_2^2}{\lambda f\tan\phi}\right)\times\left[\sum_{n=1}^{N}\frac{\boldsymbol{a}_n}{2\alpha_w}\exp\left(-\frac{\beta^2}{4\alpha_w}\right)-\sum_{n=1}^{N}\frac{\boldsymbol{a}_n}{2\alpha_v}\exp\left(-\frac{\beta^2}{4\alpha_v}\right)\right] , \qquad (7)$$

where

$$\alpha_w=\frac{n}{w_0^2}+\frac{\mathrm{i}\pi}{\lambda f\tan\phi} , \qquad (8)$$

$$\alpha_v=\frac{n}{v_0^2}+\frac{\mathrm{i}\pi}{\lambda f\tan\phi} , \qquad (9)$$

$$\beta=\frac{2\pi r_2}{\lambda f\sin\phi} . \qquad (10)$$

In the above derivation, we use the integral formula as follows:

$$\int_0^{\infty}\exp(-\alpha x^2)J_0(\beta x)x\mathrm{d}x=\frac{1}{2\alpha}\exp\left(-\frac{\beta^2}{4\alpha}\right) . \qquad (11)$$

The intensity distribution of the CDHB on the fractional Fourier transform plane is

$$I_p(r_2,\theta_2)=|\boldsymbol{E}_p(r_2,\theta_2)|^2 . \qquad (12)$$

Eq.(7) and Eq.(12) are the analytical formulas of CDHBs through the FrFT optical systems.

The propagation properties of a CDHB through an FrFT optical system are studied by using Eqs.(7)−(10). In the following calculations, we assume that the wavelength λ = 632.8 nm. Fig.3 represents the normalized intensity distribution of the CDHB with different fractional orders p in the FrFT plane. The other parameters are chosen as follows: N=5, f = 100 mm, w_0=1.5 mm and ε =0.8. The normalized intensity is given by $I/I_{\max}$, where $I_{\max}$ denotes the maximum of intensity. From Fig.3, we can see that the fractional order of the FrFT system has strong influence on the intensity distribution of the CDHB in the FrFT plane. When the fractional order meets the requirement of 0< p <1, the intensity distribution of the output beam becomes more and more convergent with increasing the fractional order. On the contrary, it can be seen that when 1< p <2, the intensity distribution of the output beam becomes more and more divergent with increasing the fractional order. Thus we can control the intensity distribution of the CDHB by choosing the fractional order p properly in the range of 0−2. What is more, the variation of normalized intensity distribution with p is periodic, and the period is 2. The case of p=1 corresponds to the ordinary Fourier transform case.

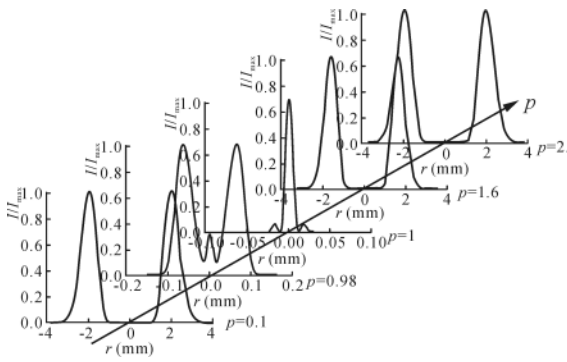

Fig.3 Normalized intensity distribution of a CDHB in the FrFT plane with different fractional orders

In Fig.4, we calculate the normalized intensity of a CDHB in the FrFT plane with different lens focal lengths f with p = 0.98, w_0=1.5 mm, λ = 632.8 nm, ε = 0.8 and N=5. In Fig.5, we calculate the normalized intensity distribution of a CDHB with different beam waists w_0 in the FrFT plane. f=100 mm and the rest of the parameters are the same as those in Fig.4. From Fig.4 and Fig.5, we can see that for a fixed p, the properties of the CDHB in the FrFT plane are closely related to lens focal length of the optical systems for performing FrFT and the initial beam waist. From Fig.4, it can be observed that with increasing focal length of the lens, the intensity distribution of the beam becomes more and more convergent. The light spot of the output beam changes gradually from annular shape to Gaussian shape. From Fig.5, it is evident to see that the intensity distribution of the output beam becomes more and more divergent with increasing the initial beam waist. For example, the beam spot at w_0=0.5 mm in the FrFT plane keeps one main peak; when w_0 further increases, the intensity distribution in the FrFT plane splits into two peaks, which means the beam spot evolves to annular shape. In particular, the distance between two peaks increases with increasing w_0, that is to say the radius of the bright ring becomes larger.

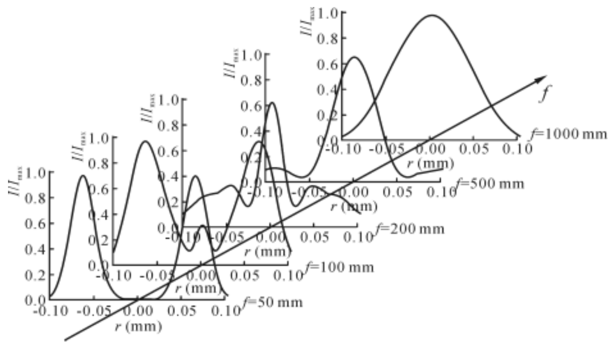

Fig.4 Normalized intensity distribution of a CDHB in the FrFT plane with different focal lengths

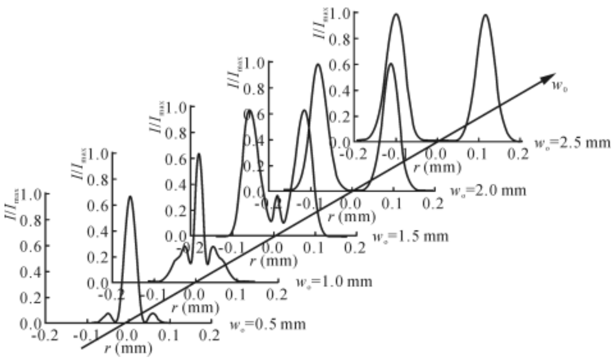

Fig.5 Normalized intensity distribution of a CDHB in the FrFT plane with different beam waists

Fig.6 investigates the influence of the beam order n and the beam parameter ε on the intensity distribution properties

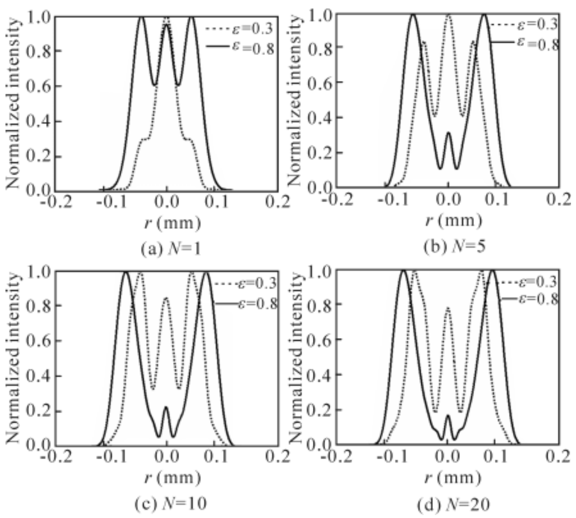

Fig.6 Normalized intensity distribution of a CDHB with different beam parameters ε in the FrFT plane for different beam orders N

of a CDHB in the FrFT plane. The focal length and the fractional order take the value of f=100 mm and p=0.98, respectively. From Fig.6, we can see that the normalized intensity distribution is related to the beam order n and beam parameter ε. For a fixed beam order N=1, the intensity distributions are quite distinct from each other for the case of ε=0.3 and 0.8, which are denoted by dot line and solid line, respectively. For a fixed beam parameter ε = 0.8, when $N \leqslant 5$, the beam order has large influence on the intensity distribution of the CDHB in the FrFT plane. However, when the beam order N is further increased, the intensity distribution almost keeps unchanged.

Fig.7 explains the influence of the fractional order p on the normalized on-axis intensity distribution properties of a CDHB in the FrFT plane. The axial irradiance is obtained from Eq.(12) when r_2=0 .The result shows that the on-axis intensity distribution of the CDHB changes with the fractional order p periodically, and the fundamental period is 2 when the beam passes through the two types of Lohmann system. The on-axis intensity has a minimum value when p=2k and a maximum value when p=2k+1.

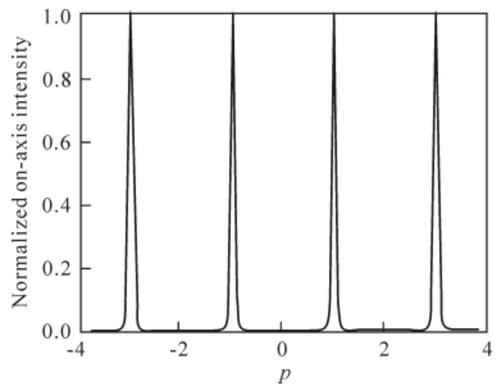

Fig.7 Evolution of the normalized on-axis intensity of a CDHB in the FrFT plane versus the fractional order p

From above results, we come to a conclusion that we can control the beam properties of a CDHB in the FrFT plane conveniently by properly choosing the fractional order of the FrFT optical system, initial beam parameter, beam order and the lens focal length of an optical system for performing FrFT. Our results have potential applications when special profiles of laser beams are required.

In conclusion, we study the propagation properties of a CDHB through an FrFT optical system. Based on the definition of FrFT in the cylindrical coordinate system, analytical formulas of the beam in the FrFT plane are derived. By using the derived formulas, the properties of a CDHB in the FrFT plane are illustrated by numerical calculations. The results show that the variation of normalized intensity distribution including the axial irradiance with the fractional order p

is periodic, and the fundamental period is 2. The properties of the intensity of the beam in the FrFT are closely related to not only the fractional order but also initial beam parameter, beam order and the lens focal length of the optical system for performing FrFT. The FrFT optical system provides a convenient way for controlling the properties of the CDHB by choosing the fractional order of the FrFT optical system properly. And the method presented in this paper is useful in the design of optical systems for beam shaping and beam analysis, which can manipulate the intensity distribution and the wave front of a laser beam.

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