Programmable quantum logic gates using teleportation with non-maximally entangled states*

LI Hui (李慧) [1,2**], **LI Chun-wen** (李春文) [1,2], **and JIANG Min** (姜敏) [1,3]

1. Department of Automation, Tsinghua University, Beijing 100084, China

2. National Laboratory for Information Science and Technology, Center for Quantum Information Science and Technology, Tsinghua University, Beijing 100084, China

3. Department of Electronics and Information, Soochow University, Suzhou 215021, China

(Received 29 October 2011)

A scheme is proposed for involving programmable quantum logic gates via teleportation, which is a unique technique in quantum mechanics. In our scheme, considering the inevitable decoherence caused by noisy environment, the quantum states are not maximally entangled. We show the implementation of single qubit quantum gates and controlled-NOT (C-NOT) gate, which are universal quantum gates. Hence, any quantum gate can be implemented by using teleportation with non-maximally entangled states. Furthermore, two schemes in differet connections of universal gates are proposed and compared, and our results show the parallel connection outperforms the cascade connection.

Document code: A **Article ID:** 1673-1905(2012)01-0063-4

DOI 10.1007/s11801-012-1035-1

Quantum computers can handle various computational tasks such as searching an unsorted database[1,2] and factoring large numbers[3,4] more efficiently than classical computers. In the practical implementation of quantum computers, Nielsen and Chuang[5] extended the concept of programmable processors to quantum domain[6-11]. They constructed a universal quantum gate which can be programmed to perform an arbitrary operation, and showed that it exists only if the gate is allowed to operate in a probabilistic fashion. Nevertheless, it is not possible to build a fixed, general purpose quantum computer which can be programmed to perform an arbitrary quantum computation. In a seminal paper, Gottesman and Chuang[12] proposed a scheme to realize universal quantum gates, by using quantum teleportation as a communicational primitive. They used single qubit operations, Bell-state measurements and maximally entangled quantum states to construct universal quantum gates. Gottesman and Chuang initiated the work of realizing programmable quantum gates, and then experiments were carried out according to their scheme. Z gate and coritrolled-NOT (C-NOT) gate were experimentally demonstrated through linear optics[13,14].

All the schemes mentioned above assume no decoherence and the qubits which are used in teleportation are maximally entangled. However, in practical application, the decoherence of qubits to noisy environment is inevitable, and then the qubits are not maximally entangled. In this paper, we propose a scheme for the case when the qubits are not maximally entangled. We show the construction of single qubit quantum gate and the construction of C-NOT gate. Single qubit quantum gates and C-NOT gate are universal quantum gates, and any unitary operation can be approximated to arbitrary accuracy by a set of universal quantum gates[15,16], so any quantum gate can be constructed.

We assume the input qubit 1 can be expressed as

$$\left|\varphi\right\rangle_{\text{in}} = a\left|0\right\rangle_1 + b\left|1\right\rangle_1 . \tag{1}$$

Taking environment influence into account, $\left|\varphi\right\rangle_{\text{p}}$ can be written as

$$\left|\varphi\right\rangle_{\text{p}} = \alpha\left|01\right\rangle_{23} + \beta\left|10\right\rangle_{23} , \tag{2}$$

where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$. Then the joint system of qubits 1, 2 and 3 reads

* This work has been supported by the National Natural Science Foundation of China (No. 60904034 and 61104002).

** E-mail:hli05@mails.tsinghua.edu.cn

$$
\begin{aligned}
|\varphi\rangle = (a|0\rangle_1 + b|1\rangle_1)(\alpha|01\rangle_{23} + \beta|10\rangle_{23}) = \\
\frac{1}{\sqrt{2}}\Big[|\phi^+\rangle_{12}(b\beta|0\rangle_3 + a\alpha|1\rangle_3) + \\
|\phi^-\rangle_{12}(-b\beta|0\rangle_3 + a\alpha|1\rangle_3) + \\
|\psi^+\rangle_{12}(a\beta|0\rangle_3 + b\alpha|1\rangle_3) + \\
|\psi^-\rangle_{12}(a\beta|0\rangle_3 - b\alpha|1\rangle_3)\Big] \quad . \qquad (3)
\end{aligned}
$$

Our scheme can be illustrated by Fig.1.

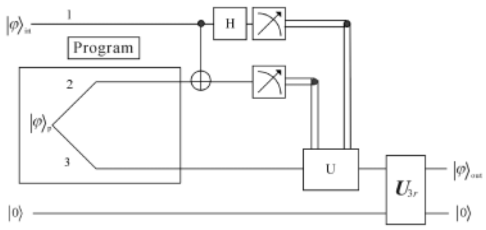

Fig.1 Quantum circuit for teleportation-based *Z* gate with non-maximally entangled qubits

In Fig. 1, time proceeds from left to right. The single wires carry qubits, and double wires carry classical bits. Qubit 1 is the input. Qubits 2 and 3 form a program register prepared in a non-maximally entangled state $|\varphi\rangle_{\rm p}$. Bell-state measurement is applied to qubits 1 and 2.

After performing Bell-state measurement on qubits 1 and 2, if the result is $|\psi^-\rangle_{12}$, qubit 3 collapses to the state as follows

$$|\varphi\rangle_{\rm out} = \frac{1}{\sqrt{2}}(a\beta|0\rangle_3 + b\alpha|1\rangle_3) \quad . \qquad (4)$$

If the result is not $|\psi^-\rangle_{12}$, but $|\phi^+\rangle_{12}$, $|\phi^-\rangle_{12}$ or $|\psi^+\rangle_{12}$, we can perform corresponding unitary transformation on qubit 3 to get the desired output.

Then we introduce auxiliary qubit r with the initial state $|0\rangle_r$, and perform unitary transformation onto qubits 3 and r as

$$\boldsymbol{U} = \begin{bmatrix} \frac{\alpha}{\beta} & 0 & 0 & \sqrt{1-\frac{\alpha^2}{\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{1-\frac{\alpha^2}{\beta^2}} & 0 & 0 & -\frac{\alpha}{\beta} \end{bmatrix} \quad . \qquad (5)$$

On the basis of $\{|00\rangle_{3r}, |01\rangle_{3r}, |10\rangle_{3r}, |11\rangle_{3r}\}$, then qubits 3 and r change to

$$|\varphi\rangle_{3r} = a\alpha|00\rangle_{3r} - b\alpha|10\rangle_{3r} + a\sqrt{\beta^2-\alpha^2}\,|11\rangle_{3r} \quad . \qquad (6)$$

Then we perform measurements on auxiliary qubit. If the result is $|0\rangle_r$, the process succeeds, and the output is

$$|\varphi\rangle_{\rm out} = a|0\rangle_3 + b|1\rangle_3 \ , \qquad (7)$$

else it fails. The successful possibility is $2|\alpha|^2$. Comparing input with output, we can see

$$|\varphi\rangle_{\rm out} = \boldsymbol{U}^z|\varphi\rangle_{\rm in} \ , \qquad (8)$$

and Z gate is constructed with non-maximally entangled states.

There are four elementary single qubit quantum gates as follows: I gate, X gate, Y gate and Z gate. The construction of Z gate with non-maximally entangled quantum states is shown above. Hence, the other three elementary single qubit quantum gates can be constructed in the similar way.

Assume the input qubits 1 and 2 are target qubit $|T\rangle$ and control qubit $|C\rangle$, respectively. They are arbitrary unkown quantum states to be teleported, which can be written as

$$|\varphi\rangle_{\rm in} = \alpha|00\rangle_{12} + \beta|01\rangle_{12} + \gamma|10\rangle_{12} + \delta|11\rangle_{12} \ , \qquad (9)$$

where α, β, γ and δ are complex numbers, and $|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2=1$. The aim of this scheme is to obtain

$$|\varphi\rangle_{\rm out} = \boldsymbol{U}^{\rm C-NOT}|\varphi\rangle_{\rm in} \ . \qquad (10)$$

Considering the environment influence, qubits 3, 4, 5 and 6 form a program register prepared in a non-maximally entangled state $|\varphi\rangle_{\rm p}$ which can be expressed as

$$|\varphi\rangle_{\rm p} = (a|00\rangle_{34} + b|11\rangle_{34})|00\rangle_{56} + (c|01\rangle_{34} + d|10\rangle_{34})|11\rangle_{56}, \qquad (11)$$

where a, b, c and d are complex numbers, and $|a|^2+|b|^2+|c|^2+|d|^2=1$. Our scheme can be illustrated by Fig.2.

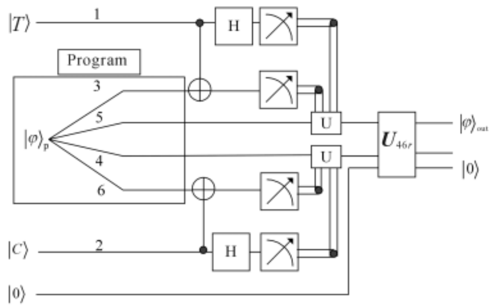

Fig.2 Quantum circuit for teleportation-based C-NOT gate with non-maximally entangled qubits

In Fig.2, time also proceeds from left to right. The single wires carry qubits, and double wires carry classical bits. The inputs consist of the target qubit $|T\rangle$ and control qubit $|C\rangle$. Qubits 3, 4, 5 and 6 form a program register prepared in a maximally entangled state $|\varphi\rangle_{\rm p}$. Bell-state measurements are applied to input qubits and qubits in the program register.

The joint system of qubits 1, 2, 3, 4, 5 and 6 can be written as

$$
\begin{aligned}
|\varphi\rangle = {} & (\alpha|00\rangle_{12}+\beta|01\rangle_{12}+\gamma|10\rangle_{12}+\delta|11\rangle_{12})(a|00\rangle_{34}+ b|11\rangle_{34})|00\rangle_{56}+(c|01\rangle_{34}+d|10\rangle_{34})|11\rangle_{56}= \\
& \frac{1}{2}\Big[|\phi^{+}\rangle_{13}|\phi^{+}\rangle_{25}(a\alpha|00\rangle_{46}+d\delta|01\rangle_{46}+b\gamma|10\rangle_{46}+c\beta|11\rangle_{46})+ \\
& |\phi^{+}\rangle_{13}|\phi^{-}\rangle_{25}(a\alpha|00\rangle_{46}-d\delta|01\rangle_{46}+b\gamma|10\rangle_{46}-c\beta|11\rangle_{46})+ \\
& |\phi^{-}\rangle_{13}|\phi^{+}\rangle_{25}(a\alpha|00\rangle_{46}-d\delta|01\rangle_{46}-b\gamma|10\rangle_{46}+c\beta|11\rangle_{46})+ \\
& |\phi^{-}\rangle_{13}|\phi^{-}\rangle_{25}(a\alpha|00\rangle_{46}+d\delta|01\rangle_{46}-b\gamma|10\rangle_{46}-c\beta|11\rangle_{46})+ \\
& |\psi^{+}\rangle_{13}|\psi^{+}\rangle_{25}(a\delta|00\rangle_{46}+d\alpha|01\rangle_{46}+b\beta|10\rangle_{46}+c\gamma|11\rangle_{46})+ \\
& |\psi^{+}\rangle_{13}|\psi^{-}\rangle_{25}(-a\delta|00\rangle_{46}+d\alpha|01\rangle_{46}-b\beta|10\rangle_{46}+c\gamma|11\rangle_{46})+ \\
& |\psi^{-}\rangle_{13}|\psi^{+}\rangle_{25}(-a\delta|00\rangle_{46}+d\alpha|01\rangle_{46}+b\beta|10\rangle_{46}-c\gamma|11\rangle_{46})+ \\
& |\psi^{-}\rangle_{13}|\psi^{-}\rangle_{25}(a\delta|00\rangle_{46}+d\alpha|01\rangle_{46}-b\beta|10\rangle_{46}-c\gamma|11\rangle_{46})+ \\
& |\phi^{+}\rangle_{13}|\psi^{+}\rangle_{25}(a\beta|00\rangle_{46}+d\gamma|01\rangle_{46}+b\delta|10\rangle_{46}+c\alpha|11\rangle_{46})+ \\
& |\phi^{+}\rangle_{13}|\psi^{-}\rangle_{25}(-a\beta|00\rangle_{46}+d\gamma|01\rangle_{46}-b\delta|10\rangle_{46}+c\alpha|11\rangle_{46})+ \\
& |\phi^{-}\rangle_{13}|\psi^{+}\rangle_{25}(a\beta|00\rangle_{46}-d\gamma|01\rangle_{46}-b\beta|10\rangle_{46}+c\alpha|11\rangle_{46})+ \\
& |\phi^{-}\rangle_{13}|\psi^{-}\rangle_{25}(-a\beta|00\rangle_{46}-d\gamma|01\rangle_{46}+b\delta|10\rangle_{46}+c\alpha|11\rangle_{46})+ \\
& |\psi^{+}\rangle_{13}|\phi^{+}\rangle_{25}(a\gamma|00\rangle_{46}+d\beta|01\rangle_{46}+b\alpha|10\rangle_{46}+c\delta|11\rangle_{46})+ \\
& |\psi^{+}\rangle_{13}|\phi^{-}\rangle_{25}(a\gamma|00\rangle_{46}-d\beta|01\rangle_{46}+b\alpha|10\rangle_{46}-c\delta|11\rangle_{46})+ \\
& |\psi^{-}\rangle_{13}|\phi^{+}\rangle_{25}(-a\gamma|00\rangle_{46}+d\beta|01\rangle_{46}+b\alpha|10\rangle_{46}-c\delta|11\rangle_{46})+ \\
& |\psi^{-}\rangle_{13}|\phi^{-}\rangle_{25}(-a\gamma|00\rangle_{46}-d\beta|01\rangle_{46}+b\alpha|10\rangle_{46}+c\delta|11\rangle_{46})\Big].
\end{aligned}
\tag{12}
$$

After performing Bell-state measurement on qubits 1 and 3, and qubits 2 and 5, respectively, if the result is $|\phi^{+}\rangle_{13}|\phi^{+}\rangle_{25}$, qubits 4 and 6 collapse to the state

$$|\varphi\rangle_{\mathrm{out}} = a\alpha|00\rangle_{46}+d\delta|01\rangle_{46}+b\gamma|10\rangle_{46}+c\beta|11\rangle_{46}, \tag{13}$$

where we do not normalize the state for convenience. If the result is not $|\phi^{+}\rangle_{13}|\phi^{+}\rangle_{25}$, we can perform unitary transformation on qubits 4 and 6 to obtain the output above.

Then we introduce auxiliary qubit r with the initial state $|0\rangle_r$, and perform the following unitary transformation onto qubits 4, 6 and r on the basis of $\{|000\rangle_{46r}, |001\rangle_{46r}, |010\rangle_{46r}, |011\rangle_{46r}, |100\rangle_{46r}, |101\rangle_{46r}, |110\rangle_{46r}, |111\rangle_{46r}\}$ as follows

$$
\boldsymbol{U}=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{a}{d} & \sqrt{1-\frac{a^2}{d^2}} & 0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{1-\frac{a^2}{d^2}} & \frac{a}{d} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{a}{b} & \sqrt{1-\frac{a^2}{b^2}} & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{1-\frac{a^2}{b^2}} & \frac{a}{b} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{a}{c} & \sqrt{1-\frac{a^2}{c^2}} & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{1-\frac{a^2}{c^2}} & \frac{a}{c} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}. \tag{14}
$$

Then measure the auxiliary qubit. If the result is $|0\rangle$, the process succeeds, and we get the desired output

$$|\varphi\rangle_{\mathrm{out}} = \alpha|00\rangle_{46}+\delta|01\rangle_{46}+\gamma|10\rangle_{46}+\beta|11\rangle_{46}. \tag{15}$$

Compare the input and output, and we can see they satisfy Eq.(10), and we get C-NOT gate.

It has been proved that any unitary operation can be approximated to arbitrary accuracy by a set of universal quantum gates. Single qubit quantum gates and C-NOT gate are a set of universal quantum gates. Therefore, the method proposed in this paper can construct any quantum gates, in consideration of inevitable decoherence of quantum states caused by environment.

However, a simple combination of the elementary gates is not optimal. For example, we want to get XZ gate. The simple combination of the elementary gates can be illustrated in Fig.3.

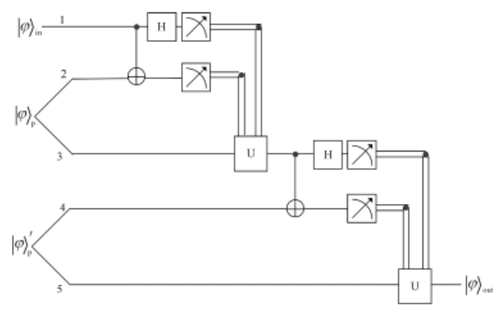

Fig.3 Quantum circuit for teleportation-based XZ gate in cascade connection

The circuit can be divided into two parts, and they are in cascade connection. The output of the first part serves as the input of the second part. Qubit 1 is the input as follows

$$|\varphi\rangle_{\mathrm{in}} = a|0\rangle_1 + b|1\rangle_1, \tag{16}$$

where a and b are complex numbers, and $|a|^2+|b|^2=1$. Qubits 2 and 3, and qubits 4 and 5 form program registers. Bell-state measurements are applied to qubits 1 and 2, qubits 3 and 4, respectively. By programming the entangled quantum states $|\varphi\rangle_p$ and $|\varphi\rangle'_p$, we can obtain the output

$$|\varphi\rangle_{out} = -b|0\rangle_5 + a|1\rangle_5 \ . \tag{17}$$

Comparing input with output, we can see

$$|\varphi\rangle_{out} = \boldsymbol{U}^{xz}|\varphi\rangle_{in}, \tag{18}$$

and get XZ gate.

However, if the circuit is constructed in parallel connection as Fig.4, both the time and fidelity can be improved.

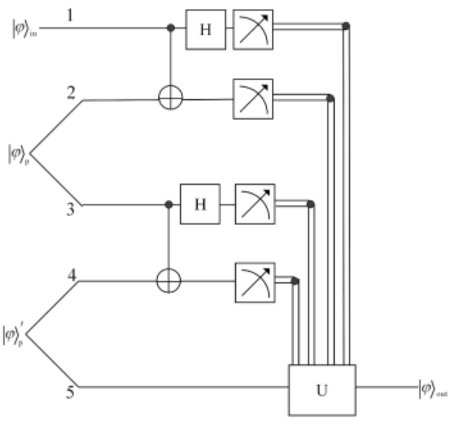

Fig.4 Quantum circuit for teleportation-based XZ gate in parallel connection

In Fig.4, $|\varphi\rangle_{in}$ is the input, and with properly programmed $|\varphi\rangle_p$ and $|\varphi\rangle'_p$, we can obtain the output, which satisfies

$$|\varphi\rangle_{out} = \boldsymbol{U}^{xz}|\varphi\rangle_{in}. \tag{19}$$

Obviously, time consumed in the parallel connection is decreased to half of that in cascade connection. Also, the fidelity is improved. If the qubits in the program registers are not maximally entangled which can be expressed as follows

$$|\varphi\rangle_p = \alpha|01\rangle_{23} + \beta|10\rangle_{23} \ , \tag{20}$$

$$|\varphi\rangle'_p = \alpha|01\rangle_{23} + \beta|10\rangle_{23} \ , \tag{21}$$

where α and β are complex numbers, and $|\alpha|^2+|\beta|^2=1$, the fidelity will be improved from $(2|\alpha|^2)^2$ to $2|\alpha|^2$.

To summarize, we present a general scheme of constructing quantum logic gates based on teleportation, taking account of the inevitable decoherence of qubits to noisy environment. In these cases, the quantum states are no longer maximally entangled states, but non-maximally entangled quantum states. To solve this problem, we introduce an auxiliary qubit and apply unitary transformation to get the desired result. In the architecture of connecting quantum logic gates, we analyze two methods and compare the time consumed and fidelity. The results show that the parallel connection outperforms cascade connection. As for the future work, it can be carried out by optical experiments[17,18].

References

[1] Shor P W, Algorithm for Quantum Computation: Discrete Logarithms and Factoring, 35 th Annual Symposium on Foundations of Computer Science, 124 (1994).

[2] Lu C Y, Browne D E, Yang T and Pan J W, Phys. Rev. Lett. **99**, 250504 (2007).

[3] Grover L K, Phys. Rev. Lett. **79**, 325 (1997).

[4] Younes A, Rowe J and Miller J, Physica D **237**, 1074 (2008).

[5] Nielson M A and Chuang I L, Phys. Rev. Lett. **79**, 321 (1997).

[6] Hanneke D, Home J P, Jost J D, Amini J M, Leibfried D and Wineland D J, Nature Physics **6**, 13 (2010).

[7] Hillery M, Ziman M and Bu•ek V, Phys. Rev. A **73**, 022345 (2006).

[8] Mièuda M, Je•ek M, Dušek M and Fiurášek J, Phys. Rev. A **78**, 062311 (2008).

[9] Kubota S and Watanabe M, Applied Optics **48**, 302 (2009).

[10] Knill E, Leibfried D, Reichle R, Britton J, Blackestad R B, Jost J D, Langer C, Ozeri R, Seidelin S and Wineland D J, Phys. Rev. A **77**, 012307 (2008).

[11] Ladd T D, Jelezko F, Laflamme R, Nakamura Y, Monroe C and O'Brien J L, Nature **464**, 45 (2010).

[12] Gottesman D and Chuang I L, Nature **402**, 390 (1999).

[13] Slodièka L, Fiurášek J and Je•ek M, Phys. Rev. A **79**, 050304 (2009).

[14] Goebel A M, Wagenknecht C, Zhang Q, Chen Y U and Pan J W, Teleportation-Based Controlled-NOT Gate for Fault-Tolerant Quantum Computation, e-print arXiv: 0809, 3583 (2008).

[15] Barenco A, Bennett C H, Cleve R, DiVincenzo D P, Margolus N, Shor P, Sleator T, Smolin J A and Weinfurter H, Phys. Rev. A **52**, 3457 (1995).

[16] Ziman M and Bu•ek V, Acta Physica Hungarica **26**, 277 (2006).

[17] Chattopadhyay T and Bhattacharyya P, Optoelectronics Letters **7**, 186 (2011).

[18] Lv S F, Montrosset I, Gioannini M, Song S and Ma J, Optoelectronics Letters **7**, 122 (2011).