Design and realization of random measurement scheme for compressed sensing*

XIE Cheng-jun(解成俊)** and **XU Lin**(徐林)
Computer Science & Technology Institute, Beihua University, Jilin 132021, China

(Received 11 August 2011)

Design and realization of random measurement scheme for compressed sensing (CS) are presented in this paper, and lower limits of the measurement number are achieved when the precise reconstruction is realized. Four kinds of random measurement matrices are designed according to the constraint conditions of random measurement. The performance is tested employing the algorithm of stagewise orthogonal matching pursuit (StOMP). Results of the experiment show that lower limits of the measurement number are much better than the results described in Refs.[13-15]. When the ratios of measurement to sparsity are 3.8 and 4.0, the mean relative errors of the reconstructed signals are 8.57×10^{-13} and 2.43×10^{-14}, respectively, which confirms that the random measurement scheme of this paper is very effective.
Document code: A **Article ID:** 1673-1905(2012)01-0060-3
DOI 10.1007/s11801-012-1097-0

Compressed sensing (CS)[1] redefines the limits deduced from Shannon's theorem and Nyquist sampling frequency. CS theory brings the reform of signal sampling theory, and it is hoped to have wide application prospects[2]. A much lower coding rate can be achieved with the help of CS[3]. A single-pixel camera is bought to birth by Baraniuk et al[4]. The two important improvements brought by CS in radar imaging filed should be: the pulse compression match filter can be omitted at the receiving end, and thanks to the avoidance of a direct sampling of the original signal, the bandwidth requirement of the analog to digital converter (ADC) can be lowered[5]. The CS theory is used in the acquirement of the data of synthetic aperture radar images[6], which greatly lowers the calculation cost of satellite image processing. Compressed sensing is also used in medical imaging field, like sparse magnetic resonance (MR) imaging[7], and 3D magnetic resonance spectroscopic imaging (MRSI)[8]. Compared with traditional uniform sampling process, the reconstruction precision of CS is better when the sampling randomization is implemented more thoroughly, and the reconstruction quality should be better in the favor of a better reconstruction algorithm. The design of random measurement scheme should be the critical step for compressed sensing coding, as well as for increasing the efficiency of sampling compression, and a measurement scheme that is easy for hardware realization is also a key factor for its practical application. In this paper, the random measurement matrix is designed according to the constraint conditions of random measurement and lower limits of the measurement number for a precise reconstruction, and simultaneously the stagewise orthogonal matching pursuit (StOMP) is used in the signal reconstruction simulation to confirm the rightness and effectiveness of our random measurement scheme.

There are principally three stages in CS: signal sparseness, random measurement and reconstruction. Assume that the original signal is $z \in R^N$, in which N is the length of the signal, and $\boldsymbol{\Psi}$ is the sparseness expression of z, $x = \boldsymbol{\Psi} z$. The goal of the random measurement lies in acquiring observables whose number is M ($M << N$). $y = \boldsymbol{\Phi} x = \boldsymbol{\Phi}\boldsymbol{\Psi} z$, $y \in R^M$. $\boldsymbol{\Phi} \in R^{M \times N}$ is the random measurement matrix. Since $M << N$, $y = \boldsymbol{\Phi} x = \boldsymbol{\Phi}\boldsymbol{\Psi} z$ should be underdetermined, and the multiple resolutions exist. x can be got from

$$\min \| x \|_0 \ s.t. \ y = \boldsymbol{\Phi} x \ . \tag{1}$$

But the optimization problem in Eq.(1) is NP one, so it can be converted to l_1 convex set optimization problem[9]

$$\min \| x \|_1 \ s.t. \ y = \boldsymbol{\Phi} x \ . \tag{2}$$

The StOMP algorithm is used here to solve Eq.(2) to get x, and then the original signal z is got through the reverse process of sparseness.

Here, we construct a random measurement matrix $\boldsymbol{\Phi} \in R^{M \times N}$. Employing $y = \boldsymbol{\Phi} x$, we can obtain the measurement value y_i, whose number is M. Here each line of $\boldsymbol{\Phi}$ can

* This work has been supported by the National Natural Science Foundation of China (Nos.61072111 and 60672156), and the Project of Science and Technology Commission of Jilin Province (Nos.20100503 and 20110360).
** E-mail: xcjciom@yahoo.com.cn

be taken as a sensor, and multiply it with the signal, so the random projection can keep the essential information of the original signal.The core of CS is a non-correlated measurement process, which is different from uniform sampling, and the measurement approach is non-adaptivity, which means that the design of the measurement matrix doesn't depend on the original signal x. The constraints of the measurement matrix are as follows: designs of $\boldsymbol{\Phi}$ should follow uniform uncertainty principle (UUP)[10]; designs of $\boldsymbol{\Phi}$ should meet restricted isometry property (RIP)[11]; designs of $\boldsymbol{\Phi}$ should meet incoherence between sparseness matrix $\boldsymbol{\Psi}$ and random measurement matrix $\boldsymbol{\Phi}$[12].

For a certain K, if M satisfies certain condition, the probability of achieving precise reconstruction is really high. In CS process, the measurement number for reconstruction greatly affects the quality of the reconstructed image, while the quality of the reconstructed image almost has nothing to do with the specific measurement value. That is to say, every measurement value contributes to reconstruction with the same power, and the precision of reconstruction largely depends on the measurement number. Generally speaking, lower limits of measurement number M for precise reconstruction are given by the following expression:

$$M \geq CK\log(N/K), \tag{3}$$

$$M \geq K\log(N), \tag{4}$$

$$M \geq 2K\log(N-K)+K, \tag{5}$$

where the sampling factor is $C = \log(N/K+1)$.

According to the above requirements of measurement matrix, the design method of random measurement matrix is implemented by designing a pseudo random noise matrix first, which means the $M \times N$ random variables with independent distribution are taken as the elements of the matrix. This kind of measurement matrix $\boldsymbol{\Phi}$, which is characterized by random and arbitary fixed sparse matrix $\boldsymbol{\Psi}$, can ensure that $\boldsymbol{\Phi}\boldsymbol{\Psi}$ meets the constraints in large probability. In the following part, 4 kinds of specific designs of measurement matrix will be considered.

Using the uniform random projection measurement matrix $\boldsymbol{\Phi}_1$, the concrete implementation includes generating a random matrix $\boldsymbol{A}$ for given row and column (m, n). Then an SVD decomposition is applied on $\boldsymbol{A}$, and we have $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\mathrm{T}}$. Let $\boldsymbol{\Phi}=\boldsymbol{V}^{\mathrm{T}}$, and the Euclidean normalization is implemented for every column of the measurement matrix $\boldsymbol{\Phi}$, and then the uniform random projection measurement matrix has been got now, e.g.,

$$\underset{\boldsymbol{A}}{\begin{bmatrix} 0.6500 & 0.9265 & 0.2930 & 0.9554 \\ 0.7829 & 0.7586 & 0.4861 & 0.7168 \\ 0.2136 & 0.2753 & 0.6365 & 0.1791 \end{bmatrix}} \rightarrow$$

$$\underset{\boldsymbol{V}}{\begin{bmatrix} -0.4870 & 0.9265 & -0.8715 \\ -0.5795 & 0.1583 & 0.2864 \\ -0.3303 & -0.9221 & -0.1603 \\ -0.5638 & 0.3520 & 0.3644 \end{bmatrix}} \rightarrow$$

$$\underset{\boldsymbol{\Phi}}{\begin{bmatrix} -0.4876 & -0.8708 & 0.3328 & -0.7438 \\ 0.0296 & 0.7586 & -0.9291 & 0.4643 \\ -0.8726 & 0.4303 & 0.1615 & 0.4808 \end{bmatrix}}.$$

Using the Euclidean normalized random measurement matrix $\boldsymbol{\Phi}_2$, the concrete implementation includes generating a random matrix for given row and column (m,n). Then the Euclidean normalized measurement matrix $\boldsymbol{\Phi}$ is got, which is the simplest and the most direct way to generate the random measurement matrix.

Using the random measurement matrix $\boldsymbol{\Phi}_3$ based on fast Fourier transform (FFT), FFT can be expressed as $\boldsymbol{F}=\boldsymbol{W}\boldsymbol{f}$, in which $\boldsymbol{W}_{n,i} = \frac{1}{\sqrt{N}} \mathrm{e}^{-\mathrm{j}2\pi\frac{ni}{N}}, 0 \leq i,n \leq N-1$. The generating steps of the random measurement matrix are as follows: generate a standard matrix $\boldsymbol{Q}$ of the pseudo random uniform distributed sequence; employ $\boldsymbol{Q}$ to realize the core position exchange of Fourier transform; generate a standard matrix $\boldsymbol{P}$ of the pseudo random uniform distributed sequence; use $\boldsymbol{P}$ to realize the core position exchange of Fourier transform, and we have $\boldsymbol{\Phi}=\boldsymbol{P}\boldsymbol{F}\boldsymbol{Q}$; implement Euclidean normalization for every column of the measurement matrix $\boldsymbol{\Phi}$ to get the random measurement matrix.

Using the symbolic random measurement matrix $\boldsymbol{\Phi}_4$, the concrete implementation includes generating a +/- symbolic matrix for given row and column(m,n). Then every column of the matrix is normalized, and we have the symbolic measurement matrix.

In this paper the StOMP algorithm is used to realize reconstruction of the measurement data to testify the correctness and effectiveness of the random measurement scheme. At the same time, four kinds of random measurement matrices are compared and analyzed. Segmentation is mentioned in condition that s is given. The StOMP algorithm is then implemented in stage s to build an approximate sequence (x_1,…, x_s) from a residual vector sequence (r_1, r_2, …) employing a structure that aims at removing residual, mark coordinate values of the non-zero elements of x_s with I_s. The initial solution is $x_0 = 0$, and I_0 is blank. The concrete steps of the StOMP algorithm are as follows:

Step1: Set up the maximal step for iteration, and solve the maximal iteration error e, s=1;

Step2: Implement random measurement for wavelet sparse matrix y, and the following equation is then derived: $\Theta_s = \boldsymbol{\Phi}^{\mathrm{T}} r_{s-1}$;

Step3 : Implement normalization for Θ_s: $\Theta_s(i)=\sqrt{n}\times\Theta_s(i)/\sqrt{\sum_1^l(\Theta_s(i))^2}$, in which l is the signal length of Θ_s ;

Step4: Implement threshold processing for Θ_s : $\sum_s=\{j:|\Theta_s(j)|>\lambda_s\}$;

Step5: Combine the newest two coordinate indices, and implement uniformization for set I_s, and then the following equation is tenable: $I_s=I_{s-1}\cup\sum_s$

Step6: Solve the linear equations: min ($\| x_s \|_1$), *s.t.* $y=\boldsymbol{\Phi}x_s$, and calculate: $(x_s)_{I_s}=(\boldsymbol{\Phi}_{I_s}^{\mathrm{T}}\boldsymbol{\Phi}_{I_s})^{-1}\boldsymbol{\Phi}_{I_s}^{\mathrm{T}}y$;

Step7: Calculate the residual part: $r_s=y-x_s$;

Step8: Decide the precision of spare solution: if $r_s<e$, end, and we have $x_0=x_s$; otherwise $s=s+1$, and return to Step3.

To evaluate the signal reconstruction quality, here the frequently-used relative error analyzing method is employed. The relative error can be expressed as: $error=\|\hat{x}-x\|_2/\|x\|_2$, and we take $\bar{e}$ as the average error of 10 times of reconstructions.

To conveniently compare with other scholars' experimental results, in this paper the following signal is used to generate available original test data:

$$z=\sin(10t+\pi/4)+\sin(2t+\pi/6)+\sin(t+\pi/8)\ . \quad (6)$$

To obtain ($K-$ sparse signal), the following form is used

$$x=[z(1{:}K);\ zeros\ (N-K,\ 1)]\ , \quad (7)$$

where an available signal with length K can be generated, while other elements of the test signal are zero. Tab.1 shows the results.

Tab.1 Reconstructed signals using different measurement matrices and *M* values

(a) M = 150, N = 1000, K = 50, M/K = 3.0, $M/K\log(N/K)$ = 1.001				
Measurement matrix	$\boldsymbol{\Phi}_1$	$\boldsymbol{\Phi}_2$	$\boldsymbol{\Phi}_3$	$\boldsymbol{\Phi}_4$
$\bar{e}$	2.58	9.51	3.26	8.79
(b) M = 170, N = 1000, K = 50, M/K = 3.4, $M/K\log(N/K)$ = 1.135				
Measurement matrix	$\boldsymbol{\Phi}_1$	$\boldsymbol{\Phi}_2$	$\boldsymbol{\Phi}_3$	$\boldsymbol{\Phi}_4$
$\bar{e}$	4.42e−12	4.46	1.45e−12	2.77
(c) M = 190, N = 1000, K = 50, M/K = 3.8, $M/K\log(N/K)$ = 1.268				
Measurement matrix	$\boldsymbol{\Phi}_1$	$\boldsymbol{\Phi}_2$	$\boldsymbol{\Phi}_3$	$\boldsymbol{\Phi}_4$
$\bar{e}$	1.16e−13	3.38e−12	1.03e−13	1.72e−13
(d) M = 200, N = 1000, K = 50, M/K = 4.0, $M/K\log(N/K)$ = 1.335				
Measurement matrix	$\boldsymbol{\Phi}_1$	$\boldsymbol{\Phi}_2$	$\boldsymbol{\Phi}_3$	$\boldsymbol{\Phi}_4$
$\bar{e}$	1.14e−14	3.52e−14	9.72e−15	4.07e−14

From Tab.1 we can see that the measurement matrix design of this paper fully meets the lower limit requirements of the precise reconstruction. Still it is far superior to lower limit condition described in Refs.[13-15]. Based on the data in Tab.1, using the sampling factor C and Eqs.(3-5), it can be concluded that $M\geqslant 456$, $M\geqslant 345$ and $M\geqslant 736$. The acquired data shows that the three formulas result in rather strict limiting conditions, and a relative error of 1.0e-8 can be considered as the standard of a precise reconstruction. According to the experiment data in Tab.l, take $M\geqslant 190$, and then the precise reconstruction requirement can be met. Since the specific reconstruction algorithm is StOMP, the introduced reconstruction difference needs to be considered. From a whole view of the compressed sensing process, the random measurement results in not all of the signals need to be measured, which means that the positions of important coefficients are not of importance. In addition, the decoding end is rather robust to missing signals. Since every random measurement coefficient has the same importance, there should be less influence even if any loss happens, which is the greatest advantage of random measurement with compressed sensing.

Different random measurement matrices can bring great influence on the quality of reconstructed signal. The measurement matrix design shouldn't break the constraints. The lower limit of measurement number should vary according to different random measurement matrices and reconstruction algorithms. The random measurement matrix should be chosen according to specific application environment.

References

[1] Donoho D, IEEE Transactions on Information Theory **52**, 1289 (2006).

[2] Candès E, Romberg J and Tao T, IEEE Transactions on Information Theory **52**, 489 (2006).

[3] Romberg J, IEEE Signal Processing Magazine **25**, 14 (2006).

[4] Duarte M F, Davenport M A, Takhar D, Sun T, Kelly K F and Baraniuk R G, IEEE Signal Processing Magazine **25**, 83 (2008).

[5] Baraniuk R and Steeghs P, Compressive Radar Imaging, IEEE: Proceedings of the Radar Conference, Washington D.C., 128 (2007).

[6] Bhattacharya S, Blumensath T, Mulagrew B and Davies M, Fast Encoding of Synthetic Aperture Radar Raw Data Using Compressed Sensing, IEEE Proceedings of Statistical Signal Processing, Washington D. C., 448 (2007).

[7] Lustig M, Donoho D L and Pauly J M, Magnetic Resonance in Medicine **58**, 1182 (2007).

[8] Hu S, Lustig M, Chen A P, Crane J, Kerr A, Kelley D AC, Hurd R, Kurhanewicz J, Nelson S J, Pauly J M and Vigneron D B, Journal of Magnetic Resonance **192**, 258 (2008).

[9] Candès E and TAO T, IEEE Transaction on Information Theory **51**, 4203 (2005).

[10] Candès E and J. Romberg, Found. Comput. Math. **6**, 227 (2006).

[11] Candès E and Wakin M B, IEEE Signal Processing Magazine **25**, 21 (2008).

[12] Gemmeke J F and Cranen B, Using Sparse Representations for Missing Data Imputation in Noise Robust Speech Recognition, European Signal Processing Conf. (EUSIPCO), Lausanne, Switzerland, 987 (2008).

[13] R Baraniuk, IEEE Signal Processing Magazing **24**, 118 (2007).

[14] SHI Guang-ming, LIU Dan-hua, GAO Da-hua, LIU Zhe, LIN Jie and WANG Liang-jun, Acta Electronica Sinica **37**, 1070 (2009). (in Chinese)

[15] Gao Rui, Matching Pursuit Algorithm for Image Reconstruction Based on Compressive Sensing, Beijing:Beijing Jiaotong University, 2 (2009). (in Chinese)