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Consistent Riccati expansion and exact solutions of the Kuramoto–Sivashinsky equation

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ABSTRACT

A consistent Riccati expansion (CRE) method is developed for a special Kuramoto–Sivashinsky (KS) equation and we prove the general KS equation is non-CRE solvable. Furthermore, we obtain the soliton–cnoidal wave interaction solution of the special KS equation.

1. Introduction

With the development of nonlinear science, nonlinear evolution equations have been used to describe certain phenomena in the fluid mechanics, plasma physics, optical fibers, and solid state physics [1,2]. Many effective methods have been proposed to find the exact solutions of the nonlinear systems, such as the Darboux transformation, the Hirota bilinear form, Painlevé analysis, symmetry group analysis, the variable separation approach and so on. Recently, the authors proposed a simple effective method in [3], the consistent Riccati expansion (CRE) method, which is based on the symmetry reductions with nonlocal symmetries. The CRE method can be used to identify CRE solvable systems and it is a more generalized but much simpler method to look for new interaction solutions between a soliton and other types of nonlinear excitations [4–7].

We first focus on a special form of the Kuramoto–Sivashinsky (KS) equation

$$u_t + uu_x + \alpha u_{xx} = 0. \tag{1}$$

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It is a model partial differential equation frequently encountered in the study of continuous media which exhibits a chaotic behavior. Eq. (1) describes the fluctuation of the position of a flame front, the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium. The KS equation also arises from the minimal ingredients necessary to observe interesting bifurcations in a simplified equation for a complex amplitude in fluid dynamics [8,9]. The solitary wave solutions of KS equation (1) was given in [10] by the extended tanh expansion and its Painlevé integrability and Bäcklund transformation have been studied in [11]. In this paper, we will prove the KS equation (1) is CRE solvable and obtain different nonlinear interaction wave solutions including the soliton and cnoidal wave solutions. Then we prove the non-CRE solvable for the general KS equation with the CRE definition in Section 4. Last section presents summary and discussion.

2. Consistent Riccati expansion for KS equation (1)

For a given nonlinear partial differential equation,

$$P(\mathbf{x}, t, \mathbf{u}) = 0, \quad \mathbf{x} = \mathbf{x}(x_1, x_2, \ldots, x_n),\ \mathbf{u} = \mathbf{u}(u_1, u_2, \ldots, u_n), \tag{2}$$

we aim to look for the following possible truncated expansion solution

$$u = \sum_{i=0}^{n} u_i R(\omega)^i, \tag{3}$$

and $R(\omega)$ is a solution of the Riccati equation

$$R_\omega = a_0 + a_1 R + a_2 R^2, \tag{4}$$

where ω is an undetermined function of $\mathbf{x}$ and t, n is determined from the leading order analysis of (2) and all the expansion coefficient functions u_i should be determined by vanishing all the coefficients of the power of $R(\omega)$ after substituting (3) with (4) into (2).

Definition 2.1. If the system for $u_i, (i = 1, 2, \ldots, n)$ and ω obtained by vanishing all the coefficients of the powers of $R(\omega)$ after substituting (3) with (4) into (2) is consistent, or not over-determined, we call the expansion (3) is a consistent Riccati expansion (CRE) and the nonlinear system (2) is CRE solvable [3].

We will apply the CRE method to the KS equation and find more interaction wave solutions of KS equation. The leading order analysis leads to $n = 1$, so the possible expansion expression of (3) has the following form

$$u = u_0 + u_1 R(\omega). \tag{5}$$

Substituting (5) with (4) into (1), we obtain

$$\begin{aligned}&(2\alpha u_1\omega_x^2a_2^2 + u_1^2\omega_x a_2)R^3 + (u_0u_1\omega_x a_2 + \alpha u_1\omega_{xx}a_2 + u_1^2a_1\omega_x + u_1a_2\omega_t + u_1u_{1x}\\&\quad + 3\alpha u_1a_1a_2\omega_x^2 + 2\alpha a_2u_{1x}\omega_x)R^2 + (\alpha u_1\omega_{xx}a_1 + \alpha u_1\omega_x^2a_1^2 + u_1\omega_t a_1 + u_1u_{0x} + u_1^2a_0\omega_x\\&\quad + 2\alpha u_{1x}\omega_x a_1 + 2\alpha u_1a_2a_0\omega_x^2 + u_0u_1\omega_x a_1 + u_{1t} + u_0u_{1x} + \alpha u_{1xx})R + \alpha u_1\omega_{xx}a_0\\&\quad + u_1\omega_t a_0 + u_{0t} + u_0u_1\omega_x a_0 + \alpha u_1\omega_x^2a_1a_0 + u_0u_{0x} + 2\alpha u_{1x}\omega_x a_0 + \alpha u_{0xx} = 0.\end{aligned} \tag{6}$$

Setting the coefficients of different powers of R to zero in (6), we have four over-determined equations for only three undetermined functions u_0, u_1, ω. It is fortunate that the over-determined system may be consistent. From the coefficients of R^3, R^2, we can simply find

$$u_1 = -2\alpha a_2\omega_x, \tag{7}$$

$$u_0 = -\frac{\alpha\omega_{xx} + \alpha a_1\omega_x^2 + \omega_t}{\omega_x}. \tag{8}$$

The coefficient of R becomes identically zero by using (7) and (8). Then from the coefficient of R^0, we find the ω equation

$$\begin{aligned}&4\alpha\omega_{xx}\omega_{xt}\omega_x - \omega_{tt}\omega_x^2 + \alpha^2\delta\omega_x^4\omega_{xx} + 2\omega_x\omega_t\omega_{xt} - 2\alpha\omega_x^2\omega_{xxt} - \alpha^2\omega_{xxxx}\omega_x^2 - 3\alpha^2\omega_{xx}^3 - \omega_t^2\omega_{xx}\\&\quad + 4\alpha^2\omega_{xx}\omega_{xxx}\omega_x - 4\alpha\omega_{xx}^2\omega_t + 2\alpha\omega_x\omega_t\omega_{xxx} = 0.\end{aligned} \tag{9}$$

Eq. (9) can be rewritten as follows:

$$\alpha^2 S_x + C_t + 2\alpha C_{xx} - CC_x + \alpha^2\delta\omega_x\omega_{xx} = 0, \tag{10}$$

where

$$C = \frac{\omega_t}{\omega_x}, \qquad S = \frac{2\omega_x\omega_{xxx} - 3\omega_{xx}^2}{2\omega_x^2}, \qquad \delta = a_1^2 - 4a_0a_2.$$

It is evident that the condition in Definition 2.1 is satisfied, so the KS equation is CRE solvable. In summary, we have the following theorem:

Theorem 2.1. *If ω is a solution of*

$$\alpha^2 S_x + C_t + 2\alpha C_{xx} - CC_x + \alpha^2\delta\omega_x\omega_{xx} = 0, \tag{11}$$

with

$$C = \frac{\omega_t}{\omega_x}, \qquad S = \frac{2\omega_x\omega_{xxx} - 3\omega_{xx}^2}{2\omega_x^2}, \qquad \delta = a_1^2 - 4a_0a_2,$$

then

$$u = -\frac{\alpha\omega_{xx} + \alpha a_1\omega_x^2 + \omega_t}{\omega_x} - 2\alpha a_2\omega_x R_\omega, \tag{12}$$

is the solution of the KS equation with R_ω being the solution of the Riccati equation (4).

3. Soliton–cnoidal wave interaction solution of the KS equation (1)

The next thing is to find the interaction wave solution of Eq. (9) with respect to ω. From Ref.[3], we know the ω solutions charactering the interactions between a soliton and a cnoidal wave for the KS equation possess a form

$$\omega = k_1x + \omega_1t + W(k_2x + \omega_2t), \tag{13}$$

where

$$W(k_2x + \omega_2t) = W(\varepsilon) \equiv W,$$

satisfies

$$W_{1\varepsilon}^2 = C_0 + C_1W_1 + C_2W_1^2 + C_3W_1^3 + C_4W_1^4, W_1 = W_\varepsilon, \tag{14}$$

with

$$C_0 = -\frac{-\alpha^2k_2^2k_1^4\delta + \alpha^2k_2^4k_1^2C_2 - 2\alpha^2k_2^5k_1C_1 + k_2^2\omega_1^2 + \omega_2^2k_1^2 - 2\omega_2k_2\omega_1k_1}{3\alpha^2k_2^6}, \tag{15}$$

$$C_3 = -\frac{k_2^3C_1 - 2k_2^2k_1C_2 - a_1k_1^3 - 3a_1k_1^3 + 16a_2a_0k_1^3}{k_2k_1^2}, \tag{16}$$

$$C_4 = \delta, \tag{17}$$

while all the other constants α, k_1, k_2, ω_1, ω_2, C_1, C_2 and δ are free under the condition of $k_1\omega_2 = k_2\omega_1$, and $a_1^2 = a_1$.

Obviously, Eq. (14) is an equation for the definition of the elliptic functions which can be expressed by many elliptic functions such as the Jacobi elliptic functions $\mathrm{cn}(\xi)$, $\mathrm{sn}(\xi)$, and $\mathrm{dn}(\xi)$. So we can write down the soliton–cnoidal wave interaction solution with

$$\omega = k_1 x + \omega_1 t + AE_\pi(\mathrm{sn}(k_2 x + \omega_2 t, m), \nu, \kappa), \tag{18}$$

where $\mathrm{sn}(z, m)$ is the usual Jacobi elliptic sine function and $E_\pi(\varsigma, n, m)$ is the third type of incomplete elliptic integral.

Substituting (18) into Eq. (13) by fixing $\kappa = 1$ and setting the coefficients of $\mathrm{cn}(\xi)$, $\mathrm{sn}(\xi)$ and $\mathrm{dn}(\xi)$ equal to zero, we can find the constants solution

$$A = \frac{1}{\sqrt{1-4a_0a_2}}, \qquad k_1 = \frac{k_2}{\sqrt{1-4a_0a_2}}, \qquad w_2 = \frac{k_2 w_1}{k_1}, \qquad m = 1, \qquad \mu = 0, \tag{19}$$

where k_2, w_1 are two arbitrary non-zero constants. From Eq. (18), we can get

$$\omega = \frac{1}{\sqrt{1-4a_0a_2}} k_2 x + \omega_1 t + \frac{1}{\sqrt{1-4a_0a_2}} \mathrm{arctanh}[\tanh(k_2 x + \sqrt{1-4a_0a_2}\, w_1 t)]. \tag{20}$$

Substituting Eq. (20) into Eq. (12) with

$$R = \tanh\left[\frac{\sqrt{\delta}(k_1 x + \omega_1 t)}{2} + \frac{\sqrt{\delta} E_f(\mathrm{sn}(\lambda k_1 x + \lambda \omega_1 t, m), 1)}{2}\right], \tag{21}$$

then we obtain the soliton–cnoidal wave interaction solution for u of Eq. (12)

$$u = -\frac{(w_1\lambda^2 + 2\alpha k_2^2)\tanh[\frac{\lambda}{2}(w_1 t + \mathrm{arctanh}(\tanh(k_2 x + \lambda w_1 t)))] + 4\alpha a_2 k_2^2}{k_2 \lambda}, \tag{22}$$

where

$$\lambda = \sqrt{1-4a_0a_2}. \tag{23}$$

4. Non-CRE solvable for the general KS equation

In this section, non-CRE solvability for the general KS equation is proved directly because of its non-integrability. The general KS equation is given in the form

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxxx} = 0, \tag{24}$$

where α, β are nonzero constants. The general KS equation (24) is a canonical nonlinear evolution equation arising in long waves on thin films, long waves on the interface between two viscous fluids, unstable drift waves in plasmas, reaction diffusion systems and flame front instability [12–15].

The Riccati expansion of Eq. (3) for the general KS equation (24) gives

$$u = u_0 + u_1 R + u_2 R^2 + u_3 R^3, \tag{25}$$

from the leading order analysis. Substituting Eq. (25) with Eq. (3) into (24) yields

$$(3u_3^2 a_2 \omega_x + 360\beta a_2^4 u_3 \omega_x^4) R^7 + \sum_{j=0}^{6} K_j R^j = 0, \tag{26}$$

where $K_j (j = 0, \ldots, 6)$ are complicated ω-dependent but R-independent functions. Setting the coefficients of R^7, R^6, R^5, R^4 to zero in Eq. (26), we can easily find that

$$\begin{aligned} u_3 &= -120\beta a_2^3 \omega_x^3, \\ u_2 &= -180\beta a_2^2 \omega_x (\omega_{xx} + a_1 \omega_x^2), \\ u_1 &= -60\beta a_2 (\omega_{xxx} + a_1^2 \omega_x^3) - 180\beta a_1 a_2 \omega_x \omega_{xx} - 120\beta a_0 a_2^2 \omega_x^3 - \frac{60}{19} a_2 \alpha \omega_x, \\ u_0 &= \frac{1}{19\omega_x^3} [570\beta \omega_x \omega_{xx} \omega_{xxx} - 1140\beta a_0 a_1 a_2 \omega_x^6 - 285\beta a_1^2 \omega_x^4 \omega_{xx} - 2280\beta a_0 a_2 \omega_x^4 \omega_{xx} \\ &\quad - 570\beta a_1 \omega_x^3 \omega_{xxx} - 30\alpha a_1 \omega_x^4 - 30\alpha \omega_x^2 \omega_{xx} - 285\beta \omega_{xx}^3 - 19\omega_x^2 \omega_t - 285\beta \omega_x^2 \omega_{xxxx}]. \end{aligned} \tag{27}$$

Then substituting Eqs. (27) into the coefficients of R^3, R^2, R, R^0, we find the coefficients of R^3, R^2, R, R^0 are not zero automatically but the functions S, C, ω should satisfy the condition

$$\begin{aligned} &2166\beta^2 \omega_x^4 S_{xx} + 722\beta C_x \omega_x^4 - 380\alpha\beta\omega_x^4 S - 1444\beta^2 \omega_x^4 S^2 + 190\alpha\beta\delta\omega_x^6 \\ &\quad - 361\beta^2 \delta^2 \omega_x^8 + 3610\beta^2 \omega_x^5 \omega_{xxx} = 0. \end{aligned}$$

So we prove the general KS equation (24) is not CRE solvable and the soliton–cnoidal interaction wave solutions could not be constructed by the same way in last section.

5. Summary and discussion

In summary, a new interaction wave solution for a special KS equation (1) is obtained by the means of the CRE method and the general KS equation (24) is proved to be non-CRE solvable. The CRE method can help us to find interaction solutions between solitons and any other nonlinear waves for many integrable systems. It is also shown that the CRE method can be applied to other kinds of integrable systems, especially for supersymmetric models and discrete ones to find different kinds of nonlinear interaction solutions. More interaction wave solutions among different kinds of nonlinear excitations are worthy of study further.

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