Wavefront fitting of Interferogram with Zernike polynomials based on SVD

Chang Liping, Wei Zihua, Shen Weixing, Lin Zunqi
Chinese Academy of Shanghai Institute of Optics and Fine Mechanics
P. O. Box 20, Jiading, Shanghai, China, 201800
E-mail: changliping@mail.siom.ac.cn

ABSTRACT

As the resolution of the interferometer is rapidly increasing, the higher requirement is proposed to the interferometry. For avoiding losing of the useful information of the actual interferogram, an algorithm is presented based on singular value decomposition (SVD) which is more stable than other algorithms to some extent. The weight coefficients of the orthogonal polynomials can be worked out directly, which can eliminate the computational error. Then an evaluation criterion is developed to choose the optimum Zernike polynomial number. The computer simulations of this algorithm are also made. The simulation experiments have proved that it is an efficient algorithm and can reconstruct the wavefront accurately and stably.

Keywords: Subject terms: Wavefront fitting; High resolution; SVD; Polynomial Number estimate

1. INTRODUCTION

As the optical testing technique progresses, interferometry is also developing in the direction of high speed and high resolution. The real-time interferometric measurement is also a trend in the course of optical elements polishing and in the running of high-power laser. This demands the wavefront fitting algorithm more accurate. And there are many classic algorithms with Zernike polynomials[1], such as the least-squares method, the Grame-schmidt orthogonalization (GSO), the covariance matrix method, etc.

Heretofore almost all the papers recommend the Grame-schmidt orthogonalization[2] (GSO) as the best choice among all the algorithms for wavefront fitting. But the problem of GSO is that although the method can avoid bringing in the terrible ill-conditioned matrix when constructing the normal equations, it will produce the severe correlation when constructing the normalized orthogonal functions. Therefore GSO cannot also obtain a stable solution[3]. The other key problem is that once the interferogram contains more frequency components, this algorithm will not be still suitable because of the system becoming unstable.

In this paper an algorithm is presented based on singular value decomposition (SVD). SVD has good stability and can always obtain satisfactory solution[4]. It obtains the coefficients of Zernike polynomials directly, which can avoid the computational error introduced by constructing normal equations.

The other section stated is how to determine the Polynomials Number rudely. Previously most analyzing software always uses fixed mode number of Zernike polynomials. But now for the interferogram with high frequency, the former polynomials number will not be enough to fit the wavefront accurately. For example if the typical 36 Zernike polynomials is used, the reconstructed wavefront will lose some useful information. An evaluation criterion is presented to determine the Zernike polynomial number.

2. ALGORITHM AND CALCULATION PROCEDURE

Wavefront fitting can be considered as interpolation between the sampling points to estimate the complete wavefront shape. To do so, one requires that the set of series obey the following properties: it is an orthogonal system over the whole interested domain so that any function can be approximated by the linear combinations of the series with any degree of the accuracy. Most lens apertures are circular, thus, we usually use Zernike polynomials that are separable in angle and radius and that form an orthonormal basis. Let us use the definition as follow[5]. This set of polynomials is convenient from the point of view of a statistical analysis.

2nd International Symp. on Adv. Optical Manufac. and Testing Tech.: Optical Test and Measurement Tech. and Equip., X. Hou, J. Yuan, J. C. Wyant, H. Wang, S. Han, Eds., Proc. of SPIE Vol. 6150, 61500G, (2006) · 0277-786X/06/$15 · doi: 10.1117/12.677984

$$z_n^l = R_n^l(\rho)e^{il\theta} = R_n^l(\rho)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(n-2m)\theta \qquad (1)$$

$$m = \frac{n-l}{2} \qquad (2)$$

where the radial polynomials are defined as

$$R_n^{n-2m}(\rho) = \sum_{s=0}^{m}(-1)^s \frac{(n-s)!}{s!(m-s)!(n-m-s)!}\rho^{n-2s} \qquad (3)$$

And sine function is used when n>2m and the cosine function is used when n⩽2m.
Any wavefront function can be transformed to a linear combination of L Zernike polynomials. Now we present a wavefront

$$w(x,y) = \sum_{j=1}^{L} k_j z_j(x,y) \qquad (4)$$

Here z_j denotes the j'th Zernike polynomial, while the coefficient is k_j.

Giving a set of N discrete data points in the interferogram with coordinates (x_i , y_i), i=1,2, …,N. In general, there are more data points than unknown coefficient N>L, which is over determined system. So rewrite Equation (4) in the matrix form of vectors as:

$$W = [Z_1, \cdots, Z_L]K \qquad (5)$$

where W is the vector formed by N measured wavefront data. Z_j is the vector formed by N values computed by j'th Zernike polynomial at N data points , and K is the vector formed by L coefficients .

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_N \end{bmatrix} \qquad Z_j = \begin{bmatrix} Z_j(x_1, y_1) \\ \vdots \\ Z_j(x_N, y_N) \end{bmatrix} \qquad K = \begin{bmatrix} K_1 \\ \vdots \\ K_L \end{bmatrix} \qquad (6)$$

Singular value decomposition (SVD) will deduce the closest possible job in the least squares sense. In other words, through SVD Equation (5) can find

K which minimizes $\chi^2 \equiv |Z \cdot K - W|^2$

The number χ^2 is called the residual of the solution.

SVD is one of the most elegant algorithms in numerical algebra for providing quantitative information about the structure of a system of linear equations. The algorithm performs SVD to the data matrix made up by $Z_1, \cdots, Z_L$ algebra vectors, which can acquire

$$[Z_1, \cdots, Z_L] = U \cdot diag(\sigma_j) \cdot V^T \qquad (7)$$

where U and V is the $N \times L$ and $L \times L$ orthogonal matrices. And the diagonal matrix has the order as $\sigma 1 \geq \sigma 2 \geq \cdots \geq \sigma_L \geq 0$. So the solution vector can be obtained as

$$K = \sum_{i=1}^{L} \left(\frac{U_{(i)}{}^T \cdot W}{\sigma_i}\right) V_{(i)} \qquad (8)$$

Although in general, the matrix formed by the L Zernike polynomials vectors will not be singular and no $\sigma_i{}' s$ will need to be set to zero[6]. However there might exist that some singular values is nonzero but very small. So we should also define its reciprocal to be zero.

And to test the stability of the system, we introduce the condition number of Z

$$cond(Z) = \sigma_{\max} / \sigma_{\min} . \qquad (9)$$

3. APPLICATION ON MEASUREMENTS

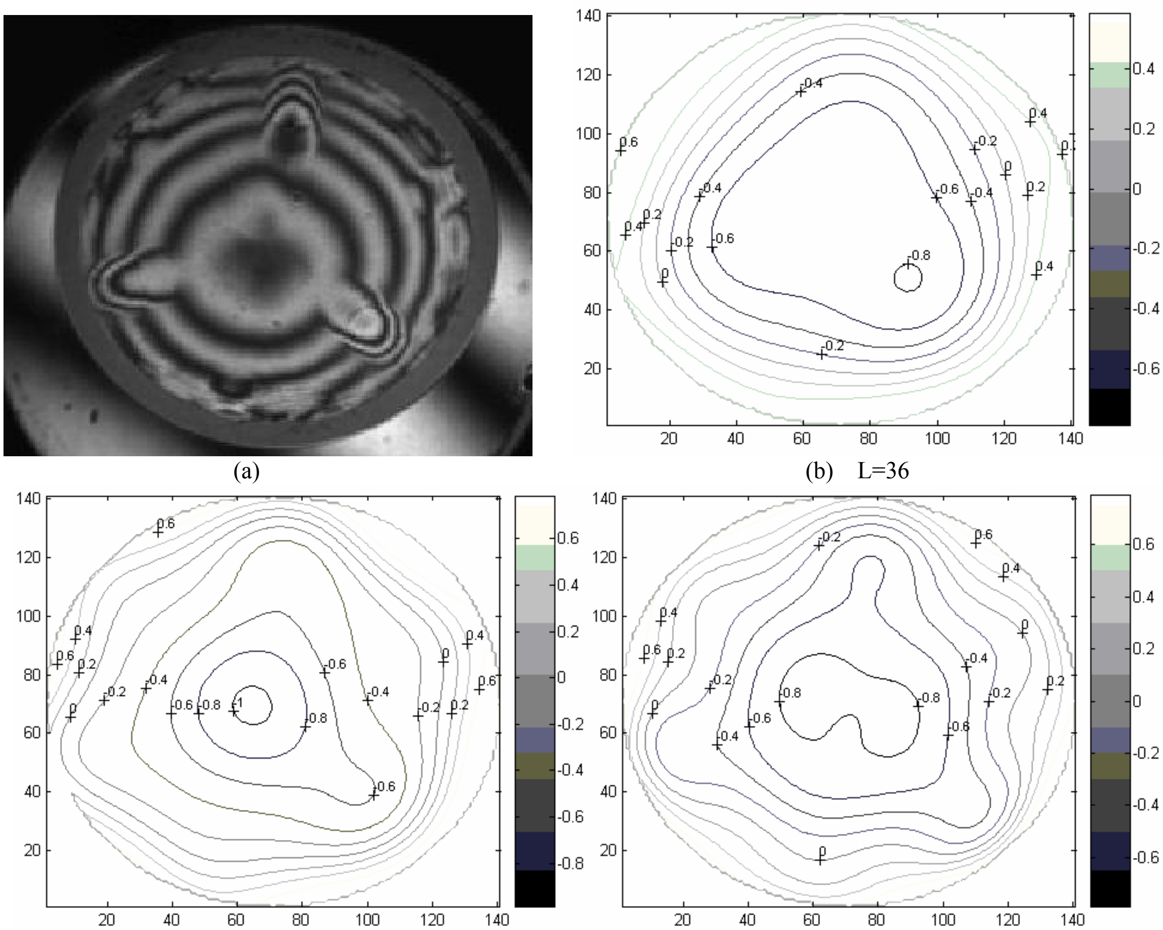
The fitting was done by the phase reduction of the interferogram shown in Fig. 1(a). We pick a circular area with the radius of 140 pixels in the center of the interferogram. Obviously the resolution is much higher than before.

Generally, in the wavefront fitting of the interferogram, one uses a determined number of the Zernike polynomials. For example, 15 or 36 items of Zernike polynomials are widely used at present measurement. But as the CCD's resolution of the interferometer becomes higher and higher, the corresponding interferogram will contain more frequency components. A determined number as before will not satisfy this condition. If we still use the fixed number to analyze the interferogram, the fitting wavefront will not be accurate. Fig. 1 is the contour plot when the Zernike polynomial number changes from 36 to 144. Form the plot, we can see that when L=36, the contour plot couldn't represent the wavefront at all, and the rms of the fitting error is 0.1064λ, which has obviously lost some useful information of origin wavefront. And if the root mean square (rms) of fit error is not divergent, the more Zernike polynomial number is, the more accurately we fit the wavefront. Therefore the main problem is "how many Zernike polynomials is optimum". Here we give a criterion to determine rudely the Zernike polynomial number:

Suppose the Zernike mode number we used in the wavefront fitting is L. Then we define the ratio between the rms of the fitting error and the surface error as an evaluation criterion[7].

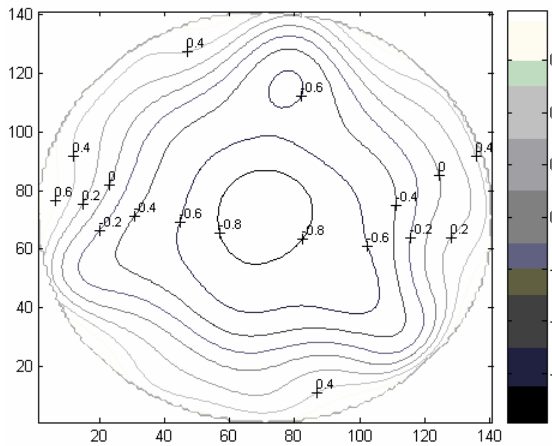
$$G = \frac{rms(W - \sum_{j=1}^{L} k_j z_j)}{rms(\sum_{j=1}^{L} k_j z_j)} \tag{10}$$

This ratio can evaluate whether the fitting result is good so as to judge whether the Zernike polynomial number is enough to fit the wavefront accurately.

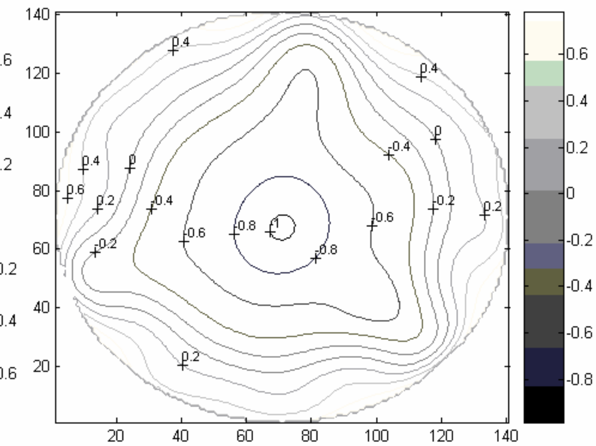


(a) (b) L=36

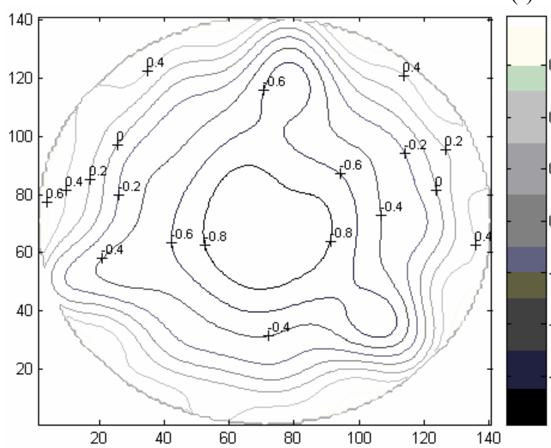
(c) L= 48 (d) L=72



(e) L= 90



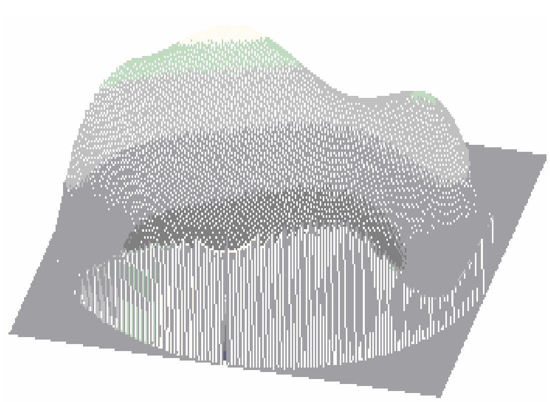
(f) L=108



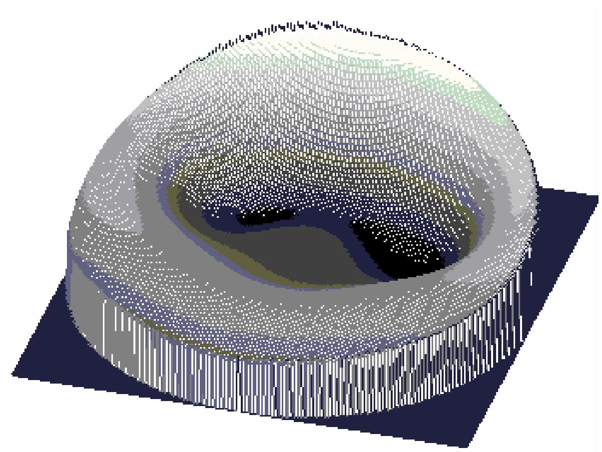
(g) L=144

Fig.1 (a) Interferogram; the others are the compared results of different L

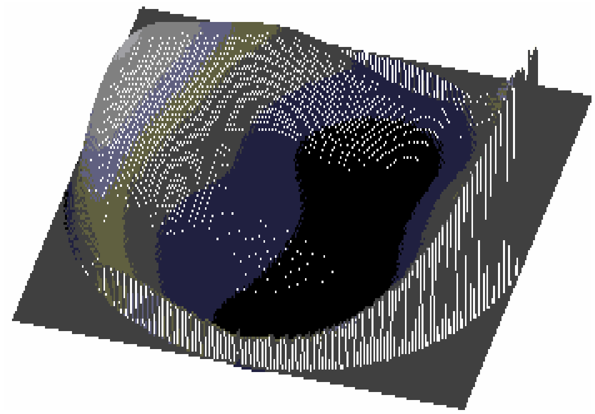
Here is three elements' three- dimensional surface map shown in Fig. 2. Fig. 3 is the fitting results obtained according to Equation (11).



(1)



(2)



(3)

Fig. 2 Three dimensional surface map of different elements

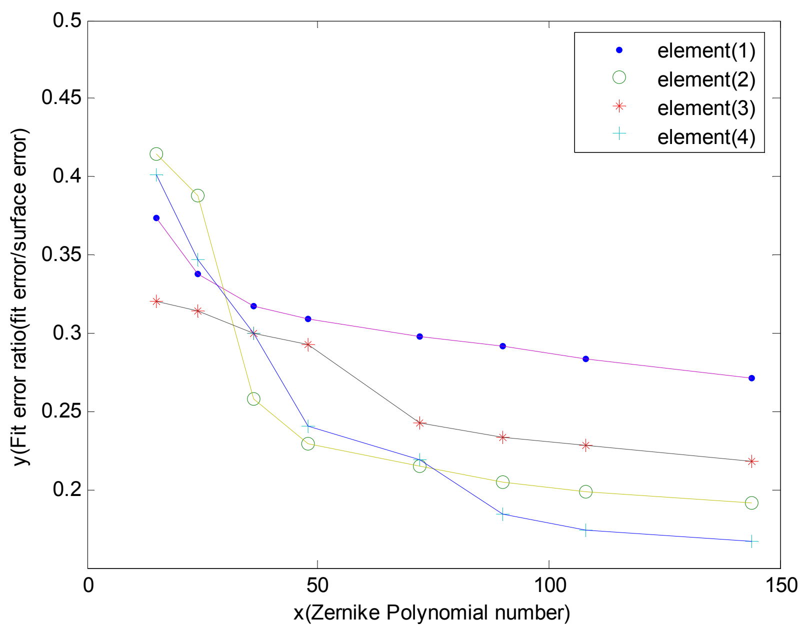


Fig. 3 Fit errors of different elements

From the Fig. 2 and Fig. 3, we can see that, element (1) is a comparably quadratic curved surface, and 24 Zernike polynomials number is enough to fit this wavefront.; element (2) is a surface with the concave hole centered, which can be fitted accurately through 48 Zernike polynomials, that is the order of the fitting polynomial is 12; element (3) is a surface with different radius of curvature in its two direction, we have to use 72 polynomials to reach our requirement. The order of the polynomial is 16

The interferogram of element (4) is shown in Fig. 1 (a). From the Fig. 3 we can see the optimum mode is 90 Zernike polynomials, and this wave aberration can be represented well. Meantime we also can see the same conclusion from Fig.1. The rms of the surface error found by SVD of Fig.1 (e), (f) and (g) are 0.3437λ, 0.3433λand 0.3438λ, and the fitting errors are 0.0582λ, 0.0542λ and 0.0533λ separately. There is no significant difference. So 90 Zernike polynomials number is enough.

Meanwhile we compute the condition number of the matrix Z when the polynomial number changes from 0 to 180. We can see from the Fig. 4, all the values of condition number is under 8, so we can induce that in the condition of high order of Zernke polynomial and great number of data points, the algorithm SVD is very stable. And this is the unique advantage of SVD to other algorithms. So SVD is an algorithm for the wavefront fitting with high stability and accuracy.

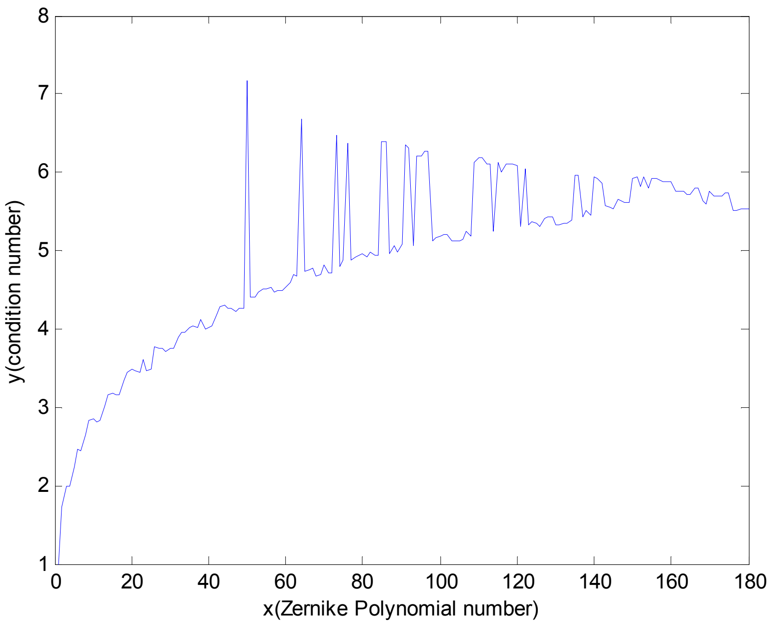


Fig. 4 Plot of the condition number of matrix Z

4. CONCLUSIONS

As the resolution of the interferometer is rapidly enhancing, many new problems have appeared gradually. The traditional algorithms will not still satisfy requirements. The algorithm is introduced, based on SVD which has better stability than other algorithms when there are more testing data than unknown coefficients. Besides their same fitting precision with other algorithms, SVD is characteristic of the better stability, less calculation and simpler computation process. These studying results have proved that SVD is an ideal algorithm for the wavefront fitting. Meanwhile in the fitting process the fixed mode number is used and the polynomial number is determined through an evaluation criterion we defined. The wavefront expressed by using this optimum mode can be exactly reconstructed. Generally speaking, it is recommend that in the algorithm SVD techniques is used instead of using the normal equation when dealing with the least-square problem. Its only disadvantage is a little slower than solving the normal equations. However, its outstanding advantage is that the algorithm cannot fail theoretically, except that it is low speed. And it follows that Zernike polynomial can be replaced by other orthogonal polynomials such as Legendre functions, "Square_Radial" functions or Zernike-Mahajan polynomials as all the lens apertures are not circular, SVD is suited for wavefront fitting of a different choice of basis function[8].

REFERENCE

1. Yan Jingzhou, Lei Fan, Zhou Bi-Fang et al.. Algorithms for wavefront fitting using Zernike polynomial. Opt. Prec. Eng., 1999, 7(5): 119~128
2. Daniel Malacara. Wavefront fitting with discrete orthogonal polynomials in a unite radius circle. Opt. Eng. 1990, 29(6), 672~675
3. Mo Weidong. Application and methods to fit optical interfere wave surface with quantizing technique. Chinese Journal of Computational physics. 2003, 20(6): 514~520
4. Zhang Xianda. Matrix analysis and applications. Beijing: Tsinghua University Press, 2004: 341~358
5. Daniel Malacara. Optical Shop Testing. John Wiley and Sons, New York (1978).
6. illiam H.Press, Saul A.Teuko. Numerical recipes in C: the art of scientific computing. Cambridge: Cambridge University Press(1992).
7. Liu Jianfeng, Long Funian, Zhang Wei. Frequency Domain Analysis of Surface Figure Fitting Based On Zernike Polynomials. Acta Optica Sinica. 25(8): 1062~1066
8. Ge Aiming, Chen Jinbang, Mathematical vector model for interferogram of arbitrary apertures of the fitting process, Infrared and Laser Engineering. 29(6): 42~46.