Article ID: 1004-4213(2010)04-0739-5

Phase Probability Distribution and Wigner Function of the Superposition of Two Arbitrary Coherent States

ZHANG Ai-ping, SHI Yi-min, WEI Nuo

(*Faculty of Science, Xi'an University of Architecture and Technology, Xi'an* 710055, *China*)

Abstract: The quantum statistical properties of the superposition state, which involve two arbitrary coherent states $|\beta\rangle$ and $|m\beta e^{i\delta}\rangle$, are investigated from two aspects, such as the phase distribution and Winger function. The results show that the non-classical properties of the superposition state depend on β^2, amplitude coefficient m, the phase difference between the coherent states δ, and the phase difference between the superposition coefficients ϕ. When selected suitable parameters the superposition state exhibits quantum effect. The phase distribution and Wigner function of an equally populated mixture of two coherent states are derived, too. By comparison the results show that the presence of the coherence terms makes the superposition state with full quantum-mechanical behavior.

Key words: Superposition state; Phase distribution; Wigner function; Mixed state; Quantum effect

CLCN: O431.2 **Document Code**: A **doi**: 10.3788/gzxb20103904.0739

0 Introduction

Since Glauber's[1] pioneering work in 1963, the coherent state has solved the mathematical difficulties from which the people study the light field by using quantum electrodynamics, and greatly promoted the development of quantum optics. At present, the theory and application about the coherent state have become important field of research in physics. Linear superposition and nonlinear superposition of the coherent states and their deformation q superposition state have become important in the preparation of different nature quantum states of light field[2-6]. Ma Zhimin et al[7] have introduced superposition of two arbitrary coherent states, and studied squeezing effect or anti-bunching effect of the superposition state. However, the phase distribution and the Wigner function of the state have not been reported.

In this paper, we study the phase distribution and the Wigner function of the superposition state. Because the superposition state, upon interaction with the environment, is degenerated to a mixed state, in order to compare we calculate the phase distribution and the Wigner function of an equally populated mixture state, too.

1 The definition of superposition of two arbitrary coherent states

According to the literature [7], the superposition state is defined as

$$|\Psi\rangle = a|\beta\rangle + be^{i\phi}|m\beta e^{i\delta}\rangle \tag{1}$$

where $|\beta\rangle$ and $|m\beta e^{i\delta}\rangle$ are coherent states, both of them satisfy $\hat{a}|\beta\rangle = \beta|\beta\rangle$, $\hat{a}$ is the annihilation operator; m, which is the amplitude multiple of the two coherent states, represents two coherent states with different average photon number; a, b, m and β are real numbers; δ represents the phase difference between the coherent states, while ϕ represents the phase difference between the superposition coefficient. Normalized condition is

$$a^2+b^2+2ab\exp\left[-\frac{1}{2}(m^2+1)\beta^2+m\beta^2\cos\delta\right]\times \cos(\phi+m\beta^2\sin\delta)=1 \tag{2}$$

In accordance with the normalized condition Eq. (2), we can know that the coefficient a, b meet a quadratic equation of the parameters m, β, ϕ. When $a=0$ or $b=0$, the superposition state reduces to one coherent state.

The density operator associated with the superposition state is given by[8]

$$\hat{\rho}_\Psi = a^2|\beta\rangle\langle\beta| + b^2|m\beta e^{i\delta}\rangle\langle m\beta e^{i\delta}| +$$

* Supported by the Youth Foundation of Xi'an University of Architecture and Technology (QN0702)
Tel: 13363937238 Email: apzhang163@163.com
Received date: 2009-07-02 Revised date: 2009-08-27

$$abe^{i\phi}|m\beta e^{i\delta}\rangle\langle\beta|+abe^{-i\phi}|\beta\rangle\langle m\beta e^{-i\delta}| \quad (3)$$

on the other hand, the density operator for an equally populated mixture[9] of $|\beta\rangle$ and $|m\beta e^{i\delta}\rangle$

$$\hat{\rho}_M=\frac{1}{2}(|\beta\rangle\langle\beta|+|m\beta e^{i\delta}\rangle\langle m\beta e^{i\delta}|) \quad (4)$$

The two density operators differ by the presence of the coherence terms in the former, such terms being absent in the case of the mixture.

2 The phase probability distribution of the the superposition state

According to Pegg-Barnet′ theory of phase[9-10], the phase state of the quantized field is defined

$$|\varphi\rangle=\lim_{s\to\infty}(s+1)^{-\frac{1}{2}}\sum_{n=0}^{\infty}e^{in\varphi}|n\rangle \quad (5)$$

where $|n\rangle$ are the $s+1$ number states which spanan $s+1$-dimensional state space, φ is eigenvalue, $|\varphi\rangle$ is the corresponding eigenstate. A phase distribution $P(\varphi)$ with the state is defined by[9-10]

$$P(\varphi)=\lim_{s\to\infty}\frac{s+1}{2\pi}|\langle\varphi|\Psi\rangle|^2 \quad (6)$$

where $|\Psi\rangle$ is an arbitrary state of the field. Clearly, $P(\varphi)$ is always positive and can be normalized as follows

$$\int_0^{2\pi}P(\varphi)d\varphi=\int_0^{2\pi}\frac{1}{2\pi}|\langle\varphi|\Psi\rangle|^2d\varphi=\langle\Psi|\Psi\rangle=1 \quad (7)$$

Let us now consider the phase distribution of the superposition state. Considering β is a real number, the phase distribution function for the superposition states is

$$P(\varphi)=\frac{1}{2\pi}|a\langle\varphi|\beta\rangle+be^{i\varphi}\langle\varphi|m\beta e^{i\delta}\rangle|^2 \quad (8)$$

When the coherent states are strong, namely, $\bar{n}\gg1$ the photon number distribution mainly concentrated in the vicinity of the average photon number. For large $|\beta|^2$, the Poisson distribution may be approximated as a Gaussian[9-10]

$$\frac{|\beta|^{2n}\rangle}{n!}e^{-|\beta|^2}\approx(2\pi|\beta|^2)^{-\frac{1}{2}}\exp\left[-\frac{(n-|\beta|^2)^2}{2|\beta|^2}\right] \quad (9)$$

So that we can obtain an approximate form for $P(\varphi)$ as

$$P(\varphi)\approx\sqrt{\frac{2|\beta|^2}{\pi}}\{a^2\exp[-2|\beta|^2\varphi^2]+mb^2 * \exp[-2|m\beta|^2(\varphi-\delta)^2]+2\sqrt{m}ab\cos\phi * \exp[-|\beta|^2\varphi^2-|m\beta|^2(\varphi-\delta)^2]\} \quad (10)$$

Obviously, the phase probability distribution $P(\varphi)$ depends not only on the average photon number $\bar{n}$ ($\bar{n}=|\beta|^2$), but also on the amplitude multiple m, phase difference between coherent states δ and superposition coefficient phase difference ϕ. In order to visually show the relationship between them, we use numerical analysis and mapping. Here we have studied several cases on the phase distribution function of the superposition state.

For $\delta=2n\pi$ or $\delta=2n\pi+\pi$, the state vector of two coherent states is the same or the reverse at this time. Normalized condition is

$$a^2+b^2+2ab\exp\left[-\frac{1}{2}(m\mp1)^2\beta^2\right]\times\cos\phi=1 \quad (11)$$

When $\delta=2n\pi, m=1$ or $\delta=2n\pi+\pi, m=-1$, two coherent states are identical, thus

$$P(\varphi)\approx\sqrt{\frac{2|\beta|^2}{\pi}}\{\exp[-2|\beta|^2\varphi^2] \quad (12)$$

Eq. (10) is the same to the distribution of a coherent state. This is a Gaussian peaked at $\varphi=0$. Furthermore, the peak becomes narrower with increasing $\bar{n}=|\beta|^2$, as illustrated in Fig. 1.

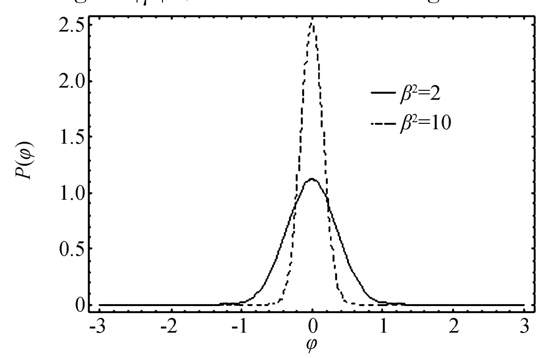

Fig. 1 Phase distribution for superposition state with $\delta=2n\pi, m=1$ or $\delta=2n\pi+\pi, m=-1$

In Fig. 2, when $\delta=2n\pi, m=-1$ or $\delta=2n\pi+\pi, m=1$, the phase distribution appears bimodal Gaussian distribution. Except for $\varphi=0$, the position of the peaks is also related to δ. The height of the peaks not only depends on the average photon number $\bar{n}$, but also depends on m and ϕ.

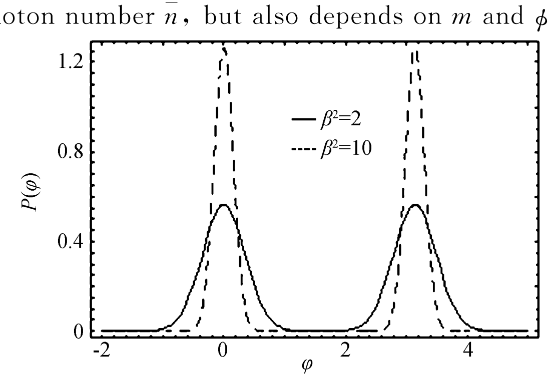

Fig. 2 Phase distribution for superposition state with $\varphi=2n\pi+0.5\pi, \delta=2n\pi, m=-1$ or $\delta=2n\pi+\pi, m=1$

For $a=b=N=[2+2\exp(-2|\beta|^2)\cos\phi]^{\frac{1}{2}}$, we have the Schrodinger cat states[11], the probability of the state is

$$P(\varphi)\approx\sqrt{\frac{2|\beta|^2}{\pi}}|N|^2\{\exp[-2|\beta|^2\varphi^2]+$$

$$\exp\left[-2|\beta|^2(\varphi-\pi)^2+2\cos\phi\times\right.$$
$$\left.\exp\left[-|\beta|^2\varphi^2-|\beta|^2(\varphi-\pi)^2\right]\right\} \quad (13)$$

In the case of $\phi=0$ and $\phi=\pi$, we obtain the even and the odd coherent states[12], respectively.

When $\delta=2n\pi$, $\varphi=2n\pi+(\pi/2)$, the relationship of the phase distributions varying with m, β are given in Fig. 3 and Fig. 4, respectively. From two figures we can see the phase distributions are the bimodal structure which is different peak height, this is due to two coherent states with different average photon number. When m, β are very small, the quantum interference effect becomes obvious. When m, β are going to infinity, the phase distribution approximates to δ function, the classical effect of the field is conspicuous.

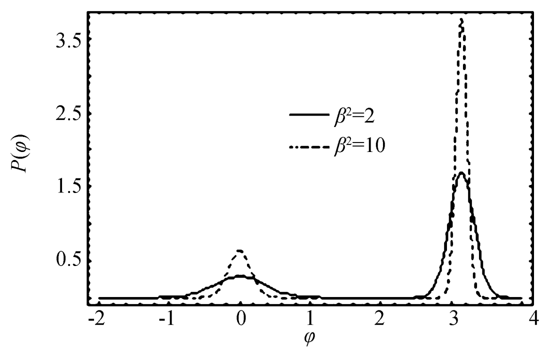

Fig. 3 Phase distribution for superposition state with $\delta=2n\pi$, $\varphi=2n\pi+0.5\pi$, $m=2$

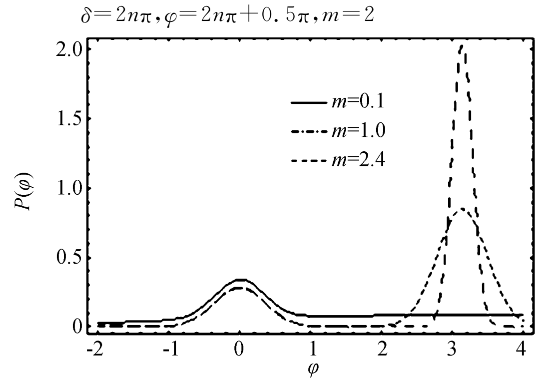

Fig. 4 Phase distribution for superposition state with $\delta=2n\pi$, $\varphi=2n\pi+0.5\pi$, $\beta=2$

In reality, if a coherent superposition state of the form of Eq. (1) is created, it should, upon interaction with the environment, quickly decoheres into a statistical mixture . More generally, for a state described by a density operator $\hat{\rho}$, the distribution of phase[9-10] is

$$P_\rho(\varphi)=\frac{1}{2\pi}\langle\varphi|\rho|\varphi\rangle \quad (14)$$

We consider the statistical mixture described by Eq. (4). In the limit of large $|\beta|^2$, the phase distribution (again with β real)

$$P_\rho(\varphi)\approx\frac{1}{2}\sqrt{\frac{2|\beta|^2}{\pi}}\left\{\exp\left[-|\beta|^2\varphi^2\right]+\right.$$
$$\left.m\exp\left[-|m\beta|^2(\varphi-\delta)^2\right]\right\} \quad (15)$$

Comparing Fig. 2 and Fig. 5, we can find that both distributions have peaks at $\varphi=0$ and $\varphi=\pi$, as expected, but only the first has an coherence term[13], which cause quantum interference between the two coherent states. Although the coherence term is very small, the Gaussians have a little overlap. The quantum interference between the two coherent states becomes intense when m and β^2 are smaller.

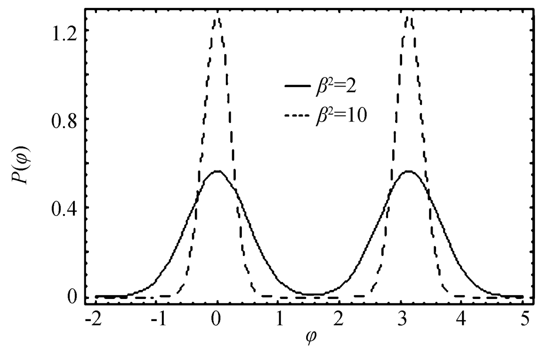

Fig. 5 The phase distribution of the mixed state with $\delta=2n\pi+\pi$ and $m=1$

3 The Wigner function for the superstition state

Since Wigner [13-14] and Cahill and Glauber [15] put forward to the quasi-probability distribution functions, they have played an important role on the statistical description of quantum states. For pure quantum states, W function is a well-behaved function. Because there exists one-to-one relationship between W function and quantum state, the quantum state can be fully determined. W function is not always positive, if negative region appears, then the corresponding state will be non-classical state. Here we calculate the Wigner function of the superposition states.

The Wigner function[13-14] is defined as

$$W(\alpha,\alpha^*)=\frac{1}{\pi^2}\int d^2\beta e^{-i\beta\alpha^*-i\beta^*\alpha}C^{(s)}(\beta,\beta^*) \quad (16)$$

where the characteristic function $C^{(s)}(\beta,\beta^*)$ is given by

$$C^{(s)}(\beta,\beta^*)=\mathrm{Tr}(e^{i\beta a^++i\beta a}\rho) \quad (17)$$

For simplicity, we only consider for the case of $\delta=2n\pi$. From Eq. (16), Eq. (17) and Eq. (3), we can obtain

$$W(\alpha,\alpha^*)=\frac{2}{\pi}\{a^2\exp(-2|\alpha-\beta|^2)+$$
$$b^2\exp(-2|\alpha-m\beta|^2)+abe^{i\varphi}e^{-2|\alpha|^2}\times$$
$$e^{-\frac{(m+1)^2}{2}|\beta|^2}\exp\left[2\beta\alpha+2m\beta\alpha^*\right]+ab\times$$
$$e^{-i\varphi}e^{-2|\alpha|^2}e^{-\frac{(m+1)^2}{2}|\beta|^2}\exp\left[2\beta\alpha^*+2m\beta\alpha\right]\} \quad (18)$$

Setting $\alpha = x + iy$, we plot the function in Fig. 6. Evidently, for given m, the Wigner function oscillates with increasing β and becomes negative over a wide region of phase space.

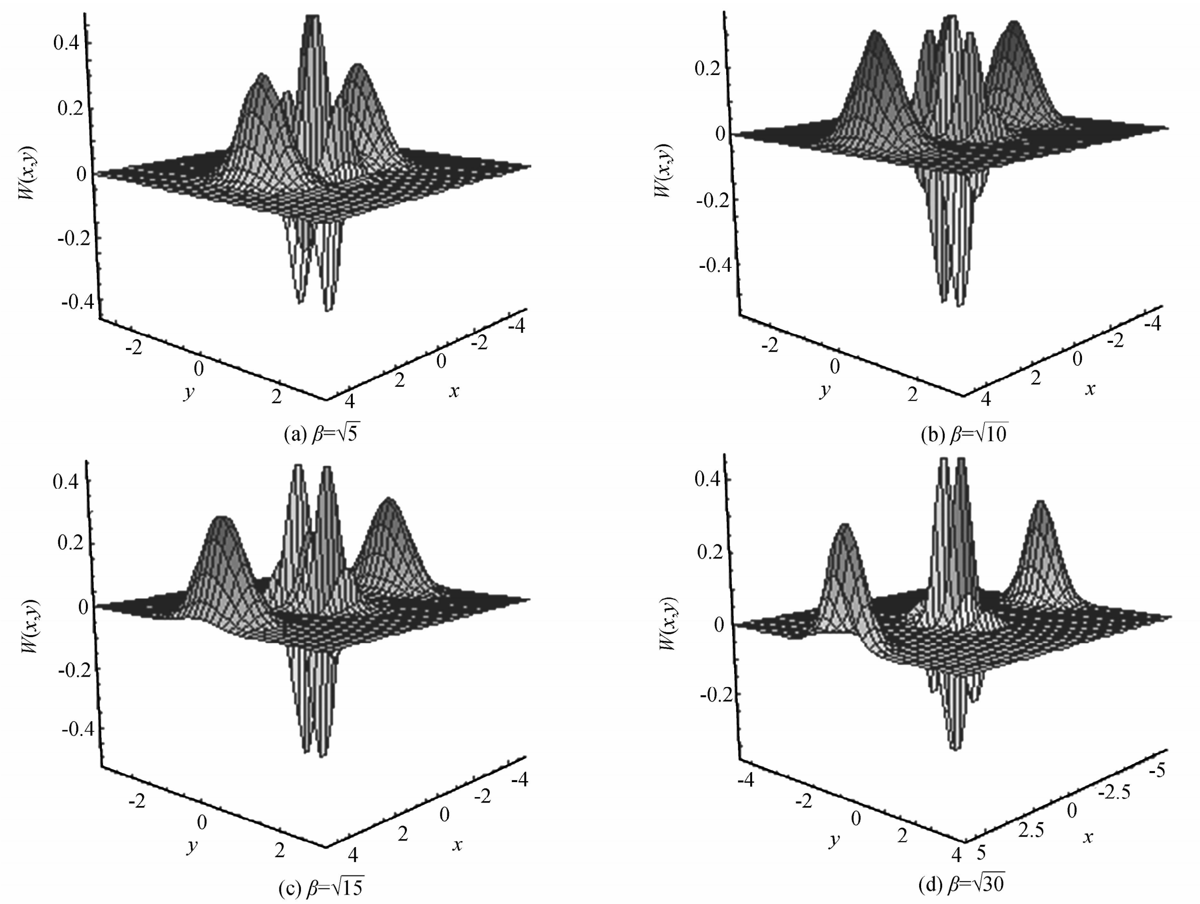

Fig. 6 Winger function for the superposition state with $m=-1$ and $\varphi = 2n\pi$

But the Wigner function for the statistical mixture given by density operator of Eq. (4), obtained from Eq. (12), is

$$W(\alpha,\alpha^*) = \frac{1}{\pi}[\exp(-2|\alpha-\beta|^2) + \exp(-2|\alpha - m\beta|^2)] \tag{19}$$

which is always positive and contains two Gaussian peaks centered at $x=\beta$ and $x=m\beta$ as shown in Fig. 7. Comparing Fig. 6 and Fig. 7, we can find that because the former has an coherence term between the two states $|\beta\rangle$ and $|m\beta e^{i\delta}\rangle$, it causes the Wigner function to become oscillatory and negative in some regions of phase space.

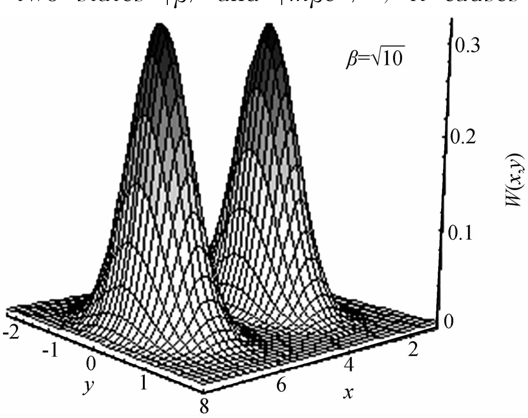

Fig. 7 Wigner function for the statistical mixture of coherent states

4 Conclusion

In this Letter, we study the phase distribution and the Wigner function of the superposition state, which involve two arbitrary coherent states $|\beta\rangle$ and $|m\beta e^{i\delta}\rangle$. The results show that the non-classical property of the superposition state depends on β^2, amplitude coefficient m, the phase difference between the coherent states δ, and the phase difference between the superposition coefficients ϕ. When selected suitable parameters the superposition state exhibits quantum effect.

Nevertheless, the quantum system always interacts with environment, and then the superposition state reduces to a mixed state. Therefore, by using the phase theory introduced by Pegg and Barnet, we derive the phase distribution and Wigner function of an equally populated mixture of two coherent states. By comparison we obtain that the presence of the coherence terms makes the superposition state with full quantum-mechanical behavior.

References

[1] GLAUBER R J. Coherent and incoherent states of radiation field[J]. *Phys Rev*, 1963, **131**(6): 2766-2788.

[2] ZHANG Min, JIANG Jun-qin. Higher power Sqeezing effects of the photon-added even and odd q-coherent states[J]. *Acta Photonica Sinica*, 2002, **31**(12): 1436-1438.

[3] WANG Ji-suo, WANG Chuan-kui. Amplitude nth-power squeezing characteristics of a new optical field superposition states[J]. *Acta Photonica Sinica*, 1994, **23**(6): 576-579.

[4] WANG Ji-suo, LIU Tang-kun, ZHAN Ming-sheng. Higher-order squeezing for generalized odd and even coherent states of a Q-deformed non-harmonic oscillator[J]. *High Energy Phys and Nucl Phys*, 2001, **25**(1): 11-15.

[5] REN Min, MA Zhi-min, MA Ai-qun, *et al*. N-th power squeezing of the field-amplitude on generalized single-parameter q-deformation coherent state of an anharmonic oscillator in a finite-dimension hilbert space. Preparation of squeezed even coherence state and its nonclassical properties [J]. *Acta Photonica Sinica*, 2004, **33**(12): 1526-1529.

[6] SCHLEICH W, PERNIGO M and KIEN F L. Nonclassical state from two pseudoclassical states[J]. *Phys Rev A*, 1991, **44**(3): 2172-2187.

[7] MA Zhi-min, MA Ai-qun, ZENG Ran, *et al*. Superposition of two arbitrary coherent states and its quantum statistical properties[J]. *Acta Photonica Sinica*, 2005, **54**(5): 2049-2058.

[8] NI Guang-jiong, CHEN Su-qing. Advanced quantum mechanics [M]. Shanghai: Fudan University Press, 2000.

[9] PEGG D T, BARNETT S M. Unitary phase operator in quantum mechanics[J]. *Europhys Lett*, 1988, **6**(6): 483-485.

[10] PEGG D T, BARNETT S M. Phase properties of the quantized single-mode electromagnetic field[J]. *Phys Rev A*, 1989, **A39**(4): 1665-1675.

[11] WANG X G. Production and detection Schrodinger cat state and its effect on the evolution of Jaynes-Cummings model[J]. *Acta Physica Sinica*, 1996, **45**(3): 389-393.

[12] XIA Yun-jie. Nonclassical properties of even and odd coherent states[J]. *Phys Lett A*, 1998, **136**(6): 281-283.

[13] MARLAN O. SCULLY, ZUBAIRY M. Quantum optics[M]. Cambridge: Cambridge University Press, 1997.

[14] WINGER E. On the quantum correction for thermodynamic equilibrium[J]. *Phys Rev*, 1932, **40**(5): 749-759.

[15] CAHILL K. E, GLAUBER R. J. Ordered expansions in Boson amplitude operators[J]. *Phys Rev*, 1969, **177**(5): 1857-1881.

任意两个相干态的叠加态的相位分布和 Wigner 函数

张爱萍，史毅敏，魏诺

（西安建筑科技大学 理学院，西安 710055）

摘　要：从相位分布和 Wigner 函数两个方面研究了任意两个相干态 $|\beta\rangle$ and $|m\beta e^{i\delta}\rangle$ 的叠加态的量子统计性质．结果表明这种叠加态的非经典特性与 β^2，振幅系数 m，相干态间的位相差 δ 以及叠加系数间的位相差 ϕ 都有关．当参量选择合适，这种叠加态存在着量子效应．计算了两个相干态等几率混合系综的相位分布和 Wigner 函数，经过与前者比较，结果表明由于相干项的存在，使得叠加态具有很好的量子力学行为．

关键词：叠加态；相位分布；Wigner 函数；混合态；量子效应

ZHANG Ai-ping　was born in 1973. She received her M. S. degree from Chongqing University. Now she is a lecture, and her research interests focus on the quantum iformation and quantum computation.