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Density of Phase Branch Points for a Light Wave Propagation in Atmospheric Turbulence

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Abstract: Under Rytov approximation and geometrical optics approximation, a formula of the variance of the log-amplitude derivative was deduced for the case of a plane wave propagating through turbulence. It was clarified that the main factors, which determined the variance, were the Rytov variance, the turbulence inner scale and the Fresnel size. And, based on the formulae deduced by Voitsekhovich, the expression for density of phase branch points was modified. The relationship between the density and the above mentioned parameters was analyzed thoroughly, which indicates that the density increases with Rytov variance increasing and decreases with turbulence inner scale and Fresnel size increasing under the condition of Rytov variance less than 1.

Key words: Atmospheric turbulence; Phase branch points; Density; Variance of the log-amplitude derivative

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0 Introduction

When a light wave propagates through a turbulent medium, not only amplitude fluctuation but also phase fluctuation would occur [1]. A lot of earlier works were on the statistics of scintillation (turbulence-induced log-amplitude fluctuation) under Rytov approximation [2-4], but relatively less on phase fluctuation.

Recently, with the development of astrophysical studies as well as adaptive optics technology, the research on phase singularities becomes significant. In 1974, Nye and Berry [5] firstly introduced the conception of these singularities attempting to understand radio echoes from the bottom of the Antarctic ice sheet. J. Krása [6] found the dependence of wavefront dislocation density on the development of turbulence of the backward self-excited ionization waves in a neon discharge by means of experimental study.

Fried and Vaughn [7] proved the existence of phase dislocations (an alternative terminology for branch points) when light waves propagate through the turbulent medium. Later extensive theoretical and experimental works [8-10] on atmospheric-compensation and phase reconstruction algorithms came forth. It has been evidently verified that the compensation technology could not work efficiently under the condition where the phase branch points present [8].

It will have great practical importance to determine the density of branch points, showing an expected number of branch points to be found inside the unit area in terms of turbulence and light propagation parameters. Voitsekhovich et al attempted to establish the relation between the density of phase branch points and the turbulence and the propagation parameters under Rytov approximation [11]. However, they did not present an explicit formula for the variance of the log-amplitude derivative, and used the simulation results of this parameter in theoretically calculating the density of phase branch points.

In this paper we derive the formula for the variance of the log-amplitude derivative under Rytov approximation in the case of the plane wave propagation. It's found that the main factors which determine the variance are the Rytov index, and the inner scale of turbulence together with the Fresnel size of light waves. Additionally, a modified expression for the density of phase branch points with the same parameters is established based on the formulae of Ref. [11]. The relationship between the density and above mentioned factors is analyzed thoroughly.

1 Variance of the log-amplitude derivative

Under Rytov approximation turbulence-induced log-amplitude fluctuation, interpreted in terms of a two-dimensional Fourier expansion of log-amplitude fluctuation in the plane

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perpendicular to the propagation direction, has the form [2]

$$\chi(\boldsymbol{\rho},L)=\int_{-\infty}^{+\infty} e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\mathrm{d}\varphi(\boldsymbol{\kappa}) \tag{1}$$

where $\boldsymbol{\rho}=(\boldsymbol{x},\boldsymbol{y})$ is the vector spatial coordinate. $\boldsymbol{\kappa}$ denotes the two-dimensional spatial wave number. L is the propagation distance. The term $\mathrm{d}\varphi(\boldsymbol{\kappa})$ represents the random spectral amplitude of the wave log-amplitude. So the partial derivation of the log-amplitude can be expressed

$$\chi_x'=\frac{\chi(\boldsymbol{\rho}_0+\Delta\boldsymbol{\rho})-\chi(\boldsymbol{\rho}_0)}{\Delta x}\Big|_{\Delta\boldsymbol{\rho}\to 0}$$

One can derive its ensemble mean square in the isotropy turbulent medium (In the isotropy turbulent medium, it is satisfied that $D_\chi(\Delta\boldsymbol{\rho})\big|_{\Delta\boldsymbol{\rho}\to 0}=\partial^2 D_\chi(\rho)\big|_{\rho=0}$). That is

$$\langle\chi_x'^2\rangle=\frac{D_\chi(\Delta\boldsymbol{\rho})}{(\Delta x)^2}\Big|_{\Delta\boldsymbol{\rho}\to 0}=\frac{\partial^2 D_\chi(\rho)}{\partial^2 x}\Big|_{\rho=0} \tag{2}$$

Thus the variance of the log-amplitude derivative is related with the structure function of log-amplitude, $D_\chi(\rho)$, and one can similarly get

$$\langle\chi_x'^2\rangle=\langle\chi_y'^2\rangle=\frac{1}{2}\langle\chi_\rho'^2\rangle \tag{3}$$

Under Roytov approximation for a plane wave with wave number $k=2\pi/\lambda$, where λ is wavelength. The expression of the structure function of log-amplitude can be obtained in the following way [4]

$$D_\chi(\rho)=2(2\pi k)^2\int_0^L \mathrm{d}z\int_0^{+\infty}[1-J_0(\kappa\rho)]\cdot \sin^2\left(\frac{L-z}{2k}\kappa^2\right)\Phi_n(\kappa,z)\kappa\mathrm{d}\kappa \tag{4}$$

where $\Phi_n(\kappa,z)$ denotes turbulence refractive index spectrum, and $J_0(\kappa\rho)$ is the zero-order Bessel function. Here we adopt the turbulence spectrum Φ_n in form of Tatarskii modified model

$$\Phi_n(\kappa,z)=0.033C_n^2(z)\kappa^{-11/3}\exp(-\kappa^2/\kappa_m^2) \tag{5}$$

where $\kappa_m=5.92/l_0$, l_0 being the inner scale. The refractive index structure constant, C_n^2, is a constant for uniform level propagation path in the current case.

Upon substituting Eq. (5) into Eq. (4) and combining Eq. (2) and Eq. (3), we find that the variance of the log-amplitude derivative for a plane wave is expressed by

$$\sigma_{\chi_x'}^2=\frac{1}{2}(2\pi k)^2\times 0.033C_n^2\int_0^L \mathrm{d}z\int_0^{+\infty}\cdot \sin^2\left(\frac{L-z}{2k}\kappa^2\right)\kappa^{-2/3}\exp(-\kappa^2/\kappa_m^2)\mathrm{d}\kappa \tag{6}$$

Under the geometrical optics approximation performing the integration in Eq. (6) and using Mellin transform, we can obtain

$$\sigma_{\chi_x'}^2=\frac{0.1324\beta_0^2\sqrt{\pi}2^{1/6}\Gamma\left[\frac{7}{12}\right]}{5K_m^4L_{\mathrm{Fr}}^6\Gamma\left[\frac{11}{12}\right]\sin\left[\frac{\pi}{12}\right]\left[\frac{K_m^4L_{\mathrm{Fr}}^4+1}{K_m^4L_{\mathrm{Fr}}^4}\right]^{7/12}}\Bigg\{-6K_m^2L_{\mathrm{Fr}}^2\sin\left[\frac{7}{6}\arctan\frac{1}{K_m^2L_{\mathrm{Fr}}^2}\right]\sin\left[\frac{5\pi}{12}\right]+3K_m^2L_{\mathrm{Fr}}^2\left[\frac{K_m^4L_{\mathrm{Fr}}^4+1}{K_m^4L_{\mathrm{Fr}}^4}\right]^{1/2}$$
$$\sin\left[\frac{\pi}{12}\right]\left[\cos\left[\frac{1}{6}\arctan\frac{1}{K_m^2L_{\mathrm{Fr}}^2}\right]-K_m^2L_{\mathrm{Fr}}^2\sin\left[\frac{1}{6}\arctan\frac{1}{K_m^2L_{\mathrm{Fr}}^2}\right]\right]+3\cos\left[\frac{7}{6}\arctan\frac{1}{K_m^2L_{\mathrm{Fr}}^2}\right]$$
$$\sin\left[\frac{5\pi}{12}\right](1-K_m^4L_{\mathrm{Fr}}^4)+5K_m^{13/3}L_{\mathrm{Fr}}^{13/3}\left[\frac{K_m^4L_{\mathrm{Fr}}^4+1}{K_m^4L_{\mathrm{Fr}}^4}\right]^{7/12}\sin\left[\frac{\pi}{6}\right]\Bigg\} \tag{7}$$

where $L_{\mathrm{Fr}}=\sqrt{L/k}$ is the Fresnel size and $\beta_0^2=1.23C_n^2k^{7/6}L^{11/6}$ is the Rytov variance. As can be seen from Eq. (7), the variance of the log-amplitude derivative depends on the Rytov index, the turbulence inner scale and the Fresnel size. From the physical point view, Eq. (7) always comes into existence because of the following reason. For most practical problem of light propagation through atmospheric turbulence we usually come down to the diffraction region, i. e. $l_0\leqslant\sqrt{\lambda L}\leqslant L_0$, where L_0 is the outer scale of turbulence, which is supposed infinite in our case. Thus κ_m^{-1} is much less than L_{Fr}. Therefore $\sigma_{\chi_x'}^2$ is always larger than zero in the diffraction region.

From Eq. (7), it is easily found that the variance of the log-amplitude derivative is proportional to the Rytov index for a given Fresnel size under the Rytov approximation. That is numerically plotted in Fig. 1. Three cases are shown for the turbulence inner scale being 1 mm, 2 mm and 5 mm respectively and the Fesnel zone size being supposed to 10.053 mm in our case. It can be also noted that the variance of the log-amplitude derivative is inversely proportional to the Fresnel size for a given Rytov index, which is shown in Fig. 2, where we suppose inner scale being 0.5 mm and Rytov index being 0.5, 0.8 and 1 respectively.

From Fig. 1 we can see that the variance of the log-amplitude derivative decreases with the turbulence inner scale for the constant Rytov index case and the constant L_{Fr} case. Fig. 3 shows more clearly the relationship between $\sigma_{\chi_x'}^2$ and l_0 for the case of $\beta_0^2=0.5; 0.8; 1$ and $L_{\mathrm{Fr}}=10.053$ mm The reason for this is that the main contribution to $\sigma^2_{\chi'_x}$ comes from a vast number of small refractive index inhomogeneity [12]. This influence becomes more significant with smaller l_0.

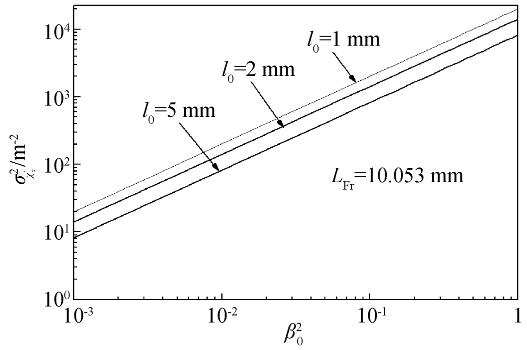

Fig. 1 Variance of the log-amplitude derivative versus Rytov index for the constant Fresnel size case

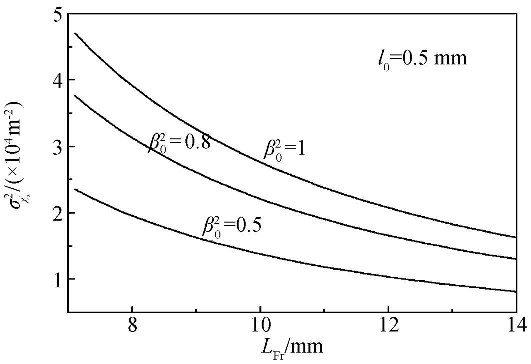

Fig. 2 Variance of the log-amplitude derivative as a function of the Fresnel size

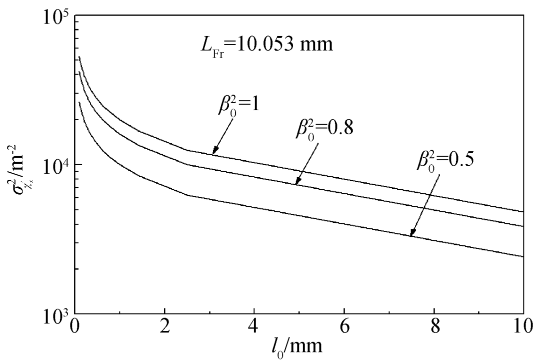

Fig. 3 Behavior between the variance of the log-amplitude derivative and inner scale of turbulence

2 Density of branch points

It has been reported that the density of turbulence-induced phase branch points, showing an expected number of branch points to be found inside the unit area, can be expressed as a function of the variance of the log-amplitude derivative. Here we modify it as a function of β_0^2, K_m^2, $\sigma^2_{\chi'_x}$ and L_{Fr} for a plane wave case

$$D_{\mathrm{bp}}=\frac{(K_m^2L_{\mathrm{Fr}}^2)^{-1/12}\sigma^2_{\chi'_x}}{2\pi^2}\mathrm{erfc}\left[\frac{\pi(K_m^2L_{\mathrm{Fr}}^2)^{1/12}}{4\sigma_{\chi'_x}L_{\mathrm{Fr}}}\right] \quad (8)$$

where erfc(…) operator represents inverse error function and D_{bp} denotes the density of phase branch points. The parameter $\sigma^2_{\chi'_x}$ is derived in Eq.(7).

The relationship between the density of phase branch points and β_0^2 for different l_0 is shown in Fig. 4. Three cases are shown for inner scale being 1, 2 and 5 mm respectively. The density is nearly zero under comparatively weak fluctuation conditions for various inner scales. Only when β_0^2 is greater than 0.1 one could find a branch point in a square meter area when $l_0\leqslant 5$ mm. The density increases continuously with β_0^2 and reaches about 65 per square meter when $\beta_0^2=1$. Fig. 5 shows the relationship between D_{bp} and the Fresnel size, L_{Fr} under given Rytov variances for the constant inner scale case. The density decreases rapidly with the Fresnel size. What′s more, the density D_{bp} versus l_0 of turbulence for a given Fresnel size and various

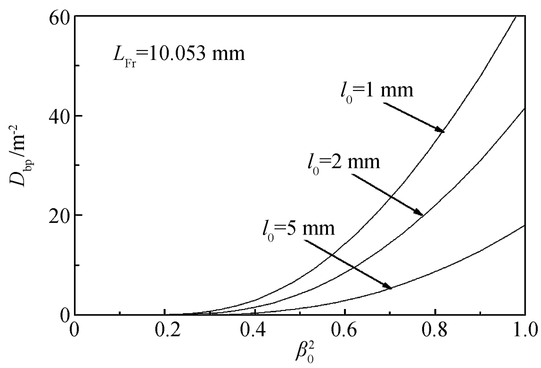

Fig. 4 Relationship between the density of phase branch points and the Rytov index for a given Fresnel size

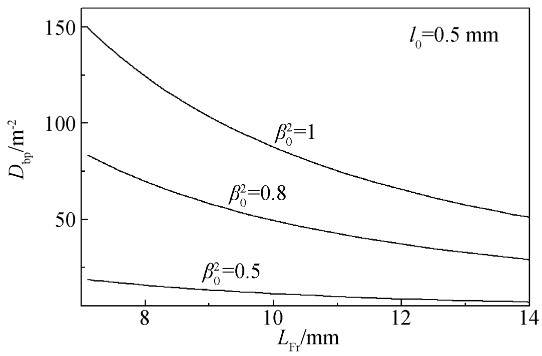

Fig. 5 Density of branch points versus the Fresnel size for the case of $l_0=0.5$ mm

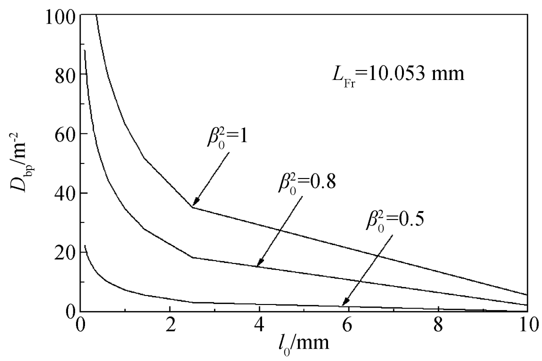

Fig. 6 Density of branch points as a function of the turbulence inner

Rytov variances is plotted with Fig. 6. For a fixed β_0^2 the density D_{bp} decreases rapidly with l_0 for l_0 less than 2 mm, then does in a relatively slower manner.

In conclusion the density of turbulence-induced phase branch points depends strongly on light propagation parameters and the turbulence conditions.

3 Conclusion

When a light wave propagates through a turbulent medium and the scintillation becomes large enough, phase branch points could appear. Its density was proved to be related to the variance of the log-amplitude derivative. However, there isn't an explicit formula for this parameter. So the density of branch points is obtained from numerical simulation at the present time.

In this paper, we have derived the expression of the variance of the log-amplitude derivative, which indicates that the variance depends on the Rytov index, the Fresnel size, and the inner scale of turbulence. Further more, the density formula for the branch points is modified under Rytov approximation The relationship between the density and the above light and turbulence parameters is analyzed respectively.

Our future research will involve numerical simulation and experimental verification.

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湍流大气中光束的相位不连续点数密度

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摘　要：基于 Rytov 近似和几何光学近似条件下，导出了对数振幅导数方差的解析表达式，证明了该参量主要取决于湍流内尺度、Rytov 方差以及 Fresnel 尺寸大小. 在此基础上，给出了光束相位不连续点数密度的修正表达式，分析了相位不连续点数密度随上述湍流参量变化的情况. 分析表明在 Rytov 方差小于 1 的湍流条件下，不连续点数密度数随 Rytov 方差的增大而增大，而随湍流内尺度和 Fresnel 尺寸的增大而减小.

关键词：湍流大气；相位不连续点；数密度；对数振幅导数方差

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