An Oscillation Analytic Solution for Single-Mode Laser Haken-Lorenz System*

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Abstract Dynamical behaviours of a single-mode laser Haken-Lorenz system were investigated. The steady state solution of the system was obtained. Stability analysis was given by using the principle of linear stability. Characteristic equation was gotten and Hopf bifurcation point μ_c was determined. A periodic oscillation analytic solution for the system at bifurcation point was obtained by using the method of series. The dynamical behaviors of the system were given by computer numerical simulation. The results show that when the bifurcation parameter μ crosses the critical value μ_c and Haken-Lorenz system gives rise to limit cycle, i. e. periodic oscillation solution, the result accords with the analytic one.

Keywords Nonlinear optics; Haken-Lorenz system; Hopf bifurcation; Periodic oscillation; Analytic solution

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0 Introduction

Oscillation and chaos behaviours have been widely investigated in many nonlinear systems belonging to different fields such as optics, biology, and chemistry[1~16]. Haken-Lorenz system, a single-mode laser system, which can be considered as a complicated periodic oscillation system[17]. According to linear stability analysis, when the bifurcation parameter crosses the critical value , the linearized operator presents a simple pair of purely imaginary eigenvalues in the system. Such a phenomenon is known for system of ordinary differential equations where it gives rise to limit cycle, i. e. periodic oscillation solution, and is called a Hopf bifurcation.

Then dynamical behaviours of the single-mode laser Haken-Lorenz system are further investigated. An analytic solution for the system at bifurcation point is obtained by using the method of series. The analytic results are also verified by numerical simulation studies.

1 Steady-state solutions and stability

The dynamic equations of the single-mode laser Haken-Lorenz system are written as follows

$$\begin{cases}(\mathrm{d}x/\mathrm{d}t)=\sigma(y-x)\\(\mathrm{d}y/\mathrm{d}t)=(\mu-z)x-y\\(\mathrm{d}z/\mathrm{d}t)=xy-bz\end{cases}\tag{1}$$

where x,y,z are variables, σ,b,μ are parameters.

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The steady-state solutions $S(x_0,y_0,z_0)$ of system (1) can be easily found by solving the three equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}z}{\mathrm{d}t}=0\tag{2}$$

which lead to

$$\begin{cases}\sigma(y_0-x_0)=0\\(\mu-z_0)x_0-y_0=0\\x_0y_0-bz_0=0\end{cases}\tag{3}$$

It can be easily verified into three steady-state solutions

$$S_0(0,0,0),S_{\pm}(\pm\sqrt{b(\mu-1)},\pm\sqrt{b(\mu-1)},\mu-1)\tag{4}$$

The stability of the steady-state solution S_+ or S_- is analyzed by linearizing system (1) at the steady-state solution S_+ or S_-. Under the linear transform $u=x-x_0$, $v=y-y_0$, $w=z-z_0$, system (1) becomes

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}=\sigma(v-u)\\\dfrac{\mathrm{d}v}{\mathrm{d}t}=(\mu-z_0)u-v-x_0w\\\dfrac{\mathrm{d}w}{\mathrm{d}t}=y_0u+x_0v-bw\end{cases}\tag{5}$$

Its Jacobi matrix is

$$L=\begin{bmatrix}-\sigma & \sigma & 0\\\mu-z_0 & -1 & -x_0\\y_0 & x_0 & -b\end{bmatrix}\tag{6}$$

and its corresponding characteristic equation is

$$\lambda^3+(\sigma+b+1)\lambda^2+(\mu+\sigma)b\lambda+2b\sigma(\mu-1)=0\tag{7}$$

Letting

$$\mu=\mu_c=\sigma\frac{\sigma+b+3}{\sigma-b-1}\tag{8}$$

Obviously, Eq. (5) has one pair of purely imaginary conjugate roots

$$\lambda=\pm i\sqrt{b(\sigma+\mu_c)} \tag{9}$$

Therefore, when $\mu=\mu_c$, system (1) has Hopf bifurcation at S_+ and S_-.

2 Analytic solution for Haken-Lorenz equations

The decomposition $u=x-x_0$, $v=y-y_0$, $w=z-z_0$ were inserted into the Eq. (1), keeping this time the nonlinear contributions in x, y and z, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}u\\v\\w\end{bmatrix}=L_c\begin{bmatrix}u\\v\\w\end{bmatrix}+\begin{bmatrix}U\\V\\W\end{bmatrix} \tag{10}$$

where

$$U=0$$
$$V=-(x_0-x_{0c})w-(z_0-z_{0c})u-uw$$
$$W=(y_0-y_{0c})u+(x_0-x_{0c})v-uv$$

and L_c is the linearized operator evaluated at the bifurcation point (x_{0c}, y_{0c}, z_{0c})

$$L_c=\begin{bmatrix}-\sigma & \sigma & 0\\ \mu_c-z_{0c} & -1 & -x_{0c}\\ y_{0c} & x_{0c} & -b\end{bmatrix} \tag{11}$$

Let $\frac{2\pi}{\Omega}$ be the period of the solution and λ the eigenvalue of the linear operator, and introduce the scaled variable $\tau=\Omega t$ and expand $u, v, w, \Omega, x_0, y_0, z_0$ in powers of a parameter ε

$$\begin{bmatrix}u\\v\\w\end{bmatrix}=\varepsilon\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}+\varepsilon^2\begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}+\cdots\cdots$$

$$\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix}=\begin{bmatrix}x_{0c}\\y_{0c}\\z_{0c}\end{bmatrix}+\varepsilon\begin{bmatrix}\alpha_1\\\beta_1\\\gamma_1\end{bmatrix}+\varepsilon^2\begin{bmatrix}\alpha_2\\\beta_2\\\gamma_2\end{bmatrix}+\cdots\cdots \tag{12}$$

$$\Omega=\delta+\varepsilon\lambda_1+\varepsilon^2\lambda_2+\cdots\cdots$$

where δ is the imaginary part of λ. Substituting Eq. (12) into Eq. (10) and equating equal powers of ε, it obtain

$$\delta\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}u_n\\v_n\\w_n\end{bmatrix}-L_c\begin{bmatrix}u_n\\v_n\\w_n\end{bmatrix}=-\sum_{k=1}^{n-1}\lambda_k\begin{bmatrix}u_{n-k}\\v_{n-k}\\w_{n-k}\end{bmatrix}+\begin{bmatrix}g_n\\h_n\\j_n\end{bmatrix}$$

$(n\geqslant 1)$ (13)

The first few coefficients are

$g_1=0, h_1=0, j_1=0$;

$g_2=0, h_2=-\alpha_1 w_1-u_1\gamma_1-u_1w_1$,

$j_2=-u_1\beta_1+\alpha_1 v_1-u_1v_1$; ……

For $n=1$, according to expression (13), it gets

$$\delta\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}-L_c\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}=0 \tag{14}$$

Eq. (14) has the same form as the linearized equation around the bifurcation. Thus, $\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}$ is proportional to the eigenvector of L_c with the eigenvalue λ, then obtain

$$\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}=\begin{bmatrix}1\\c_2^+\\c_3^+\end{bmatrix}e^{i\tau} \tag{15}$$

Substituting Eq. (15) into Eq. (14), it gets

$$i\delta\begin{bmatrix}1\\c_2^+\\c_3^+\end{bmatrix}=\begin{bmatrix}-\sigma & \sigma & 0\\ \mu_c-z_{0c} & -1 & -x_{0c}\\ y_{0c} & x_{0c} & -b\end{bmatrix}\begin{bmatrix}1\\c_2^+\\c_3^+\end{bmatrix} \tag{16}$$

and the solution is as follows

$$\begin{cases}c_2^+=Pe^{i\theta}\\ c_3^+=Qe^{i\zeta}\end{cases} \tag{17}$$

The real solution of $\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}$ now reads

$$\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}=\begin{bmatrix}\cos\Omega t\\ P\cos(\Omega t+\theta)\\ Q\cos(\Omega t+\zeta)\end{bmatrix} \tag{18}$$

For $n=2$, according to Eq. (13), it gets

$$\delta\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}-L_c\begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}=-\lambda_1\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}+\begin{bmatrix}g_2\\h_2\\j_2\end{bmatrix} \tag{19}$$

Eq. (19) is an inhomogeneous linear equation. Fredholm alternative[18] requires that the right-hand side of the equation to be orthogonal to the vectors $\begin{bmatrix}u^*\\v^*\\w^*\end{bmatrix}$, that is

$$\int_0^{2\pi}[u^*\ v^*\ w^*](-\lambda_1\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}+\begin{bmatrix}g_2\\h_2\\j_2\end{bmatrix})\mathrm{d}\tau=0 \tag{20}$$

$\begin{bmatrix}u^*\\v^*\\w^*\end{bmatrix}$ is the eigenvector of L_c^*, which is the adjoin operator of L_c. According to expression (20), that is

$$\begin{cases}\alpha_1=0, \beta_1=0, \gamma_1=0\\ \lambda_1=0\end{cases} \tag{21}$$

Meanwhile, expanding u_2, v_2, w_2 in Fourier series, that is

$$\begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}=\sum_{m=-\infty}^{\infty}\begin{bmatrix}K_m\\N_m\\O_m\end{bmatrix}e^{im\tau} \tag{22}$$

Substituting Eq. (22) into Eq. (19), that is

$$im\delta\sum_{m=-\infty}^{\infty}\begin{bmatrix}K_m\\N_m\\O_m\end{bmatrix}e^{im\tau}-L_c\sum_{m=-\infty}^{\infty}\begin{bmatrix}K_m\\N_m\\O_m\end{bmatrix}e^{im\tau}=\begin{bmatrix}g_2\\h_2\\j_2\end{bmatrix} \tag{23}$$

The coefficients are

$g_2=0$

$$h_2 = -Q\cos\tau\cos(\tau+\zeta) = -\frac{1}{2}Q[\cos\zeta+\frac{1}{2}(e^{i(\zeta+2\tau)}+e^{-i(\zeta+2\tau)})]$$

$$j_2 = -\frac{1}{2}P[\cos\zeta+\frac{1}{2}(e^{i(\zeta+2\tau)}+e^{-i(\zeta+2\tau)})]$$

Comparing the coefficients of the equal powers of $e^{im\tau}$, it obtains a set of relations

when $m\neq 0, \pm 2$

$$\begin{bmatrix} K_m \\ N_m \\ O_m \end{bmatrix} = 0 \tag{24}$$

when $m=0$

$$-\begin{bmatrix} -\sigma & \sigma & 0 \\ \mu_c - z_{0c} & -1 & x_{0c} \\ y_{0c} & x_{0c} & -b \end{bmatrix} \begin{bmatrix} K_0 \\ N_0 \\ O_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}Q\cos\zeta \\ -\frac{1}{2}P\cos\zeta \end{bmatrix} \tag{25}$$

that is

$$\begin{cases} \sigma K_0 - \sigma N_0 = 0 \\ -(\mu_c - z_{0c})K_0 + N_0 - x_{0c}O_0 = -\frac{1}{2}Q\cos\zeta \\ -y_{0c}K_0 - x_{0c}N_0 + bO_0 = -\frac{1}{2}P\cos\zeta \end{cases} \tag{26}$$

and it gets

$$\begin{cases} K_0 = \dfrac{\frac{1}{2}\cos\zeta(bQ + x_{0c}P)}{x_{0c}(y_{0c}+x_{0c}) + b(\mu_c - z_{0c} - 1)} \\ N_0 = \dfrac{\frac{1}{2}\cos\zeta(bQ + x_{0c}P)}{x_{0c}(y_{0c}+x_{0c}) + b(\mu_c - z_{0c} - 1)} \\ O_0 = \dfrac{1}{2b}\cos\zeta[-P + \dfrac{(bQ + x_{0c}P)(y_{0c}+x_{0c})}{x_{0c}(y_{0c}+x_{0c}) + b(\mu_c - z_{0c} - 1)}] \end{cases} \tag{27}$$

when $m=\pm 2$,

$$i2\delta\begin{bmatrix} K_2 \\ N_2 \\ O_2 \end{bmatrix} - \begin{bmatrix} -\sigma & \sigma & 0 \\ \mu_c - z_{0c} & -1 & x_{0c} \\ y_{0c} & x_{0c} & -b \end{bmatrix} \begin{bmatrix} K_2 \\ N_2 \\ O_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4}Qe^{i\zeta} \\ -\frac{1}{4}Pe^{i\zeta} \end{bmatrix} \tag{28}$$

$$-i2\delta\begin{bmatrix} K_{-2} \\ N_{-2} \\ O_{-2} \end{bmatrix} - \begin{bmatrix} -\sigma & \sigma & 0 \\ \mu_c - z_{0c} & -1 & x_{0c} \\ y_{0c} & x_{0c} & -b \end{bmatrix} \begin{bmatrix} K_{-2} \\ N_{-2} \\ O_{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4}Qe^{-i\zeta} \\ -\frac{1}{4}Pe^{-i\zeta} \end{bmatrix} \tag{29}$$

According to Eq. (28) and Eq. (29), it can get the expression of $\begin{bmatrix} K_2 \\ N_2 \\ O_2 \end{bmatrix}$ and $\begin{bmatrix} K_{-2} \\ N_{-2} \\ O_{-2} \end{bmatrix}$, therefore, the real solution of $\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix}$ is

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} K_0 + K\cos(2\Omega t + \phi) \\ N_0 + N\cos(2\Omega t + \rho) \\ O_0 + O\cos(2\Omega t + \eta) \end{bmatrix} \tag{30}$$

Thus, the final analytic solution of Haken-Lorenz equations reads

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = (\frac{x_0 - x_{0c}}{\alpha_2})^{1/2} \begin{bmatrix} \cos\Omega t \\ P\cos(\Omega t + \theta) \\ Q\cos(\Omega t + \zeta) \end{bmatrix} + (\frac{x_0 - x_{0c}}{\alpha_2}) \begin{bmatrix} K_0 + K\cos(2\Omega t + \phi) \\ N_0 + N\cos(2\Omega t + \rho) \\ O_0 + O\cos(2\Omega t + \eta) \end{bmatrix} + \cdots \tag{31}$$

The coefficients P, Q, K_0, K, N_0, N, θ, $\zeta\cdots$ are explicit functions of the given parameters. The periodic solution of the Haken-Lorenz system and the time series of $x(t)$ are shown in Fig. (1) and (2) with $\sigma = 10$, $b = 8/3$, $\mu_c = 24.74$. They are accordant with theoretical analysis.

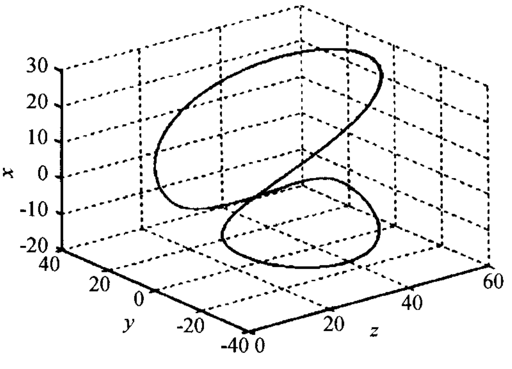

Fig. 1 The periodic solution of the Haken-Lorenz system

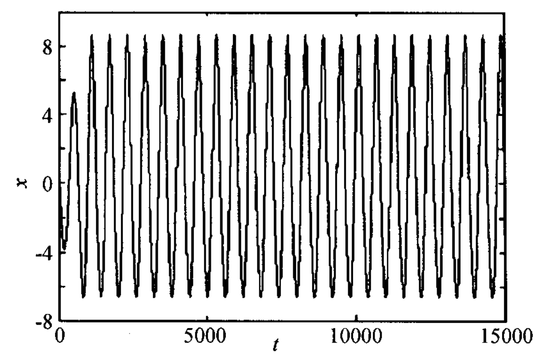

Fig. 2 The time series

3 Conclusion

In this paper dynamical behaviours of a single-mode laser Haken-Lorenz system were investigated. The steady state solution of the system was obtained, stability analysis was given by using the principle of linearzed stability and characteristic equation was gotten. And Hopf bifurcation point μ_c was determined . A periodic oscillation analytic solution for the system at bifurcation point was obtained by using the method of series. The simulation results show that when the bifurcation parameter μ crosses the critical value μ_c, Haken-Lorenz system gives rise to limit cycle, i. e. periodic oscillation solution and it is accordant with the analytic one.

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单模激光 Haken-Lorenz 系统的振荡解析解

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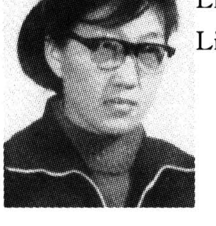
摘 要 研究了单模激光 Haken-Lorenz 系统在 Hopf 分歧点处的动力学行为.求出了 Haken-Lorenz 系统的定态解,采用线性稳定性原理对定态解进行了稳定性分析,获得了本征值方程,进而确定了系统的 Hopf 分歧点 μ_c.利用级数法求出了系统在分歧点处的时间周期振荡解的解析表达式.通过计算机对系统分歧点处的动力学行为进行了数值模拟,结果表明,系统在分歧点处存在一个极限环,即时间周期振荡解.与理论分析的解析结果相一致.

关键词 非线性光学;Haken-Lorenz 系统;Hopf 分歧;周期振荡;解析解



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