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Analyzing the Axial Intensity of Plane Waves Diffracted Nonparaxially by a Small Circular Aperture*

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Abstract A simply and analytical formula of the axial light intensity distribution behind a circular aperture is derived by using the Helmhotz-Kirchhoff integral theorem and the Kirchhoff's boundary conditions. It is studied the nonparaxial on-axis intensity distribution throughout the whole space behind a circular aperture. An accurate formula to calculate the Fresnel number of circular aperture is presented and the validity of usual Fresnel number formula is reexamined. By using the analytical formula and diffraction integral formula, some numerical simulation comparisons are done, and it is shown that the results of the two methods are completely coincident.

Keywords Scalar diffraction; Nonparaxial intensity distribution; Circular aperture; Fresnel number

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0 Introduction

Diffraction of circular aperture is an elementary diffraction phenomenon. When the diffraction distance is smaller than the diameter of aperture or when the radius is small to certain range, the diffraction beam no longer is the paraxial one and the paraxial approximation is invalid. For these cases a rigorous nonparaxial theory is necessary. The finite-difference time-domain method is employed to simulate the nonparaxial field distribution of circular aperture[1]. But this method is very complex and quite time consuming. Recently, Zhou et al[2,3] studied the axial intensity distribution of small circular aperture by using the angular of spectrum of plane waves under the condition of nonparaxial approximation. Wang et al[4] used the vector diffraction theory to investigate the design of circular pupil filters. However, the expressions in the Ref. [2～4] above are in an integrated form but not in an analytical one. The diffraction integration, especially for large diffraction distance, is quite time consuming. Ref. [5] use a complex Gaussian expansion for a hard-edge aperture function and the stationary phase method to the propagation of elliptical Gaussian beams diffracted by a circular aperture. In this paper, a simply analytical formula for the nonparaxial axial intensity distribution throughout the whole space behind a circular aperture is obtained and a new formula to calculate the Fresnel number of circular aperture is presented, which are based on the Helmhotz-Kirchhoff integral and Kirchhoff's boundary conditions.

1 Analytical expression for axial intensity distribution

It is known that one form of the Helmhotz-Kirchhoff integral theorem is expressed as[6]

$$U_2 = \frac{1}{4\pi}\iint_{\Sigma}\left\{U_1 \frac{\partial}{\partial n}\left[\frac{\exp(ikl)}{l}\right] - \frac{\exp(ikl)}{l}\frac{\partial U_1}{\partial n}\right\}\mathrm{d}S \tag{1}$$

where U_1 represents the complex amplitude emerging from the diffracting aperture Σ, U_2 at the observation plane, and l is the distance from an arbitrary point in the diffracting aperture to an arbitrary point in the observation plane as illustrated in Fig. 1. When the aperture is illuminated normally by the monochromic planar wave, under the Kirchhoff's boundary conditions, the complex amplitude in the observation plane can

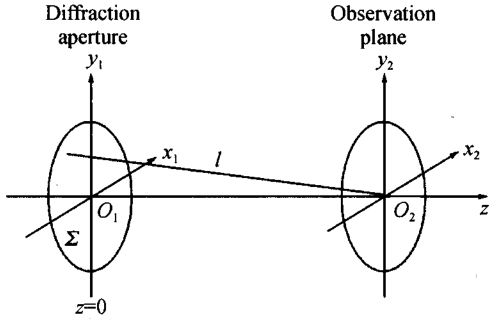

Fig. 1 Geometric relationship between the diffracting aperture and the observation space

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be written as

$$U_2(r_2;z)=\frac{A}{2\pi}\iint_{\Sigma}t(r_1)\frac{\exp(ikl)}{l}\left[ik-\frac{1}{l}\right]\cdot \cos(n,l)r_1\mathrm{d}r_1\mathrm{d}\theta \quad (2)$$

where A is the illuminated amplitude, $t(r_1)$ is the circular aperture function, and the cosine function is the incline factor. Setting $r_2=0$ in Eq. (2), the incline factor will be $\cos(n,l)=z/l$, then the axial diffracting complex amplitude is given by

$$U_2(z)=\frac{A}{2\pi}\int_0^R\int_0^{2\pi}\frac{\exp(ikl)}{l}\left[ik-\frac{1}{l}\right]\cdot\frac{z}{l}r_1\mathrm{d}r_1\mathrm{d}\theta= A\int_0^R\frac{\exp(ikl)}{l}\left[ik-\frac{1}{l}\right]\cdot\frac{z}{l}r_1\mathrm{d}r_1 \quad (3)$$

The change of integral variable (from r_1 to l) also affects the limits of integration; hence

$$U_2(z)=iAkz\int_z^{z\sqrt{a}}\frac{\exp(ikl)}{l}\mathrm{d}l-Az\int_z^{z\sqrt{a}}\frac{\exp(ikl)}{l^2}\mathrm{d}l \quad (4)$$

where $a=1+R^2/z^2$. By inserting the following results of two integrals[7]

$$ik\int_a^b\frac{\exp(ikl)}{l}\mathrm{d}l=\frac{\exp(ikl)}{l}\sum_{m=0}^{\infty}\frac{m!}{(ikl)^m}\Bigg|_a^b \quad (5)$$

$$\int_a^b\frac{\exp(ikl)}{l}\mathrm{d}l=\frac{\exp(ikl)}{l}\sum_{m=0}^{\infty}\frac{m!}{(ikl)^m}\Bigg|_a^b \quad (6)$$

into Eq. (4), and after simply calculating, it can obtained

$$U_2(z)=A\left[\frac{\exp(ikz\sqrt{a})}{\sqrt{a}}-\exp(ikz)\right] \quad (7)$$

Thus the axial intensity distribution throughout the whole space behind a circular aperture can be written as

$$I(z)=|U_2(z)|^2=I_0\left\{\frac{a+1}{a}-\frac{2}{\sqrt{a}}\cdot\cos\left[kz(\sqrt{a}-1)\right]\right\} \quad (8)$$

where $I_0=|A|^2$ is the irradiance incident upon the aperture. No any approximations are made in the above calculations; hence the simple expression of Eq. (8) is valid for all values of $z\gg\lambda$. When $z=0$, $I=I_0$, as expected.

2 Numerical simulation and comparison

According to the result in Ref. 3, the axial complex amplitude is

$$U_2(z)=kR\int_0^{\infty}J_1(k\alpha R)\exp(ik\sqrt{1-\alpha^2}z)\mathrm{d}\alpha \quad (9)$$

where J_1 is the first-order Bessel function. The axial intensity reads as

$$I(z)=\left|kR\int_0^{\infty}J_1(k\alpha R)\exp(ik\sqrt{1-\alpha^2}z)\mathrm{d}\alpha\right|^2 \quad (10)$$

Fig. 2 shows a log-log plot of the normalized axial intensity distributions (I/I_0) throughout the whole space behind a circular aperture. N is the Fresnel number. It is assumed that a circular aperture of $R=10\ \mu$m is illuminated by a unit amplitude normally incident plane wave of $\lambda=0.6328\ \mu$m. The log-log plot is illustrated with the axial position expressed in terms of aperture radius. From the Fig. 2, it is found that the dashed curve obtained by using analytical formula coincides exactly with the solid curve found by using the numerical diffraction integral calculation. But it should be pointed out that the numerical calculation efficiency using the analytical formula is over 1000 times that using the numerical integral calculation directly.

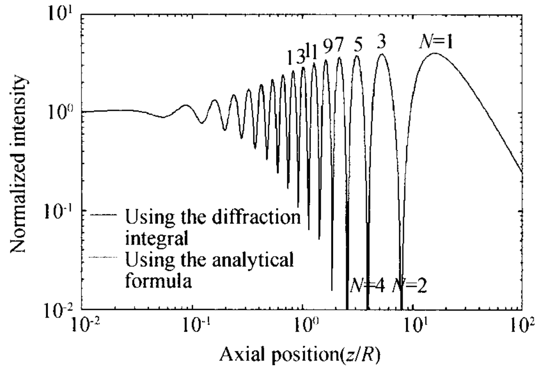

Fig. 2 Axial intensity distribution behind a circular aperture

The oscillations in Fig. 2 that are due to the interference between the edge-diffracted wave and the direct illumination, and its modulation increases throughout the near field, asymptotically approaching a maximum peak value of four times the incident irradiance. After reaching the far field or the Fraunhofer region, the axial intensity decreases steadily.

3 Fresnel number of circular aperture

The Fresnel number is an important parameter for the design and analysis of laser cavities and resonators[8]. To obtain the relationship between the axial position of maximum or minimum intensity and the Fresnel number, one should obtain the roots of the equation

$$\partial I(z)/\partial z=0 \quad (11)$$

However, in many cases of interest an exact analytical result cannot be obtained by this method. From Eq. (8) it is noted that the oscillation of the axial intensity is mainly caused from the cosine factor. Starting with this cosine factor and after some calculations, obtain a simply analytical relationship between the axial position of maximum or minimum intensity and the Fresnel number N, at any distances from a circular aperture, giving

$$z=\frac{1}{N\lambda}\left[R^2-\left(\frac{N\lambda}{2}\right)^2\right] \qquad (12)$$

At large distance from a circular aperture, i. e., throughout the entire Fraunhofer region, the Fresnel number is smaller less than unity. Moving closer to the aperture, passing from the Fraunhofer region into the Fresnel region, the axial intensity increases and reached a maximum at a Fresnel number of exactly unity. A further reduction in z results in the aperture to consist of more than one Fresnel zone, and destructive interference diminish the axial intensity. It reaches a minimum at the axial position which corresponds to precisely two Fresnel zones. The oscillatory characteristic in Fig. 2 is thus completely consistent with a qualitative discussion of constructive (destructive) interference of an odd (even) number of Fresnel zones. Fig. 2 illustrates only a portion of the above irradiance distribution with a few discrete Fresnel numbers indicated by N.

We note that when $2R\gg N\lambda$, Eq. (12) reduces to

$$z=R^2/N\lambda \qquad (13)$$

which is the well-known result in the paraxial approximation[6,8].

Fig. 3 plots the axial positions as a function of the Fresnel number for a circular aperture of $R=5\lambda$, where the solid curve is the exact result by using Eq. (11), the dashed curve is the case by using Eq. (12), and the dotted curve is the case by using Eq. (13). It is seen that the dashed curve is basically coincident with the solid one and the dotted one produces, especially for large Fresnel number, an obvious departure from the solid one. Therefore, Eq. (12) is valid for any axial distance from the circular aperture.

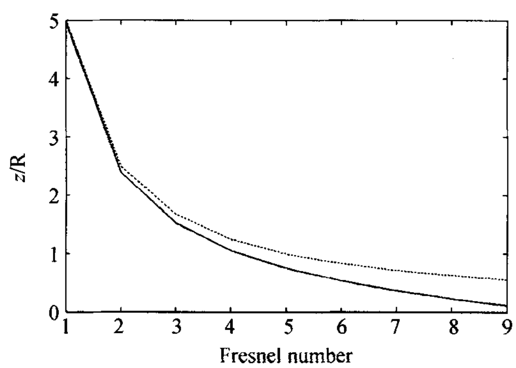

Fig. 3 Axial position as a function of the Fresnel number for $R=5\lambda$

4 Conclusions

This paper derived a simple and accurate formula to study axial intensity distribution of circular aperture by using the Helmhotz-Kirchhoff integral theorem and the Kirchhoff's boundary conditions. Since no limit of paraxial condition, it becomes possible to study the nonparaxial propagation of plane wave passing through a circular aperture by the use of the analytical formula. In addition, an accurate formula to calculate the Fresnel number of circular aperture illuminated by plane wave and compare it with the usual formula of Fresnel number was obtained. It is found that the usual formula of Fresnel number is invalid for small aperture or in the near field region.

It needs to be indicated that the presented analytical results are based on the Kirchhoff's boundary conditions. Recall that the Kirchhoff's boundary conditions are valid only for $z\gg\lambda$. If one considers a factor 10 to qualify as much greater, the calculated values for the intensity might be suspect for axial position less that 6.328 μm ($\lambda=0.6328$ μm). On the other hand, if a factor of 5 qualifies as being much greater, then only those calculated values for axial position less than 3.164 μm are suspect[9]; however, the value of unity at the extreme axial position of zero is of course correct. Hence the range of suspect values is small.

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平面波经小圆孔非傍轴衍射的轴上光强解析分析

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摘 要 用亥姆霍兹-基尔霍夫积分定理和基尔霍夫边界条件,推导出了平面波经小圆孔非傍轴衍射时轴上强度的简单解析表达式,研究了平面波经小圆孔后整个衍射空间非傍轴的轴上光强分布.给出了计算圆孔菲涅尔数的精确公式,重新检查了通常的菲涅尔数公式的有效性.数值计算显示,应用解析表达式所得的结果与应用衍射积分公式所得的结果完全一致.

关键词 标量衍射;非傍轴光强分布;圆形孔径;菲涅尔数

Zhang Yaoju was born in Dengzhou city of Henan Province in July of 1960. He graduated and received his M. S. degree from Department of Physics, Henan Normal University, Xingxiang, China, in June, 1990. His interesting research area is diffractive optics and binary optics, microscopy, optical data storage, and laser applications. He obtained the title of a professor in 2004. For his research achievements, he has published over 40 scientific papers on scientific journal where 19 papers are indexed by SCI and EI.