

AB 系统在椭圆函数背景下的多呼吸子

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摘要 呼吸子泛指一类具有周期演化或周期分布结构的非线性波,它已成为非线性光学系统的研究热点之一。研究多种呼吸子的时空动力学特征具有实际的物理意义。本文基于孤子理论中的 Darboux 变换方法构造出 AB 系统在椭圆函数背景下的多呼吸子解,并进一步分析了呼吸子的非线性动力学特征。首先,基于 AB 系统的 Lax 对,通过修正的平方波函数法与行波变换法获得该系统的椭圆函数周期解以及 Lax 对的基本解。然后,通过椭圆函数积分公式和 Darboux 变换迭代算法,推导出 dn 和 cn 两种椭圆函数背景下的一阶和二阶呼吸子解的解析表达式。另外,讨论了三种呼吸子的动力学特征,主要包括一般呼吸子(GB)、Kuznetsov-Ma 呼吸子(KMB)和 Akhmediev 呼吸子(AB)。最后,研究了在 dn 背景下 GB 和 KMB 相互作用的时空结构以及在 cn 背景下两个 GB 相互作用的时空结构。本文研究结果将有助于理解在非线性光学领域中周期背景下的呼吸子动力学特征和相互作用行为。

关键词 非线性光学; 呼吸子; Darboux 变换; AB 系统

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1 引言

众所周知,非线性演化方程可以用来描述许多自然科学和工程技术领域中的非线性物理现象。因此,寻求其精确解在非线性科学中具有十分重要的意义。非线性演化方程的非线性局域解主要包括孤子、呼吸子和怪波。近年来,呼吸子和怪波的动力学行为引起了人们的广泛关注^[1]。呼吸子动力学的研究已成为非线性物理学科中的一个重要课题。大量的实验证实,有许多新的非线性物理现象与呼吸子有关^[2]。例如,通过控制激光脉冲在各种非线性介质中的传播来激发呼吸子^[3],从而探究呼吸子的动力学性质。就目前的非线性物理系统而言,非线性光学是实验科学上发展成熟的非线性实验平台^[4-6]。例如,文献^[7]基于非线性薛定谔(NLS)方程对周期背景下的几种典型呼吸子的相关理论和实验研究进行了论述,展示了其时域和频域空间中的动力学演化,报道了在非线性和光学系统中基本呼吸子的相互作用。因此,对呼吸子的理论研究在很大程度上推动了对呼吸子的实验探索,进一步可以提高呼吸子在实际物理应用方面的可能性。

呼吸子是一类特殊的非线性局域波解,在时空环境中显示出有规律的周期性振荡现象。1977 年, Kuznetsov^[8]在等离子体中发现了一种特殊的孤子解,其空间分布呈现出显著的局域性,而在时间维度上则

展现出周期性特点。这种波的截面图类似于正弦波形,波峰波谷交替出现,呈现出典型的周期性振荡行为。于是,这种特殊的孤子解被称为呼吸子。几乎同时, Ma^[9]在含立方项的 NLS 方程中再次发现具有这一特性的呼吸子解,即 Kuznetsov-Ma 呼吸子(KMB)。1986 年, Akhmediev 等^[10]在研究 NLS 方程时找到了一类与 KMB 时空完全相反的呼吸子解,这类呼吸子在空间分布上呈现周期性振荡行为,而沿着时间方向表现出典型的局域性,这类特殊的呼吸子解现在被称为 Akhmediev 呼吸子(AB)。与多孤子解的构造类似,多呼吸子解同样可以通过单呼吸子来构造。此外,1988 年, Its 等^[11]利用 Darboux 变换方法发现了 NLS 方程的一般呼吸子(GB)。自此,呼吸子可以被分为三大类,即 KMB、AB 和 GB。

本文对呼吸子的研究主要是针对 AB 系统展开。AB 系统是描述地球流体中的不稳定斜压波包和非线性光学中尺度重力流的基本可积模型^[12]。AB 系统存在于物理系统中,由于其色散关系的频率由两个复共轭值组成且控制在一定参数范围内,使其线性波具有不稳定性。这样的线性波会不断增长,直到非线性效应发生^[13]。AB 系统属于可积方程组,其孤子解可以用逆散射变换方法进行研究。然而,有些应用不仅需要解方程的孤子解,还需要解其周期解,本文便是在周期解的基础上展开研究。

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首先,本文借助文献[14]中求解 NLS 方程周期解的平方波函数法和行波变换法,推出 AB 系统的椭圆函数周期解;然后,借助修正的平方波函数法^[14]获得该系统 Lax 对的基本解;接下来,利用 Darboux 变换法,在椭圆函数背景下构造多呼吸子,同时分析呼吸子的非线性动力学特征。另外,通过研究两个呼吸子的相互作用,发现呼吸子间的碰撞是弹性的,即呼吸子在传播过程中并未发生形状改变。这一发现对于理解呼吸子的传播特性具有重要意义,并进一步阐述了其可以在不改变形状的情况下完成长距离传输的特性。这些理论研究结果有助于探索在非线性光学中呼吸子的实际物理意义和应用。

2 椭圆函数周期解

AB 系统为

$$\begin{cases} Q_{xt} = QS, \\ S_x = -\frac{1}{2}(|Q|^2)_t, \end{cases} \quad (1)$$

式中: x 和 t 为半特征坐标; Q 和 S 为服从归一化条件的波振幅,可进一步转化为

$$|Q_t|^2 + S^2 = 1. \quad (2)$$

AB 系统具有如下形式的 Lax 对:

$$\begin{cases} \Phi_x = U\Phi = \begin{bmatrix} F & G \\ H & -F \end{bmatrix} \Phi = \begin{bmatrix} -i\lambda & \frac{Q}{2} \\ -\frac{Q^*}{2} & i\lambda \end{bmatrix} \Phi \\ \Phi_t = V\Phi = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \Phi = \begin{bmatrix} -\frac{S}{4i\lambda} & \frac{Q_t}{4i\lambda} \\ \frac{Q_t^*}{4i\lambda} & \frac{S}{4i\lambda} \end{bmatrix} \Phi \end{cases}, \quad (3)$$

式中: $\Phi = (\psi, \phi)^T$ (上标 T 表示向量转置)是向量特征函数; λ 是任意的谱参数; Q^* 为 Q 的复共轭。由相容性条件 $\Phi_{xt} = \Phi_{tx}$ 或者零曲率方程 $U_t - V_x + (UV - VU) = 0$ 均可以推出方程式(1)。

众所周知,Lax 对的解依赖于给定的势 Q 和谱参数 λ 。假设 $\Phi_i = (\psi_i, \phi_i)^T$ ($i = 1, 2$)是 Lax 对式(3)的两个基本解,建立其平方波函数如下:

$$f = -\frac{1}{2}i(\psi_1\phi_2 + \phi_1\psi_2), \quad g = \psi_1\psi_2, \quad h = -\phi_1\phi_2. \quad (4)$$

结合式(4)中 f 、 g 和 h 的定义以及式(1)的 Lax 对,可得到关于变量 x 与 t 的导数方程组:

$$\begin{cases} \frac{\partial f}{\partial x} = -iHg + iGh, & \frac{\partial f}{\partial t} = -iCg + iBh \\ \frac{\partial g}{\partial x} = 2iGf + 2Fg, & \frac{\partial g}{\partial t} = 2iBf + 2Ag \\ \frac{\partial h}{\partial x} = -2iHf - 2Fh, & \frac{\partial h}{\partial t} = -2iCf - 2Ah \end{cases}. \quad (5)$$

由于矩阵 U 和 V 具有零迹,易证明 Wronskian 为

$\psi_1\phi_2 - \psi_2\phi_1$ 是独立于 x 和 t 存在的。显然, $f^2 - gh = -\frac{1}{4}(\psi_1\phi_2 - \psi_2\phi_1)^2$ 是一个仅依赖于 λ 的运动积分。为了得到 Lax 对在周期背景下的周期解,考虑了椭圆 Riemann 曲面^[15-17]上最简单的非平凡单相周期解:

$$P(\lambda) = f^2 - gh = \prod_{i=1}^4 (\lambda - \lambda_i) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4, \quad (6)$$

式中, λ_i 是 $P(\lambda)$ 的零点,取为

$$\lambda_1 = r + ic, \quad \lambda_2 = b + id, \quad \lambda_3 = r - ic, \quad \lambda_4 = b - id, \quad (7)$$

式中, r 、 b 、 c 、 d 都为正实数。在满足式(5)的条件下,取平方波函数如下:

$$f = \lambda^2 - f_1\lambda + aS, \quad g = \frac{Q}{2}(\lambda - \mu), \quad h = -\frac{Q^*}{2}(\lambda - \mu^*), \quad (8)$$

式中: μ 为辅助谱点,是待确定 x 和 t 的函数; f_1 和 a 为常数。为了方便起见,定义变量 ν 为

$$\nu = \frac{1}{4}|Q|^2. \quad (9)$$

将式(8)代入到式(6)与式(6)中 λ 的各阶系数进行对比,从而得到

$$s_1 = 2f_1, \quad s_2 = f_1^2 + 2aS + \nu, \quad s_3 = 2af_1S + \nu(\mu + \mu^*), \quad s_4 = a^2S^2 + \nu\mu\mu^*.$$

由上式易得

$$f_1 = \frac{s_1}{2} = \frac{1}{2} \sum_{i=1}^4 \lambda_i, \quad (10)$$

$$a^2 = s_4 = \prod_{i=1}^4 \lambda_i. \quad (11)$$

结合式(5)以及式(8)求导的结果,对比 λ 各阶系数的关系可以得到

$$\mu = -2ia \frac{Q_t}{Q}, \quad (12)$$

$$Q_x = -2i \left(\frac{s_1 Q}{2} - Q\mu \right), \quad Q_t = \frac{iQ\mu}{2a}. \quad (13)$$

在式(8)式中取 $\lambda = \mu$,则可以进一步推出

$$\begin{aligned} \frac{\partial \mu}{\partial x} &= -2if(\mu) = -2i\sqrt{p(\mu)}, \\ \frac{\partial \mu}{\partial t} &= \frac{-if(\mu)}{2a} = \frac{1}{4a} \frac{\partial \mu}{\partial x}. \end{aligned} \quad (14)$$

由式(13)~(14)可知, μ 只依赖于相位 $\xi = x + \frac{t}{4a}$,则

$$\frac{\partial \mu}{\partial \xi} = -2i\sqrt{p(\mu)}. \quad (15)$$

由行波变换法,结合式(4)、式(5)及式(12)进一步可以推出

$$Q = \exp\left(\frac{if_1 t}{2a}\right) \tilde{Q}, \quad (16)$$

式中, \tilde{Q} 满足:

$$\frac{d\tilde{Q}}{d\xi} = 2i\left(\mu - \frac{s_1}{2}\right)\tilde{Q}. \quad (17)$$

再由 $s_3 = 2af_1S + \nu(\mu + \mu^*)$, $s_4 = a^2S^2 + \nu\mu\mu^*$ 进行联立方程组, 可以推出

$$\mu = \frac{s_1^3 - 4s_1s_2 + 8s_3 + 4s_1\nu - 8\sqrt{R(\nu)}}{16\nu}, \quad (18)$$

式中, $R(\nu)$ 为

$$R(\nu) = \frac{1}{64}(s_1^3 - 4s_1s_2 + 8s_3 + 4s_1\nu)^2 - 4\nu\left(s_4 - \frac{1}{64}(s_1^2 - 4s_2 + 4\nu)^2\right).$$

$R(\nu)$ 的零点 ν_i 与 $P(\lambda)$ 的零点 λ_i 的关系为

$$\begin{aligned} \nu_1 &= -\frac{1}{4}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4) = -(2b)^2, \\ \nu_2 &= -\frac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4) = (c - d)^2, \\ \nu_3 &= -\frac{1}{4}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4) = (c + d)^2, \end{aligned}$$

显然有 $\nu_1 < \nu_2 < \nu_3$ 成立。

为了简便表示, 引入

$$\begin{aligned} q &= \frac{1}{2}s_1\left(s_2 - \frac{1}{4}s_1^2\right) - s_3, \quad p = s_2 - \frac{3}{8}s_1^2, \\ r &= s_4 + \frac{1}{16}s_1^2\left(s_2 - \frac{3}{16}s_1^2\right) - \frac{s_1s_3}{4}, \end{aligned}$$

则 $R(\nu)$ 简化为

$$R(\nu) = \nu^3 - 2p\nu^2 + (p^2 - 4r)\nu + q^2.$$

显然正变量 ν 可在 $\nu_2 \leq \nu \leq \nu_3$ 中振荡。进一步, μ 简化为

$$Q = \sqrt{\nu} \exp\left(\frac{is_1t}{4a} - iq \int_0^\xi \frac{1}{\nu} ds - \frac{is_1\xi}{2}\right) = 2 \frac{\sigma(2\xi + \omega' + \kappa)}{\sigma(2\xi + \omega')\sigma(\kappa)} \exp\left[\frac{is_1t}{4a} - \frac{is_1\xi}{2} + 4\zeta(\kappa)\xi + 2\zeta(\kappa)\omega'\right].$$

将式(21)中的 ν 退化为下面两种雅可比椭圆函数解:

1) $\nu_1 = 0$ 时, 即满足 $b = r$, 得到 dn 形式解:

$$\nu = k^2 \operatorname{dn}^2(k\xi; m), \quad (25)$$

式中: $k = c + d; m = \frac{4cd}{(c + d)^2}$.

2) $\nu_2 = 0$ 时, 即满足 $c = d$, 得到 cn 形式解:

$$\nu = k^2 \operatorname{cn}^2\left(\frac{k}{\sqrt{m}}\xi; m\right), \quad (26)$$

式中: $k = 2d; m = \frac{4d^2}{(b - r)^2 + 4d^2}$.

3 Lax 对的基本解

行波变换后原方程(1)转化为

$$\begin{cases} Q_{\xi t} = QS + \frac{Q_{\xi\xi}}{4\sqrt{s_4}} \\ S_{\xi} = -\frac{1}{2}\left[Q^*\left(Q_i - \frac{Q_{\xi}}{4\sqrt{s_4}}\right) + Q\left(Q_i^* - \frac{Q_{\xi}^*}{4\sqrt{s_4}}\right)\right]. \end{cases} \quad (27)$$

$$\mu = \frac{s}{2} - \frac{q + i\sqrt{-R(\nu)}}{2\nu}. \quad (19)$$

ν 对 ξ 的依赖关系如下所示:

$$\frac{d\nu}{d(2\xi)} = \sqrt{-R(\nu)}. \quad (20)$$

对式(20)进行积分运算得到

$$|Q^2| = 4\nu = 4\left[\nu_3 + (\nu_2 - \nu_3) \operatorname{sn}^2\left(\sqrt{\nu_3 - \nu_1} W, m\right)\right], \quad (21)$$

式中, $m = (\nu_3 - \nu_2)/(\nu_3 - \nu_1)$.

接下来将 Q 转化为 Weierstrass 椭圆函数形式。

式(9)中的 ν 转化为

$$\nu = \frac{2p}{3} - 4\varphi(2\xi + \omega'), \quad (22)$$

式中, ω' 为 φ 函数的半周期。将式(22)和式(20)代入到式(19)中, 则式(19)转化为关于 ξ 的函数:

$$\mu = \frac{s_1}{4} + \frac{q}{8\left[\varphi(2\xi + \omega') - \frac{p}{6}\right]} - \frac{i}{2\nu} \frac{d\nu}{d(2\xi)}. \quad (23)$$

将式(23)代入式(17)中进行积分求解得到

$$\tilde{Q} = \sqrt{\nu} \exp\left[-\frac{is_1\xi}{2} + \frac{iq}{8} \int_0^\xi \frac{d(2\xi)}{\varphi(2\xi + \omega') - \frac{p}{6}}\right]. \quad (24)$$

取 $\varphi(\kappa) = \frac{p}{6}$, 将式(24)代入式(16)中, 由积分运算, 可以得到以下势函数的表现形式:

对应的 Lax 对转化为

$$\begin{cases} \Phi_{\xi} = U\Phi \\ \Phi_t = \left(V + \frac{1}{4\sqrt{s_4}}U\right)\Phi. \end{cases} \quad (28)$$

下面运用修正平方波函数法, 在周期解式(25)与式(26)背景下构造出 Lax 对的基本解。首先定义 $r_i = \phi_i/\psi_i (i = 1, 2)$, 再将式(4)中的 f 改写为

$$f = \frac{i}{2}\left(\frac{h}{r_j} - gr_j\right).$$

结合方程式(4)推出

$$\begin{aligned} r_1 &= i\left[\frac{f + \sqrt{P(\lambda)}}{g}\right] = i\left[\frac{h}{f - \sqrt{P(\lambda)}}\right], \\ r_2 &= i\left[\frac{f - \sqrt{P(\lambda)}}{g}\right] = i\left[\frac{h}{f + \sqrt{P(\lambda)}}\right]. \end{aligned}$$

结合式(28)易得

$$(\ln \phi_1)_\xi = F + Gr_1, (\ln \phi_1)_\xi = H \frac{1}{r_1} - F, \quad (29)$$

$$(\ln \phi_1)_t = A + Br_1, (\ln \phi_1)_t = C \frac{1}{r_1} - A. \quad (30)$$

将式(8)代入变量 r_1 与 r_2 中, 结合式(29), 易得

$$(\ln \phi_1)_\xi = i\lambda + i \frac{C_1}{\nu(\xi) - 2\alpha_1} + \frac{1}{2} \ln[\nu(\xi) - 2\alpha_1]_\xi - \frac{is_1}{2},$$

$$(\ln \phi_1)_\xi = -i\lambda - i \frac{C_2}{\nu(\xi) - 2\alpha_2} + \frac{1}{2} \ln[\nu(\xi) - 2\alpha_2]_\xi + \frac{is_1}{2},$$

上述式子中 $C_j = -\frac{1}{8}[s_1^3 - 4s_1s_2 + 8s_3 + 8a_i(s_1 - 4\lambda)]$

以及 $\alpha_j = \frac{1}{8}[-s_1^2 + 8\lambda^2 + 4s_2 - 4s_1\lambda \mp 8\sqrt{P(\lambda)}]$ (\pm 与 \mp 符号说明: 上边符号设置对应的是 $j=1$ 时的情形, 下边符号设置则对应 $j=2$ 时的情形)。接着, 对上式关于 ξ 积分一次, 得到

$$\ln \phi_1 = \left(i\lambda - \frac{is_1}{2} \right) \xi + i \int_0^\xi \frac{C_1 ds}{\nu(s) - 2\alpha_1} + \frac{1}{2} \ln[\nu(\xi) - 2\alpha_1] + D_1(t), \quad (31)$$

$$\ln \phi_1 = \left(-i\lambda + \frac{is_1}{2} \right) \xi - i \int_0^\xi \frac{C_2 ds}{\nu(s) - 2\alpha_2} + \frac{1}{2} \ln[\nu(\xi) - 2\alpha_2] + D_2(t), \quad (32)$$

式中, $D_1(t)$ 和 $D_2(t)$ 为待定函数。为此, 将式(13)和式(19)代入到式(30)中, 得到

$$D_1(t) = -\frac{i[-2\sqrt{P(\lambda)} + s_1\lambda]t}{8\sqrt{s_4}\lambda}, \quad (33)$$

$$D_2(t) = \frac{i[2\sqrt{P(\lambda)} + s_1\lambda]t}{8\sqrt{s_4}\lambda}. \quad (34)$$

基于式(31)~(34)的结果, 求得 ψ_1 与 ϕ_1 的表达式为

$$\psi_1(\xi, t; \lambda) = B_1 \sqrt{\nu(\xi) - 2\alpha_1} \exp \theta_1,$$

$$\phi_1(\xi, t; \lambda) = B_2 \sqrt{\nu(\xi) - 2\alpha_2} \exp \theta_2,$$

$$\theta_1 = i \int_0^\xi \frac{C_1 ds}{\nu(s) - 2\alpha_1} + \left(i\lambda - \frac{is_1}{2} \right) \xi - \frac{i[-2\sqrt{P(\lambda)} + s_1\lambda]t}{8\sqrt{s_4}\lambda},$$

$$\theta_2 = -i \int_0^\xi \frac{C_2 ds}{\nu(s) - 2\alpha_2} - \left(i\lambda - \frac{is_1}{2} \right) \xi + \frac{i[2\sqrt{P(\lambda)} + s_1\lambda]t}{8\sqrt{s_4}\lambda},$$

其中 B_j 表示待定的复常数。最后再结合前面的定义 $r_1 = \phi_1/\psi_1$, 有

$$r_1(0, 0; \lambda) = i \frac{\sqrt{f(0, 0; \lambda) + y} \sqrt{h(0, 0; \lambda)}}{\sqrt{f(0, 0; \lambda) - y} \sqrt{g(0, 0; \lambda)}} \pm \frac{\sqrt{\nu(0) - 2\alpha_2}}{\sqrt{\nu(0) - 2\alpha_1}} \equiv \frac{B_2}{B_1} \frac{\sqrt{\nu(0) - 2\alpha_2}}{\sqrt{\nu(0) - 2\alpha_1}}, \quad (35)$$

式中, $y = \sqrt{p(\lambda)}$ 。不失一般性, 考虑让 $B_1 = 1$ 以及

$B_2 = 1$ 。同理, 易得 AB 系统 Lax 对的另外一组解, 如下所示:

$$\psi_2(\xi, t; \lambda) = \sqrt{\nu(\xi) - 2\alpha_2} \exp(-\theta_2),$$

$$\phi_2(\xi, t; \lambda) = \sqrt{\nu(\xi) - 2\alpha_1} \exp(-\theta_1).$$

综上所述, 本文求解出带有椭圆函数 $v(\xi)$ 的 Lax 对的基本解为

$$\Psi(x, t; \lambda) = \begin{bmatrix} \sqrt{\nu(\xi) - 2\alpha_1} e^{\theta_1} & \sqrt{\nu(\xi) - 2\alpha_2} e^{-\theta_2} \\ \sqrt{\nu(\xi) - 2\alpha_2} e^{\theta_2} & \sqrt{\nu(\xi) - 2\alpha_1} e^{-\theta_1} \end{bmatrix}. \quad (36)$$

4 呼吸子解

接下来, 本文将构造 AB 系统在椭圆函数背景下的呼吸子解。首先, 考虑一个由 Lax 对两组基本解通过线性组合得到的通解:

$$\Psi(x, t; \lambda_i) = \begin{bmatrix} \psi_1(\xi, t; \lambda_i) \\ \phi_1(\xi, t; \lambda_i) \end{bmatrix} + c_i \begin{bmatrix} \psi_2(\xi, t; \lambda_i) \\ \phi_2(\xi, t; \lambda_i) \end{bmatrix} = \begin{bmatrix} \sqrt{\nu(\xi) - 2\alpha_1} e^{\theta_1} + c_i \sqrt{\nu(\xi) - 2\alpha_2} e^{-\theta_2} \\ \sqrt{\nu(\xi) - 2\alpha_2} e^{\theta_2} + c_i \sqrt{\nu(\xi) - 2\alpha_1} e^{-\theta_1} \end{bmatrix}, \quad (37)$$

式中, c_i 为任意常数。

4.1 Darboux 变换

本文借助 Darboux 变换方法构造新解, 其中 $\Psi_1 = \psi_1 + c_i \psi_2$, $\Psi_2 = \phi_1 + c_i \phi_2$ 是式(28)新 Lax 对的线性组合解, 根据文献[18-19], AB 系统的 n 阶 Darboux 变换如下:

$$\Psi[n] = T[n] \cdots T[1] \Psi, T[n] = \begin{matrix} \lambda I - H_{n-1} \Lambda_n H_{n-1}^{-1}, \\ Q[n] = Q + 4i \frac{M}{N}, \end{matrix}$$

其中 M, N 如下所示:

$$M = \begin{bmatrix} \lambda_1^{n-1} \psi_1 & -\lambda_1^n \psi_1 & \cdots & \psi_1 & \phi_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1} \psi_n & -\lambda_n^n \psi_n & \cdots & \psi_n & \phi_n \\ \lambda_1^{*n-1} \psi_1^* & -\lambda_1^{*n} \psi_1^* & \cdots & \psi_1^* & -\psi_1^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{*n-1} \psi_n^* & -\lambda_n^{*n} \psi_n^* & \cdots & \psi_n^* & -\psi_n^* \end{bmatrix},$$

$$N = \begin{bmatrix} \lambda_1^{n-1} \phi_1 & \lambda_1^{n-1} \phi_1 & \cdots & \psi_1 & \phi_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1} \phi_n & \lambda_n^{n-1} \phi_n & \cdots & \psi_n & \phi_n \\ \lambda_1^{*n-1} \phi_1^* & -\lambda_1^{*n-1} \phi_1^* & \cdots & \psi_1^* & -\psi_1^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{*n-1} \phi_n^* & -\lambda_n^{*n-1} \phi_n^* & \cdots & \psi_n^* & -\psi_n^* \end{bmatrix},$$

$$H_{l-1} = \begin{pmatrix} \psi_1[l-1] & \phi_1^*[l-1] \\ \psi_1[l-1] & -\phi_1^*[l-1] \end{pmatrix}, \Lambda_l = \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l^* \end{pmatrix},$$

ψ_i 和 $\phi_i (1 \leq i \leq n)$ 对应的是谱参数 $\lambda = \lambda_i$ 时的特解。

4.2 在 dn 和 cn 背景下的单呼吸子解

为了得到 dn 背景下的单呼吸子解, 将式(37)中的线性通解代入到 AB 系统的 Darboux 变换中。首先, 取

$i = 5, c_5 = 1$, 推导出较为紧凑的单呼吸子解的表达式为

$$Q[1] = Q - 4i(\lambda_5 - \lambda_5^2) \frac{\Psi_1 \Psi_2^*}{(|\Psi_1|^2 + |\Psi_2|^2)}, \quad (38)$$

式中, Q, Ψ_1 以及 Ψ_2 如下所示:

$$Q = \sqrt{\nu} \exp\left(\frac{is_1 t}{4a} - iq \int_0^\xi \frac{1}{\nu} ds - \frac{is_1 \xi}{2}\right),$$

$$\Psi_1 = \sqrt{\nu(\xi) - 2\alpha_1} e^{\theta_1} + \sqrt{\nu(\xi) - 2\alpha_2} e^{-\theta_2},$$

$$\Psi_2 = \sqrt{\nu(\xi) - 2\alpha_2} e^{\theta_2} + \sqrt{\nu(\xi) - 2\alpha_1} e^{-\theta_1},$$

$$\theta_1 = i \int_0^\xi \frac{C_1 ds}{\nu(s) - 2\alpha_1} + \left(i\lambda_5 - \frac{is_1}{2}\right) \xi - \frac{i[-2\sqrt{P(\lambda_5)} + s_1 \lambda_5] t}{8\sqrt{s_4} \lambda_5},$$

$$\theta_2 = -i \int_0^\xi \frac{C_2 ds}{\nu(s) - 2\alpha_2} - \left(i\lambda_5 - \frac{is_1}{2}\right) \xi + \frac{i[2\sqrt{P(\lambda_5)} + s_1 \lambda_5] t}{8\sqrt{s_4} \lambda_5},$$

$$C_j = -\frac{1}{8} [s_1^3 - 4s_1 s_2 + 8s_3 + 8a_j (s_1 - 4\lambda_5)],$$

$$\alpha_j = \frac{1}{8} \left[-s_1^2 + 8\lambda_5^2 + 4s_2 - 4s_1 \lambda_5 \mp 8\sqrt{P(\lambda_5)} \right],$$

$$j = 1, 2,$$

上式 Ψ_1 与 Ψ_2 为 $\lambda = \lambda_5$ 时的特解, 其中 $P(\lambda_i), s_i$ 和 ν_i 的值均由 λ_i 决定。因此, λ_i 的值决定了 Lax 对的通解, 且进一步影响了呼吸子解及其动态行为。通过改变 λ_i 的值, 得到三种呼吸子解, 即 GB、AB 和 KMB。

接下来, 选取特定参数, 呈现出 dn 背景下的单呼吸子解。满足 $r=b=0.4, c=0.3, d=0.7, \lambda_5=i$ 时, 可得到一阶 GB, 其最大值为 5, 位于原点, 如图 1(a) 所示。取 $r=b=0.2, c=0.1, d=0.6, \lambda_5=0.05i$ 时, 可得到一阶 AB, 它的所有峰值位于直线 $t=0$ 上, 最大值为 0.9, 位于原点。如图 1(b) 所示, 呈现出该呼吸子在空间分布方向上具有周期性结构, 而在时间演化方向上具有局部性结构的特点。取 $r=b=0.6, c=0.5, d=1, \lambda_5=i$ 时, 可得到一阶 KMB, 它将在直线 $x=0$ 中传播, 其最大值为 5.5, 也位于原点。如图 1(c) 所示, 呈现出该呼吸子在空间分布方向上具有局域性结构, 而在时间演化方向上具有周期性结构的特点。图 1(d)~(f) 分别呈现了图 1(a)~(c) 的等高线图。

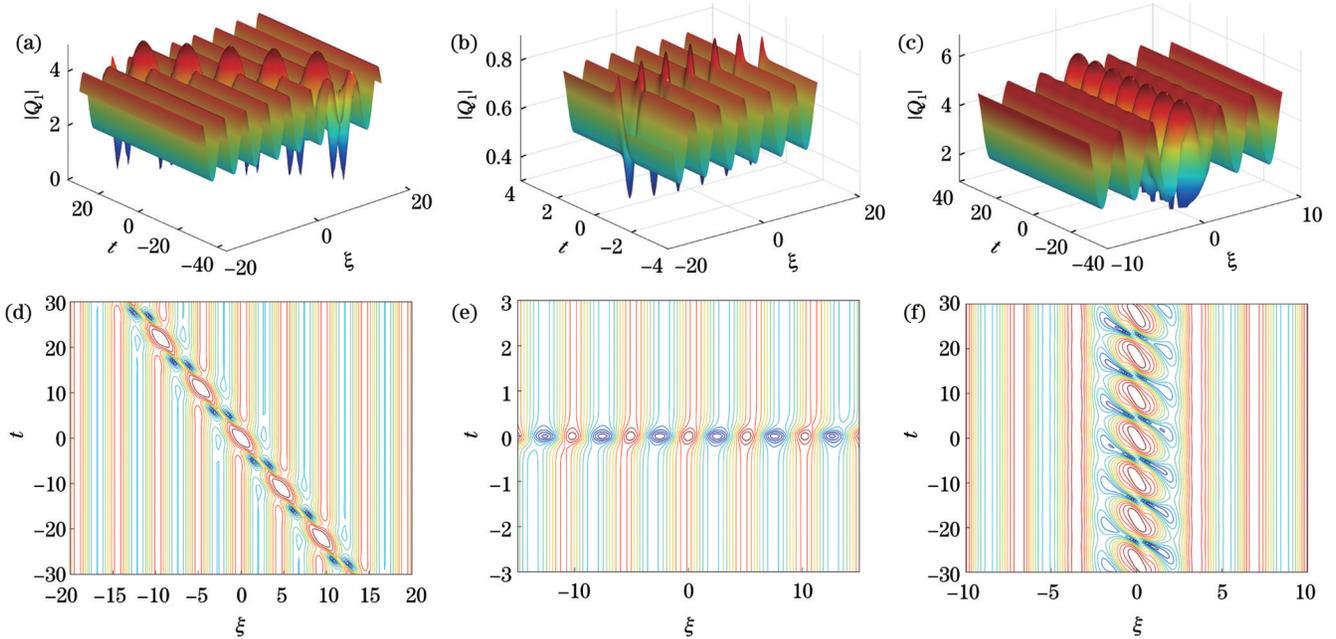


图 1 dn 周期波背景下的单呼吸子。(a) GB; (b) AB; (c) KMB; (d)、(e)、(f) 对应的等高线图

Fig. 1 Single breather on dn periodic wave background. (a) GB; (b) AB; (c) KMB; (d), (e), (f) corresponding contour plots

与 dn 背景下的呼吸子类似, 本文在 cn 背景下也构造出三种呼吸子, 即 GB、KMB 和 AB。对于 cn 背景, 参数取 $r=0.5, b=1.5, c=d=0.1, \lambda_5=0.2i$ 时, 可得到一阶 GB, 其最大值为 1, 位于原点, 如图 2(a) 所示。取 $r=0.3, b=1, c=d=0.3, \lambda_5=0.08i$ 时, 可得到一阶 AB, 最大值为 0.92, 位于原点。如图 2(b) 所示, 显然可以发现, AB 在时间上经历了一个单一的增长—回归循环, 并在空间上表现出一个周期结构。取 $r=0.35, b=1, c=d=0.15, \lambda_5=0.6i$ 时, 可得到一阶

KMB, 其最大值为 2.7, 也位于原点, 如图 2(c) 所示。图 2(d)~(f) 分别呈现了图 2(a)~(c) 的等高线图。

4.3 在 dn 和 cn 背景下的双呼吸子解

为了得到 dn 背景下的双呼吸子解, 将式(37)中的线性通解代入到 AB 系统的 Darboux 变换中。取定 $i=5, c_5=1, i=6, c_6=1$, 推导出较为紧凑的双呼吸子解的表达式为

$$Q[2] = Q + 4i \frac{M}{N}, \quad (39)$$

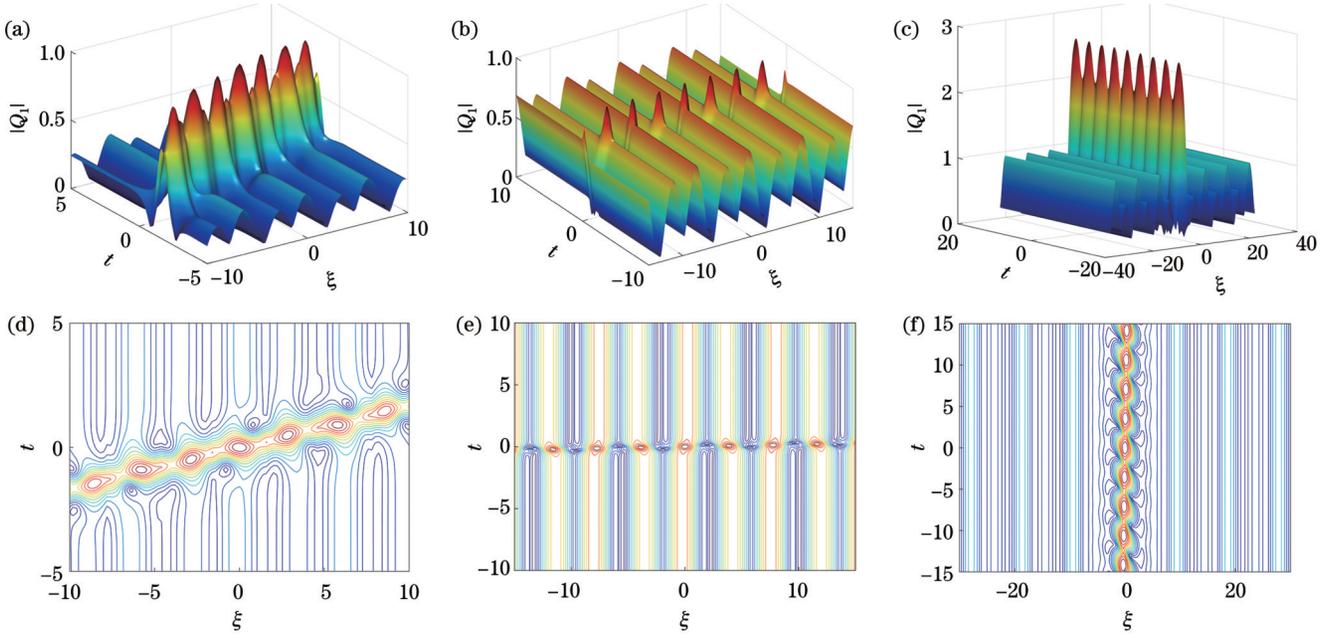


图 2 cn 周期波背景下的单呼吸子。(a) GB; (b) AB; (c) KMB; (d)、(e)、(f) 对应的等高线图

Fig. 2 Single breather on cn periodic wave background. (a) GB; (b) AB; (c) KMB; (d), (e), (f) corresponding contour plots

式中, M 和 N 如下:

$$M = \begin{pmatrix} \lambda_5 \Psi_1 & -\lambda_5^2 \Psi_1 & \Psi_1 & \Psi_2 \\ \lambda_6 \Psi_3 & -\lambda_6^2 \Psi_3 & \Psi_3 & \Psi_4 \\ \lambda_5^* \Psi_2^* & -\lambda_5^{*2} \Psi_2^* & \Psi_2^* & -\Psi_1^* \\ \lambda_6^* \Psi_4^* & -\lambda_6^{*2} \Psi_4^* & \Psi_4^* & -\Psi_3^* \end{pmatrix},$$

$$N = \begin{pmatrix} \lambda_5 \Psi_1 & \lambda_5 \Psi_2 & \Psi_1 & \Psi_2 \\ \lambda_6 \Psi_3 & \lambda_6 \Psi_4 & \Psi_3 & \Psi_4 \\ \lambda_5^* \Psi_2^* & -\lambda_5^* \Psi_1^* & \Psi_2^* & -\Psi_1^* \\ \lambda_6^* \Psi_4^* & -\lambda_6^* \Psi_3^* & \Psi_4^* & -\Psi_3^* \end{pmatrix},$$

$$Q = \sqrt{\nu} \exp\left(\frac{is_1 t}{4a} - iq \int_0^\xi \frac{1}{\nu} ds - \frac{is_1 \xi}{2}\right),$$

$$\Psi_1 = \sqrt{\nu(\xi) - 2\alpha_1} e^{\theta_1} + \sqrt{\nu(\xi) - 2\alpha_2} e^{-\theta_2},$$

$$\Psi_2 = \sqrt{\nu(\xi) - 2\alpha_2} e^{\theta_2} + \sqrt{\nu(\xi) - 2\alpha_1} e^{-\theta_1},$$

$$\theta_1 = i \int_0^\xi \frac{C_1 ds}{\nu(s) - 2\alpha_1} + \left(i\lambda_5 - \frac{is_1}{2}\right) \xi - \frac{i[-2\sqrt{P(\lambda_5)} + s_1 \lambda_5] t}{8\sqrt{s_4} \lambda_5},$$

$$\theta_2 = -i \int_0^\xi \frac{C_2 ds}{\nu(s) - 2\alpha_2} - \left(i\lambda_5 - \frac{is_1}{2}\right) \xi + \frac{i[2\sqrt{P(\lambda_5)} + s_1 \lambda_5] t}{8\sqrt{s_4} \lambda_5},$$

$$C_j = -\frac{1}{8} [s_1^3 - 4s_1 s_2 + 8s_3 + 8a_j (s_1 - 4\lambda_5)],$$

$$\alpha_j = \frac{1}{8} [-s_1^2 + 8\lambda_5^2 + 4s_2 - 4s_1 \lambda_5 \mp 8\sqrt{P(\lambda_5)}],$$

$j = 1, 2,$

$$\Psi_3 = \sqrt{\nu(\xi) - 2\alpha_1} e^{\theta_3} + \sqrt{\nu(\xi) - 2\alpha_2} e^{-\theta_4},$$

$$\Psi_4 = \sqrt{\nu(\xi) - 2\alpha_2} e^{\theta_4} + \sqrt{\nu(\xi) - 2\alpha_1} e^{-\theta_3},$$

$$\theta_3 = i \int_0^\xi \frac{C_3 ds}{\nu(s) - 2\alpha_1} + \left(i\lambda_6 - \frac{is_1}{2}\right) \xi - \frac{i[-2\sqrt{P(\lambda_6)} + s_1 \lambda_6] t}{8\sqrt{s_4} \lambda_6},$$

$$\theta_4 = -i \int_0^\xi \frac{C_4 ds}{\nu(s) - 2\alpha_2} - \left(i\lambda_6 - \frac{is_1}{2}\right) \xi + \frac{i[2\sqrt{P(\lambda_6)} + s_1 \lambda_6] t}{8\sqrt{s_4} \lambda_6},$$

$$C_j = -\frac{1}{8} [s_1^3 - 4s_1 s_2 + 8s_3 + 8a_j (s_1 - 4\lambda_6)],$$

$$\alpha_j = \frac{1}{8} [-s_1^2 + 8\lambda_6^2 + 4s_2 - 4s_1 \lambda_6 \mp 8\sqrt{P(\lambda_6)}],$$

$j = 3, 4,$

上式中 Ψ_1 与 Ψ_2 为 $\lambda = \lambda_5$ 时的线性特解, Ψ_3 与 Ψ_4 为 $\lambda = \lambda_6$ 时的特解。

为了实现 dn 背景下的双呼吸子解图, 参数选取为 $r = b = 0.8$ 、 $c = 0.48$ 、 $d = 0.8$ 、 $\lambda_5 = 1.2 + 2.5i$ 、 $\lambda_6 = i$, 绘制图 3(a)。图 3(a) 展示了 GB 和 KMB 相互作用的时空结构, 从图中可以发现, 两个呼吸子解在相互作用后没有发生波峰偏移。得到其最大值为 15.28, 也位于原点。图 3(b) 为其 $\xi = 0$ 时的截面图, 显然可以看出, GB 和 KMB 间的碰撞是弹性的, 呼吸子在传播过程中并未发生形状改变。

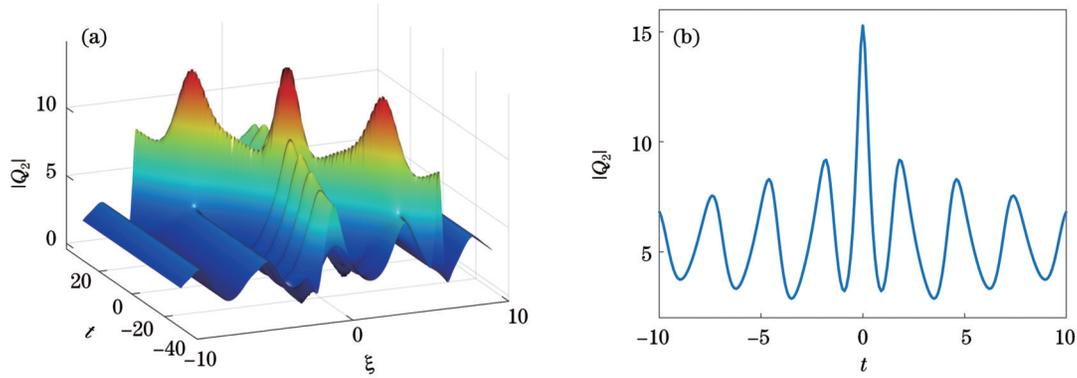


图 3 dn 周期波背景下的双呼吸子。(a)GB 和 KMB 的相互作用;(b) $\xi=0$ 时的截面图

Fig. 3 Double breathers on dn periodic wave background. (a) Interaction between GB and KMB; (b) section view at $\xi=0$

为了实现 cn 背景下的双呼吸子图,参数选取 $r=0.22$ 、 $b=0.73$ 、 $c=d=0.1$ 、 $\lambda_5=0.3i$ 、 $\lambda_6=0.8i$, 绘制图 4(a)。图 4(a)展示了两个 GB 相互作用的时空结构,从图中可以发现,两个呼吸子

解在相互作用后没有发生波峰偏移,得到其最大值为 4.6,仍位于原点。图 4(b)为其 $\xi=0$ 时的截面图,显然可以看出,GB 间的碰撞也是具有弹性的。

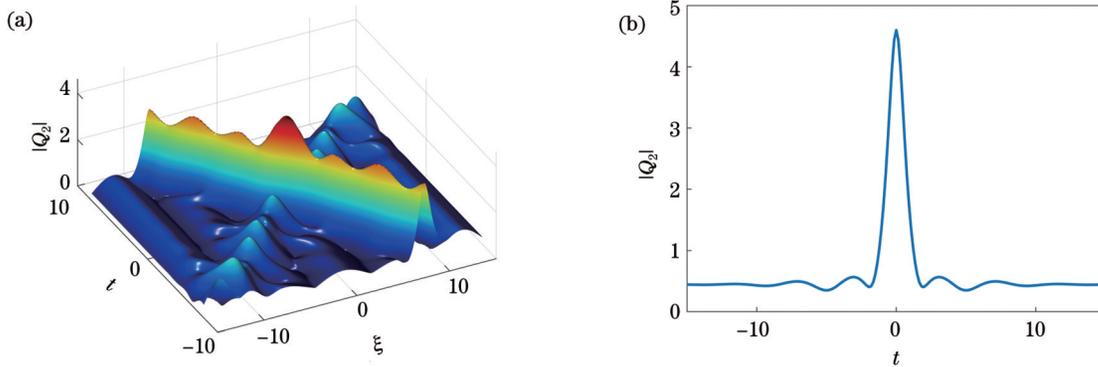


图 4 cn 周期波背景下的双呼吸子。(a)两个 GB 的相互作用;(b) $\xi=0$ 时的截面图

Fig. 4 Double breathers on cn periodic wave background. (a) Interaction between two GBs; (b) section view at $\xi=0$

5 结 论

本文主要研究了在椭圆函数背景下 AB 系统的多呼吸子的时空结构和相互作用。首先,基于 AB 系统的 Lax 对,通过行波变换法与平方波函数法获得该系统的椭圆函数周期解以及 Lax 对的基本解。随后,借助 Darboux 变换,得到其在椭圆函数背景下的多呼吸子解。通过呼吸子的时空结构图,进一步分析了其呼吸子的非线性动力学特征。研究 AB 系统解对于更好地理解非线性光学中超短光脉冲的传播具有重要意义。研究发现,AB 系统中两个 GB 间的碰撞以及 GB 和 KMB 间的碰撞都是弹性的,这意味着呼吸子能够在不改变形状的情况下长距离传输,这在光纤通信中是极为重要的。期望本文所得的结果将有助于理解从流体动力学到非线性光学领域中,周期背景下的呼吸子动力学行为和相互作用。

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Multiple Breathers of AB System Under Background of Elliptic Functions

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Abstract

Objective For current nonlinear physical systems, nonlinear optical fibers serve as a mature nonlinear experimental platform in experimental science. As a type of nonlinear wave with periodic evolution or periodic distribution structure, breathers have become one of the research hotspots in nonlinear optical systems. As the demand for long-distance and high-capacity fiber optic communication increases, the dynamic properties of breathers are receiving increasing attention. Studying the breather solutions for the AB system is of great significance for better understanding long-distance transmission without shape changing in fiber optic communication. In the context of the periodic solution of the AB system, we focus on the breathers of the system. By studying the interactions between two breathers, it is found that the collision between breathers is elastic, which means that breathers can be transported over long distances without changing their shapes. The results obtained in this article will help to understand the dynamics and interactions of breathers under periodic backgrounds in nonlinear optics.

Methods Via the Darboux transformation method in soliton theory, multi-breather solutions for the AB system were constructed under the elliptic function background. With the help of Matlab software, the spatiotemporal structure of the breathers was plotted, and the nonlinear dynamic characteristics of these breathers were further analyzed. Firstly, elliptic function solutions of the AB system were solved by the modified squared wave (MSW) function approach and the traveling wave transformation. Then, we obtained the basic solution to the Lax pair corresponding to the seed solution to the Jacobi elliptic function. Based on the elliptic function transformation formulas and the integral formulas, the potential function solution could be expressed in terms of the Weierstrass elliptic function. Secondly, by the once-iterated Darboux transformation, three types of breather solutions under the elliptic function background were constructed including the general breather (GB), the Kuznetsov-Ma breather (KMB), and the Akhmediev breather (AB). In addition, we analyzed the dynamic behaviors of these three kinds of breathers and presented their three-dimensional spatiotemporal structures. By the twice-iterated Darboux transformation, the spatiotemporal structure of the interaction between a GB and a KMB under the dn background was investigated, as well as the interaction between two GBs under the cn background.

Results and Discussions As an important integrable model, the AB system can be used to describe various nonlinear phenomena in many physical fields such as the quantum field theory, weak nonlinear dispersive water wave, and nonlinear optics. It is meaningful to solve various types of solutions of this model to describe the propagation of nonlinear waves. As

far as we know, the breather solutions for the AB system have not been constructed under the elliptic function background. In the context of the periodic solution to the elliptic function in the AB system, the basic solution to the Lax pair of the system is obtained using the MSW function. Using the Darboux transformation method, multiple breathers are constructed under the elliptic function background. Based on the expressions of the breather solutions, the dynamic characteristics of three types of breathers are discussed, including the GB, the KMB, and the AB (Figs. 1 and 2). Finally, the spatiotemporal structure of the interaction between a GB and a KMB under the dn background is investigated (Fig. 3), as well as the interaction between two GBs under the cn background (Fig. 4). It is found that collisions between breathers are elastic, which means that breathers can be transmitted over long distances without changing their shapes. These theoretical research results contribute to exploring the practical physical significance and applications of breathers in nonlinear optics.

Conclusions Based on the elliptic function formulas, we derive the explicit expressions of the first- and second-order breather solutions under the backgrounds of the dn and cn elliptic functions using the Darboux iteration algorithm. By analyzing the dynamic characteristics of three types of breathers and studying the spatiotemporal structure of multi-breather interactions under the dn and cn backgrounds, we find that the collision of GBs and the collision between GB and KMB in the AB system are both elastic, and the breathers do not undergo any shape change during their propagation. This discovery is of great significance for understanding the propagation characteristics of breathers and further elucidating their ability to complete long-distance transmission without changing their shapes. This research will help to understand the dynamics and interactions of breathers under the periodic background from fluid dynamics to nonlinear optics.

Key words nonlinear optics; breather; Darboux transformation; AB system