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非简并光学参变振荡器的不稳定性研究

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摘要 通过求得非简并光学参变振荡器的稳态解及特征值，研究了在共振与失谐两种情况下该系统的不稳定性。结果表明：共振时系统总是处于稳定状态，而有失谐时系统存在不稳定性。并通过数值模拟绘出共振情况下特征值随驱动场强度的变化曲线，及失谐时三个模量的输出信号随时间的变化情况。

关键词 非线性光学；不稳定性；失谐；非简并光学参变振荡器

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Instabilities of Nondegenerate Optical Parametric Oscillator

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Abstract The steady-state solutions and eigenvalues of the nondegenerate optical parametric oscillator are obtained, and the instabilities of the system in both cases of resonance and non-zero detuning are investigated. The results show that the system is always in a steady state when there is resonating and it has instability when there is non-zero detuning. The evolution of eigenvalues versus input field amplitude portrayed by numerical simulation with no detuning, as well as the evolution of the three output modes versus time with non-zero detuning, are proposed.

Key words nonlinear optics; instabilities; detuning; nondegenerate optical parametric oscillator

1 引 言

光学参变振荡器是在很宽的频率范围内可调的具有高相干性的辐射光源，不仅在经典领域有重要应用，在量子信息中也有着广泛的应用前景[1~6]。光学参变振荡器同时又是典型的非线性光学系统，展示出具有不稳定性的一面，因而其动力学特性的研究引起人们越来越多的关注。相关工作有简并光学参变振荡器的双稳态现象、Hopf 分岔和混沌现象等[7~10]，以及由此引发的对简并光学参变振荡器的混沌控制与反控制、混沌相位同步与反相位同步等问题的研究也有不少报道[11~14]。但对非简并光学参变振荡器的研究由于其数学处理复杂，尚不多见。

本文在求解非简并光学参变振荡器稳态解的基础上，通过求解特征方程，得出特征值并分析其稳定性来研究非简并光学参变振荡器的不稳定性。

2 稳态解

非简并光学参变振荡器是非线性光学系统，其半经典运动方程为[15]

$$\begin{aligned}\frac{\mathrm{d}\alpha_0}{\mathrm{d}t} &= -(\gamma_0 + \mathrm{i}\Delta_0)\alpha_0 - K\alpha_1\alpha_2 + \varepsilon,\\ \frac{\mathrm{d}\alpha_1}{\mathrm{d}t} &= -(\gamma_1 + \mathrm{i}\Delta_1)\alpha_1 + K\alpha_0\alpha_2^*,\\ \frac{\mathrm{d}\alpha_2}{\mathrm{d}t} &= -(\gamma_2 + \mathrm{i}\Delta_2)\alpha_2 + K\alpha_0\alpha_1^*,\end{aligned} \tag{1}$$

式中 $\alpha_i(i=0,1,2)$ 分别为基模和两个亚谐波模的复振幅，α_i^* 为各模的复共轭，γ_i 分别表示三个模的衰减系数，Δ_i 分别为三个模的失谐量，K 为耦合常数，ε 表示相干驱动场强度，这里取为正实数。下面分别讨论共振和有失谐时系统的稳态解。

2.1 共振时的稳态解

共振时，$\Delta_0=\Delta_1=\Delta_2=0$。方程(1)式变为

$$\begin{cases}\dfrac{\mathrm{d}\alpha_0}{\mathrm{d}t} = -\gamma_0\alpha_0 - K\alpha_1\alpha_2 + \varepsilon,\\ \dfrac{\mathrm{d}\alpha_1}{\mathrm{d}t} = -\gamma_1\alpha_1 + K\alpha_0\alpha_2^*,\\ \dfrac{\mathrm{d}\alpha_2}{\mathrm{d}t} = -\gamma_2\alpha_2 + K\alpha_0\alpha_1^*,\end{cases} \tag{2}$$

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用 α_0^0、α_1^0 和 α_2^0 表示稳态解，满足 $\frac{d\alpha_0^0}{dt}=0$，$\frac{d\alpha_1^0}{dt}=0$，$\frac{d\alpha_2^0}{dt}=0$，即

$$\begin{cases}-\gamma_0\alpha_0^0-K\alpha_1^0\alpha_2^0+\varepsilon=0,\\-\gamma_1\alpha_1^0+K\alpha_0^0\alpha_2^{0\,*}=0,\\-\gamma_2\alpha_2^0+K\alpha_0^0\alpha_1^{0\,*}=0.\end{cases}\tag{3}$$

由方程(3)可得到下面的稳态解

$$\begin{cases}|\alpha_0^0|^2=\dfrac{\gamma_1\gamma_2}{K^2},\\|\alpha_1^0|^2=\dfrac{-\gamma_0\gamma_2+K\varepsilon\sqrt{\gamma_2/\gamma_1}}{K^2},\\|\alpha_2^0|^2=\dfrac{-\gamma_0\gamma_1+K\varepsilon\sqrt{\gamma_1/\gamma_2}}{K^2},\end{cases}\tag{4}$$

由于 $|\alpha_i^0|^2\geqslant 0$，因此系统的驱动场应满足 $K\varepsilon\geqslant\gamma_0\sqrt{\gamma_1\gamma_2}$，即共振时非简并光学参变振荡器的阈值条件为

$$K\varepsilon_{\rm th}=\gamma_0\sqrt{\gamma_1\gamma_2}.\tag{5}$$

2.2　有失谐时的稳态解

当系统存在失谐，即方程(1)式中 $\Delta_i\neq 0$（$i=0$ 或 1 或 2）时。稳态解 α_0^0、α_1^0 和 α_2^0 满足

$$\begin{cases}-(\gamma_0+i\Delta_0)\alpha_0^0-K\alpha_1^0\alpha_2^0+\varepsilon=0,\\-(\gamma_1+i\Delta_1)\alpha_1^0+K\alpha_0^0\alpha_2^{0\,*}=0,\\-(\gamma_2+i\Delta_2)\alpha_2^0+K\alpha_0^0\alpha_1^{0\,*}=0,\end{cases}\tag{6}$$

由方程(6)式可求得此时非简并光学参变振荡器的稳态解（见附录 A）：

$$K^2|\alpha_0^0|^2=\gamma_1\gamma_2+\Delta_1\Delta_2,\tag{7}$$

$$K^2|\alpha_1^0|^2=-(\gamma_0\gamma_2-\Delta_0\Delta_2)+\sqrt{\frac{\Delta_0\gamma_2+\Delta_2\gamma_0}{\Delta_0\gamma_1+\Delta_1\gamma_0}\cdot K^2\varepsilon^2-(\Delta_0\gamma_2+\Delta_2\gamma_0)^2},\tag{8}$$

$$K^2|\alpha_2^0|^2=-(\gamma_0\gamma_1-\Delta_0\Delta_1)+\sqrt{\frac{\Delta_0\gamma_1+\Delta_1\gamma_0}{\Delta_0\gamma_2+\Delta_2\gamma_0}\cdot K^2\varepsilon^2-(\Delta_0\gamma_1+\Delta_1\gamma_0)^2},\tag{9}$$

其中(7)式为基模的稳态解，(8)式和(9)式分别为两个亚谐波模的稳态解。

由 $|\alpha_i^0|^2$ 为非负，得非共振情况下系统驱动场应满足的阈值条件

$$K\varepsilon_{\rm th}=\sqrt{(\Delta_0^2+\gamma_0^2)(\gamma_1\gamma_2+\Delta_1\Delta_2)}.\tag{10}$$

3　不稳定性分析

下面用线性化方程研究系统稳态解的稳定性问题。

3.1　共振时

共振情况下，系统各模的复振幅 α_i 及其复共轭 α_i^* 满足

$$\begin{cases}\dfrac{d\alpha_0}{dt}=-\gamma_0\alpha_0-K\alpha_1\alpha_2+\varepsilon,\\\dfrac{d\alpha_0^*}{dt}=-\gamma_0\alpha_0^*-K\alpha_1^*\alpha_2^*+\varepsilon,\\\dfrac{d\alpha_1}{dt}=-\gamma_1\alpha_1+K\alpha_0\alpha_2^*,\\\dfrac{d\alpha_1^*}{dt}=-\gamma_1\alpha_1^*+K\alpha_0^*\alpha_2,\\\dfrac{d\alpha_2}{dt}=-\gamma_2\alpha_2+K\alpha_0\alpha_1^*,\\\dfrac{d\alpha_2^*}{dt}=-\gamma_2\alpha_2^*+K\alpha_0^*\alpha_1.\end{cases}\tag{11}$$

假设 $\delta\alpha_i$ 和 $\delta\alpha_i^*$ 为稳态解附近小的扰动，则(11)式的线性化方程为

$$\frac{\partial}{\partial t}\begin{bmatrix}\delta\alpha_0\\\delta\alpha_0^*\\\delta\alpha_1\\\delta\alpha_1^*\\\delta\alpha_2\\\delta\alpha_2^*\end{bmatrix}=\begin{bmatrix}-\gamma_0&0&-K\alpha_2^0&0&-K\alpha_1^0&0\\0&-\gamma_0&0&-K\alpha_2^{0\,*}&0&-K\alpha_1^{0\,*}\\K\alpha_2^{0\,*}&0&-\gamma_1&0&0&K\alpha_0^0\\0&K\alpha_2^0&0&-\gamma_1&K\alpha_0^{0\,*}&0\\K\alpha_1^{0\,*}&0&0&K\alpha_0^0&-\gamma_2&0\\0&K\alpha_1^0&K\alpha_0^{0\,*}&0&0&-\gamma_2\end{bmatrix}\begin{bmatrix}\delta\alpha_0\\\delta\alpha_0^*\\\delta\alpha_1\\\delta\alpha_1^*\\\delta\alpha_2\\\delta\alpha_2^*\end{bmatrix}.\tag{12}$$

决定稳态解稳定性的特征值 λ 满足的特征方程为

$$\begin{vmatrix}-\gamma_0-\lambda&0&-K\alpha_2^0&0&-K\alpha_1^0&0\\0&-\gamma_0-\lambda&0&-K\alpha_2^{0\,*}&0&-K\alpha_1^{0\,*}\\K\alpha_2^{0\,*}&0&-\gamma_1-\lambda&0&0&K\alpha_0^0\\0&K\alpha_2^0&0&-\gamma_1-\lambda&K\alpha_0^{0\,*}&0\\K\alpha_1^{0\,*}&0&0&K\alpha_0^0&-\gamma_2-\lambda&0\\0&K\alpha_1^0&K\alpha_0^{0\,*}&0&0&-\gamma_2-\lambda\end{vmatrix}=0,\tag{13}$$

可解得六个特征值为（见附录 B）：

$$
\begin{cases}
\lambda_1 = 0,\\
\lambda_2 = \dfrac{1}{2}\left[-(\gamma_0+\gamma_1+\gamma_2)+\sqrt{(\gamma_0+\gamma_1+\gamma_2)^2-4K\varepsilon(\gamma_1+\gamma_2)/\sqrt{\gamma_1\gamma_2}}\right],\\
\lambda_3 = \dfrac{1}{2}\left[-(\gamma_0+\gamma_1+\gamma_2)-\sqrt{(\gamma_0+\gamma_1+\gamma_2)^2-4K\varepsilon(\gamma_1+\gamma_2)/\sqrt{\gamma_1\gamma_2}}\right],\\
\lambda_4 = \sqrt[3]{-\dfrac{q}{2}+\sqrt{\delta}}+\sqrt[3]{-\dfrac{q}{2}-\sqrt{\delta}}-\dfrac{b}{3},\\
\lambda_5 = -\dfrac{1}{2}\left(\sqrt[3]{-\dfrac{q}{2}+\sqrt{\delta}}+\sqrt[3]{-\dfrac{q}{2}-\sqrt{\delta}}\right)+\dfrac{\sqrt{3}}{2}\mathrm{i}\left(\sqrt[3]{-\dfrac{q}{2}+\sqrt{\delta}}-\sqrt[3]{-\dfrac{q}{2}-\sqrt{\delta}}\right)-\dfrac{b}{3},\\
\lambda_6 = -\dfrac{1}{2}\left(\sqrt[3]{-\dfrac{q}{2}+\sqrt{\delta}}+\sqrt[3]{-\dfrac{q}{2}-\sqrt{\delta}}\right)-\dfrac{\sqrt{3}}{2}\mathrm{i}\left(\sqrt[3]{-\dfrac{q}{2}+\sqrt{\delta}}-\sqrt[3]{-\dfrac{q}{2}-\sqrt{\delta}}\right)-\dfrac{b}{3},
\end{cases}
\tag{14}
$$

其中

$$b = \gamma_0+\gamma_1+\gamma_2,$$

$$p = K\varepsilon\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}-\frac{1}{3}(\gamma_0+\gamma_1+\gamma_2)^2,$$

$$q = 2\left(\frac{\gamma_0+\gamma_1+\gamma_2}{3}\right)^3-\frac{\gamma_0+\gamma_1+\gamma_2}{3}K\varepsilon\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}+4K\varepsilon\sqrt{\gamma_1\gamma_2}-4\gamma_0\gamma_1\gamma_2,$$

$$\delta = \left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3 = \left(\frac{\gamma_0+\gamma_1+\gamma_2}{3}\right)^3(4K\varepsilon\sqrt{\gamma_1\gamma_2}-4\gamma_0\gamma_1\gamma_2)-\frac{1}{12}\left(\frac{\gamma_0+\gamma_1+\gamma_2}{3}\right)^2K^2\varepsilon^2\left(\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}\right)^2-$$
$$\left(\frac{\gamma_0+\gamma_1+\gamma_2}{3}\right)K\varepsilon\left(\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}\right)(2K\varepsilon\sqrt{\gamma_1\gamma_2}-2\gamma_0\gamma_1\gamma_2)+\left(\frac{1}{3}K\varepsilon\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}\right)^3+(2K\varepsilon\sqrt{\gamma_1\gamma_2}-2\gamma_0\gamma_1\gamma_2)^2.$$

(14)式即为共振时系统的六个特征值。当驱动场满足 $\varepsilon\geqslant\varepsilon_{\mathrm{th}}$ 时，$\lambda_1=0$，λ_2 和 λ_3 均为负实数，λ_4、λ_5、λ_6 的符号与 δ 的正负有关，当 $\delta\leqslant0$ 时 λ_4、λ_5、λ_6 均为负实数，当 $\delta>0$ 时 λ_4 为负实数，而 λ_5、λ_6 为实部为负的一对共轭复根，因此共振时系统的稳态解是稳定的，即系统始终处于稳定状态。若参数取为 $\gamma_0=1$，$\gamma_1=0.01$，$\gamma_2=5$，$K=1$，阈值 $\varepsilon_{\mathrm{th}}\approx0.224$，当驱动场 $0.224<\varepsilon\leqslant0.4$ 时，$\delta\leqslant0$，λ_5、λ_6 为负实数，图 1 (a)、(b) 为特征值 λ_5、λ_6 随驱动场强度 ε 的变化曲线，当驱动场 $\varepsilon>0.4$ 时，$\delta>0$，λ_5、λ_6 为实部为负的一对共轭复根，图 1 (c)、(d) 为特征值 λ_5、λ_6 的实部随驱动场强度 ε 的变化曲线。

3.2 有失谐时

当非简并光学参变振荡器系统存在失谐时，其各模的复振幅 α_i 及其复共轭 α_i^* 满足

$$
\begin{cases}
\dfrac{\mathrm{d}\alpha_0}{\mathrm{d}t} = -(\gamma_0+\mathrm{i}\Delta_0)\alpha_0-K\alpha_1\alpha_2+\varepsilon,\\
\dfrac{\mathrm{d}\alpha_0^*}{\mathrm{d}t} = -(\gamma_0-\mathrm{i}\Delta_0)\alpha_0^*-K\alpha_1^*\alpha_2^*+\varepsilon,\\
\dfrac{\mathrm{d}\alpha_1}{\mathrm{d}t} = -(\gamma_1+\mathrm{i}\Delta_1)\alpha_1+K\alpha_0\alpha_2^*,\\
\dfrac{\mathrm{d}\alpha_1^*}{\mathrm{d}t} = -(\gamma_1-\mathrm{i}\Delta_1)\alpha_1^*+K\alpha_0^*\alpha_2,\\
\dfrac{\mathrm{d}\alpha_2}{\mathrm{d}t} = -(\gamma_2+\mathrm{i}\Delta_2)\alpha_2+K\alpha_0\alpha_1^*,\\
\dfrac{\mathrm{d}\alpha_2^*}{\mathrm{d}t} = -(\gamma_2-\mathrm{i}\Delta_2)\alpha_2^*+K\alpha_0^*\alpha_1.
\end{cases}
\tag{15}
$$

用线性化方程求得特征方程为(见附录 C)

$$[(\gamma_0+\lambda)^2+\Delta_0^2][(\gamma_1+\lambda)^2+\Delta_1^2][(\gamma_2+\lambda)^2+\Delta_2^2]-4|\alpha_0^0|^2|\alpha_1^0|^2|\alpha_2^0|^2+[(\gamma_0+\lambda)^2+\Delta_0^2]|\alpha_0^0|^4+$$
$$[(\gamma_1+\lambda)^2+\Delta_1^2]|\alpha_1^0|^4+[(\gamma_2+\lambda)^2+\Delta_2^2]|\alpha_2^0|^4-2[(\gamma_0+\lambda)^2+\Delta_0^2][(\gamma_1+\lambda)(\gamma_2+\lambda)+\Delta_1\Delta_2]|\alpha_0^0|^2+$$
$$2[(\gamma_1+\lambda)^2+\Delta_1^2][(\gamma_0+\lambda)(\gamma_2+\lambda)-\Delta_0\Delta_2]|\alpha_1^0|^2+2[(\gamma_2+\lambda)^2+\Delta_2^2][(\gamma_0+\lambda)(\gamma_1+\lambda)-\Delta_0\Delta_1]|\alpha_2^0|^2-$$
$$2[(\gamma_0+\lambda)(\gamma_1+\lambda)-\Delta_0\Delta_1]|\alpha_0^0|^2|\alpha_1^0|^2-2[(\gamma_0+\lambda)(\gamma_2+\lambda)-\Delta_0\Delta_2]|\alpha_0^0|^2|\alpha_2^0|^2+$$
$$2[(\gamma_1+\lambda)(\gamma_2+\lambda)+\Delta_1\Delta_2]|\alpha_1^0|^2|\alpha_2^0|^2=0. \tag{16}$$

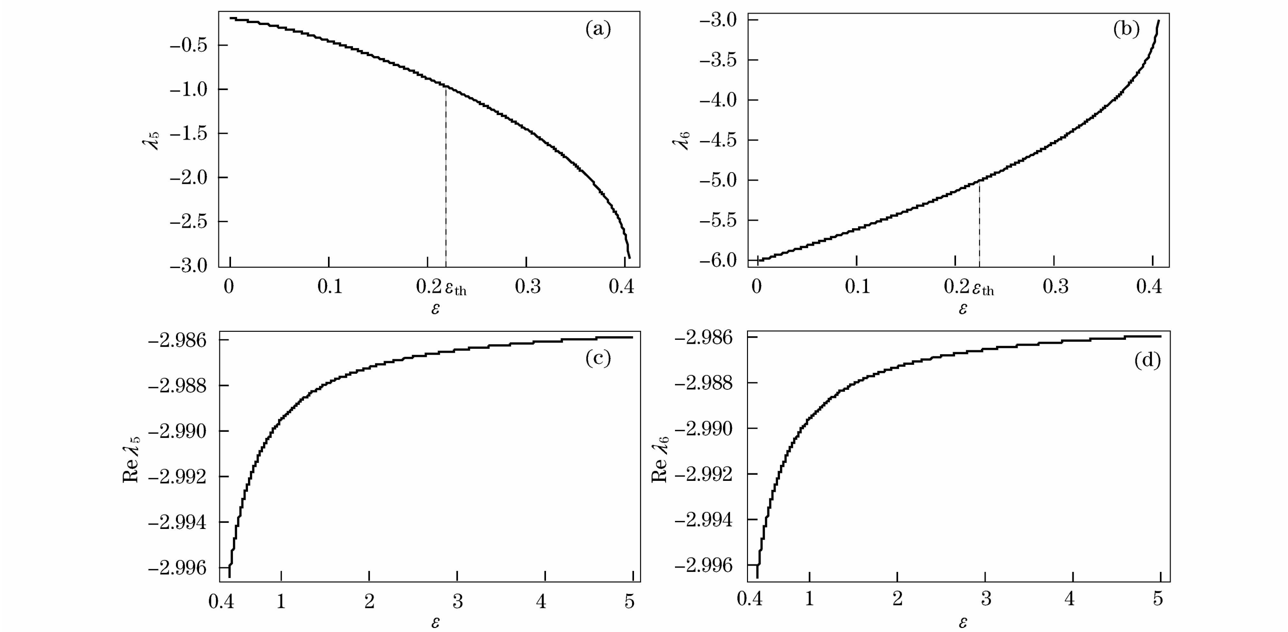

图 1　λ_5 和 λ_6 随驱动场强度 ε 的变化曲线。(a)、(b)为负实数 λ_5、λ_6 随 ε 的变化曲线(ε≤0.4)；(c)、(d)为 λ_5、λ_6 的实部随 ε 的变化曲线(ε>0.4)

Fig. 1　Evolution of λ_5 and λ_6 versus input field amplitude ε. (a)、(b) Negative real number λ_5 and λ_6 versus ε (ε≤0.4)；(c)、(d) real parts of λ_5 and λ_6 versus ε (ε>0.4)

把有失谐时的稳态解即(7)～(9)式代入特征方程(16)式，取参数 $\gamma_0=0.1$，$\gamma_1=1.0$，$\gamma_2=2.0$，$\Delta_0=1.0$，$\Delta_1=-5$，$\Delta_2=-10$，$K=1$，取驱动场 $\varepsilon=12$(阈值 $\varepsilon_{th}\approx7.25$)，可以求得其六个特征值为：$\lambda_1=0.35+i2.90$，$\lambda_2=0.35-i2.90$，$\lambda_3=0$，$\lambda_4=-1.85$，$\lambda_5=-2.52+i6.08$，$\lambda_6=-2.52-i6.08$。其中 λ_1 和 λ_2 是一对共轭复数，由于它们的实部大于零，因而此时系统的稳态解是不稳定的，即系统存在不稳定状态。

下面通过数值模拟来验证参数下系统的动力学行为。为了便于计算，需要把复数方程(1)式转换成实数方程，作如下变换：

$$\begin{cases} x=\dfrac{(\alpha_0+\alpha_0^*)}{2},\ y=\dfrac{(\alpha_0-\alpha_0^*)}{2i} \\ u=\dfrac{(\alpha_1+\alpha_1^*)}{2},\ v=\dfrac{(\alpha_1-\alpha_1^*)}{2i} \\ m=\dfrac{(\alpha_2+\alpha_2^*)}{2},\ n=\dfrac{(\alpha_2-\alpha_2^*)}{2i}, \end{cases} \tag{17}$$

则方程(1)式变换为以下实数方程组($K=1$)：

$$\begin{cases} \dfrac{dx}{dt}=-\gamma_0x+\Delta_0y-um+vn+\varepsilon, \\ \dfrac{dy}{dt}=-\gamma_0y-\Delta_0x-un-vm, \\ \dfrac{du}{dt}=-\gamma_1u+\Delta_1v+xm+yn, \\ \dfrac{dv}{dt}=-\gamma_1v-\Delta_1u-xn+ym, \\ \dfrac{dm}{dt}=-\gamma_2m+\Delta_2n+xu+yv, \\ \dfrac{dn}{dt}=-\gamma_2n-\Delta_2m-xv+yu. \end{cases} \tag{18}$$

系统参数取为与求特征值 λ 时相同的参数，驱动场强度 $\varepsilon=12$，系统的初始条件取为：$\alpha_0^0=0.1+i0.1$，$\alpha_1^0=0.1+i0.1$，$\alpha_2^0=0.1+i0.1$，图 2(a)、(b)、(c)分别为基模和两个亚谐波模的输出随时间的变

化情况,从图中可以看出基模为周期输出,而两个亚谐波模的输出是不稳定的。这与上述求得的特征值所反映的系统状态相一致。

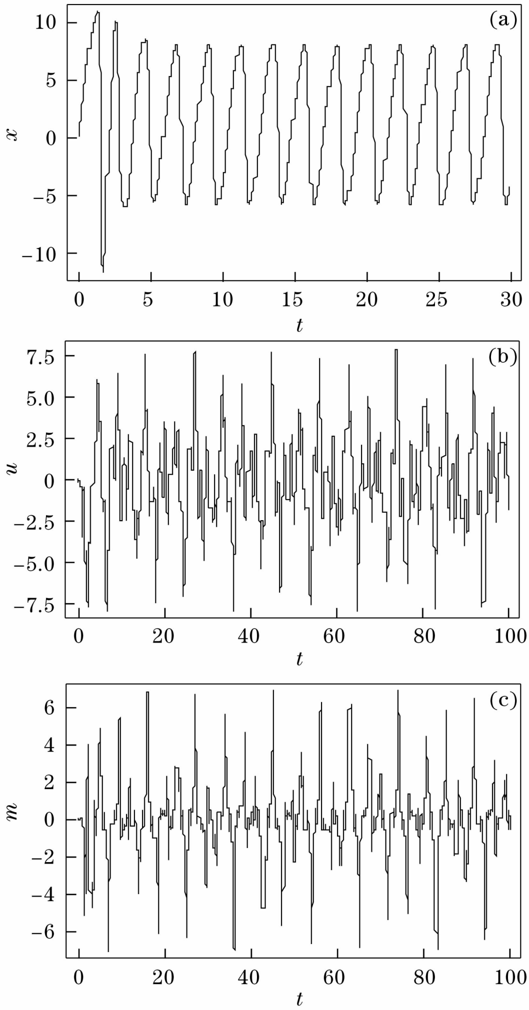

图2 基模和两个亚谐波模的输出随时间的变化曲线。(a)基模的周期输出;(b)、(c)两个亚谐波模的不稳定输出(ε=12)

Fig. 2 Evolution of the output of fundamental mode and two subharmonic modes versus time. (a) Periodic output of fundamental mode; (b)、(c) instable output of two subharmonic modes (ε=12)

4 结 论

研究了非简并光学参变振荡器的不稳定性。在共振和失谐两种情况下,分别求得了稳态解,并求出了特征值。对于共振情况,不管参数如何选取,系统总是处于稳定状态。对于失谐情况,选取的参数不同,系统存在不稳定状态。通过数值模拟,得到了基模的周期输出和两个亚谐波模的不稳定输出,这个结果与理论分析结果相一致。

参 考 文 献

1 Y. H. Shih, A. V. Sergienko, Morton H. Rubin. Two-photon entanglement in type-Ⅱ parametric down-conversion [J]. *Phys. Rev. A*, 1994, **50**(1):23~28

2 Luo Yu, Li Ying, Pan Qing. Experimental generation of entangled lights from resonating frequency doubler [J]. *Acta Optica Sinica*, 2006, **26**(2):305~310
罗 玉,李 莹,潘 庆. 利用谐振倍频产生纠缠光束[J]. 光学学报, 2006, **26**(2):305~310

3 A. G. White, D. F. James, P. H. Eberhard *et al*.. Nonmaximally entangled states: Production, characterization, and utilization [J]. *Phys. Rev. Lett.*, 1999, **83**(16):3103~3107

4 Wang Hai, Xue Chenyang, Gao Jiangrui. Quantum correlated twin beams generation through an intracavity parametric down-conversion in a pair of KTP [J]. *Acta Optica Sinica*, 1997, **17**(8):1002~1005
王 海,薛晨阳,郜江瑞. 双KTP内腔参量下转换产生量子相关孪生光子对[J]. 光学学报, 1997, **17**(8):1002~1005

5 Gao Jiangrui, Xie Changde, Peng Kunchi. Squeezing of pump light reflected from triply OPO [J]. *Acta Optica Sinica*, 1999, **19**(11):1447~1451
郜江瑞,谢常德,彭堃墀. 三模共振光学参量振荡腔抽运场的压缩[J]. 光学学报, 1999, **19**(11):1447~1451

6 K. S. Zhang, T. Coudreau, M. Martinelli *et al*.. Generation of bright squeezed light at 1.06 μm using cascaded nonlinearities in a triply resonant cw periodically-poled lithium niobate optical parametric oscillator [J]. *Phys. Rev. A*, 2001, **64**(3):033815

7 P. D. Drummond, K. J. McNeil, D. F. Walls. Non-equilibrium transitions in sub/second harmonic generation [J]. *Optica Acta*, 1980, **27**(3):321~335

8 N. P. Pettiaux, Ruo Ding Li, Paul Mandel. Instabilities of the degenerate optical parametric oscillator [J]. *Opt. Commun.*, 1989, **72**(3):256~260

9 K. Staliunas, V. J. Sánchez-Morcillo. Spatial-localized structures in degenerate optical parametric oscillators [J]. *Phys. Rev. A*, 1998, **57**(2):1454~1457

10 Axelle Amon, Marc Lefranc. Topological signature of deterministic chaos in short nonstationary signals from an optical parametric oscillator [J]. *Phys. Rev. Lett.*, 2004, **92**(9):094101

11 Ma Jun, Jin Wuyin, Li Yanlong. Chaotic signal-induced dynamics of degenerate optical parametric oscillator [J]. *Chaos, Solitons and Fractals*, 2008, **36**(2):494~499

12 Feng Xiuqin, Shen Ke. Anti-control of chaos on the degenerate optical parametric oscillator [J]. *Acta Physica Sinica*, 2006, **55**(9):4455~4459
冯秀琴,沈 柯. 简并光学参量振荡器混沌反控制[J]. 物理学报, 2006, **55**(9):4455~4459

13 Feng Xiuqin, Shen Ke. Phase synchronization and anti-phase synchronization of chaos for degenerate optical parametric oscillator [J]. *Chinese Physics*, 2005, **14**(8):1526~1532

14 He Xiangang, Zhou Ping, Cheng Xuefeng. Synchronization of degenerate optical parametric oscillator with different system parameter [J]. *Piezoel Ectectrics & Acoustooptics*, 2008, **30**(2):173~179
何先刚,周 平,程雪峰. 系数参数不同的简并光学参量振荡器之间的混沌同步[J]. 压电与声光, 2008, **30**(2):173~179

15 K. J. McNeil, C. W. Gardiner. Quantum statistics of parametric oscillation [J]. *Phys. Rev. A*, 1983, **28**(3):1560~1566

附录 A　有失谐时的稳态解

有失谐时的稳态解满足

$$-(\gamma_0+\mathrm{i}\Delta_0)\alpha_0^0-K\alpha_1^0\alpha_2^0+\varepsilon=0, \tag{A1}$$

$$-(\gamma_1+\mathrm{i}\Delta_1)\alpha_1^0+K\alpha_0^0\alpha_2^{0\,*}=0, \tag{A2}$$

$$-(\gamma_2+\mathrm{i}\Delta_2)\alpha_2^0+K\alpha_0^0\alpha_1^{0\,*}=0, \tag{A3}$$

由(A2)、(A3)式得

$$K^2\,|\,\alpha_0^0\,|^2=\gamma_1\gamma_2+\Delta_1\Delta_2, \tag{A4}$$

即为基模的稳态解。令 $\alpha_0^0=x_0+\mathrm{i}y_0$，$\alpha_1^0=x_1+\mathrm{i}y_1$，$\alpha_2^0=x_2+\mathrm{i}y_2$，则式(A1)～(A3)可写成

$$-\gamma_0x_0+\Delta_0y_0-K(x_1x_2-y_1y_2)+\varepsilon=0, \tag{A5}$$

$$-(\Delta_0x_0+\gamma_0y_0)-K(x_1y_2+x_2y_1)=0, \tag{A6}$$

$$-\gamma_1x_1+\Delta_1y_1+K(x_0x_2+y_0y_2)=0, \tag{A7}$$

$$-(\Delta_1x_1+\gamma_1y_1)+K(x_2y_0-x_0y_2)=0, \tag{A8}$$

$$-\gamma_2x_2+\Delta_2y_2+K(x_0x_1+y_0y_1)=0, \tag{A9}$$

$$-(\Delta_2x_2+\gamma_2y_2)+K(x_1y_0-x_0y_1)=0, \tag{A10}$$

解得两个亚谐波的稳态解有

$$\frac{|\,\alpha_1^0\,|^2}{|\,\alpha_2^0\,|^2}=\frac{x_1^2+y_1^2}{x_2^2+y_2^2}=\frac{\Delta_0\gamma_2+\Delta_2\gamma_0}{\Delta_0\gamma_1+\Delta_1\gamma_0}, \tag{A11}$$

由(A5)式～(A10)式整理可得

$$aX-bY=\frac{\varepsilon\gamma_0}{\Delta_0^2+\gamma_0^2}, \tag{A12}$$

$$bX+aY=-\frac{\varepsilon\Delta_0}{\Delta_0^2+\gamma_0^2}, \tag{A13}$$

其中

$$a=\frac{\gamma_2}{K(x_1^2+y_1^2)}+\frac{K\gamma_0}{\Delta_0^2+\gamma_0^2},$$

$$b=\frac{\Delta_2}{K(x_1^2+y_1^2)}-\frac{K\Delta_0}{\Delta_0^2+\gamma_0^2}, \tag{A14}$$

$$X=x_1x_2-y_1y_2,$$

$$Y=x_1y_2+x_2y_1. \tag{A15}$$

解得第一个亚谐波的稳态解为

$$K^2\,|\,\alpha_1^0\,|^2=-(\gamma_0\gamma_2-\Delta_0\Delta_2)+\sqrt{\frac{\Delta_0\gamma_2+\Delta_2\gamma_0}{\Delta_0\gamma_1+\Delta_1\gamma_0}\cdot K^2\varepsilon^2-(\Delta_0\gamma_2+\Delta_2\gamma_0)^2}, \tag{A16}$$

由(A11)式两个亚谐波的稳态解关系可得，第二个亚谐波的稳态解为

$$K^2\,|\,\alpha_2^0\,|^2=-(\gamma_0\gamma_1-\Delta_0\Delta_1)+\sqrt{\frac{\Delta_0\gamma_1+\Delta_1\gamma_0}{\Delta_0\gamma_2+\Delta_2\gamma_0}\cdot K^2\varepsilon^2-(\Delta_0\gamma_1+\Delta_1\gamma_0)^2}. \tag{A17}$$

附录 B　共振时的特征方程求解

决定稳态解稳定性的特征值 λ 满足的特征方程为

$$\begin{vmatrix} -\gamma_0-\lambda & 0 & -K\alpha_2^0 & 0 & -K\alpha_1^0 & 0 \\ 0 & -\gamma_0-\lambda & 0 & -K\alpha_2^{0\,*} & 0 & -K\alpha_1^{0\,*} \\ K\alpha_2^{0\,*} & 0 & -\gamma_1-\lambda & 0 & 0 & K\alpha_0^0 \\ 0 & K\alpha_2^0 & 0 & -\gamma_1-\lambda & K\alpha_0^{0\,*} & 0 \\ K\alpha_1^{0\,*} & 0 & 0 & K\alpha_0^0 & -\gamma_2-\lambda & 0 \\ 0 & K\alpha_1^0 & K\alpha_0^{0\,*} & 0 & 0 & -\gamma_2-\lambda \end{vmatrix}=0, \tag{B1}$$

上式可分解为

$$(\gamma_0+\lambda)(\gamma_1+\lambda)(\gamma_2+\lambda)-K^2\,|\,\alpha_0^0\,|^2(\gamma_0+\lambda)+K^2\,|\,\alpha_1^0\,|^2(\gamma_1+\lambda)+K^2\,|\,\alpha_2^0\,|^2(\gamma_2+\lambda)\pm2K^3\,|\,\alpha_0^0\,|\,|\,\alpha_1^0\,|\,|\,\alpha_2^0\,|=0, \tag{B2}$$

将(4)式的稳态解代入(B2)式，并将方程展开为

$$\lambda^3+b\lambda^2+c\lambda+d=0, \tag{B3}$$

的形式，其中

$$b=\gamma_0+\gamma_1+\gamma_2,\ c=K\varepsilon\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}},\ d=0\text{ 或 }d=4K\varepsilon\sqrt{\gamma_1\gamma_2}-4\gamma_0\gamma_1\gamma_2.$$

Ⅰ. 当 $d=0$ 时，

特征方程为

$$\lambda^3+b\lambda^2+c\lambda=0.$$

可求出相应的三个特征值为

$$\begin{aligned}
\lambda_1 &= 0,\\
\lambda_2 &= \frac{1}{2}\left[-(\gamma_0+\gamma_1+\gamma_2)+\sqrt{(\gamma_0+\gamma_1+\gamma_2)^2-4K\varepsilon(\gamma_1+\gamma_2)/\sqrt{\gamma_1\gamma_2}}\right],\\
\lambda_3 &= \frac{1}{2}\left[-(\gamma_0+\gamma_1+\gamma_2)-\sqrt{(\gamma_0+\gamma_1+\gamma_2)^2-4K\varepsilon(\gamma_1+\gamma_2)/\sqrt{\gamma_1\gamma_2}}\right].
\end{aligned} \tag{B4}$$

Ⅱ. 当 $d=4K\varepsilon\sqrt{\gamma_1\gamma_2}-4\gamma_0\gamma_1\gamma_2$ 时，令

$$\lambda = X-\frac{b}{3}, \tag{B5}$$

可将特征方程(B3)式化为

$$X^3+pX+q=0, \tag{B6}$$

的形式，式中

$$p=c-\frac{b^2}{3}=K\varepsilon\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}-\frac{1}{3}(\gamma_0+\gamma_1+\gamma_2)^2,$$

$$q=2\left(\frac{b}{3}\right)^3-\frac{b}{3}c+d=2\left(\frac{\gamma_0+\gamma_1+\gamma_2}{3}\right)^3-\frac{\gamma_0+\gamma_1+\gamma_2}{3}K\varepsilon\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}+4K\varepsilon\sqrt{\gamma_1\gamma_2}-4\gamma_0\gamma_1\gamma_2,$$

由(B5)和(B6)式求出另三个特征值为

$$\begin{aligned}
\lambda_4 &= \sqrt[3]{-\frac{q}{2}+\sqrt{\delta}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\delta}}-\frac{b}{3},\\
\lambda_5 &= -\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\delta}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\delta}}\right)+\frac{\sqrt{3}}{2}\mathrm{i}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\delta}}-\sqrt[3]{-\frac{q}{2}-\sqrt{\delta}}\right)-\frac{b}{3},\\
\lambda_6 &= -\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\delta}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\delta}}\right)-\frac{\sqrt{3}}{2}\mathrm{i}\left(\sqrt[3]{-\frac{q}{2}+\sqrt{\delta}}-\sqrt[3]{-\frac{q}{2}-\sqrt{\delta}}\right)-\frac{b}{3},
\end{aligned} \tag{B7}$$

其中

$$\delta=\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3=\left(\frac{\gamma_0+\gamma_1+\gamma_2}{3}\right)^3(4K\varepsilon\sqrt{\gamma_1\gamma_2}-4\gamma_0\gamma_1\gamma_2)-\frac{1}{12}\left(\frac{\gamma_0+\gamma_1+\gamma_2}{3}\right)^2K^2\varepsilon^2\left(\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}\right)^2-$$
$$\left(\frac{\gamma_0+\gamma_1+\gamma_2}{3}\right)K\varepsilon\left(\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}\right)(2K\varepsilon\sqrt{\gamma_1\gamma_2}-2\gamma_0\gamma_1\gamma_2)+\left(\frac{1}{3}K\varepsilon\frac{\gamma_1+\gamma_2}{\sqrt{\gamma_1\gamma_2}}\right)^3+(2K\varepsilon\sqrt{\gamma_1\gamma_2}-2\gamma_0\gamma_1\gamma_2)^2.$$

附录 C 有失谐时的特征方程求解

假设 $\delta\alpha_i$ 和 $\delta\alpha_i^*$ 为稳态解附近小的扰动，则(15)式的线性化方程为

(令 $\Gamma_0=\gamma_0+\mathrm{i}\Delta_0$，$\Gamma_1=\gamma_1+\mathrm{i}\Delta_1$，$\Gamma_2=\gamma_2+\mathrm{i}\Delta_2$)

$$\frac{\partial}{\partial t}\begin{bmatrix}\delta\alpha_0\\ \delta\alpha_0^*\\ \delta\alpha_1\\ \delta\alpha_1^*\\ \delta\alpha_2\\ \delta\alpha_2^*\end{bmatrix}=\begin{bmatrix}-\Gamma_0 & 0 & -K\alpha_2^0 & 0 & -K\alpha_1^0 & 0\\ 0 & -\Gamma_0^* & 0 & -K\alpha_2^{0*} & 0 & -K\alpha_1^{0*}\\ K\alpha_2^{0*} & 0 & -\Gamma_1 & 0 & 0 & K\alpha_0^0\\ 0 & K\alpha_2^0 & 0 & -\Gamma_1^* & K\alpha_0^{0*} & 0\\ K\alpha_1^{0*} & 0 & 0 & K\alpha_0^0 & -\Gamma_2 & 0\\ 0 & K\alpha_1^0 & K\alpha_0^{0*} & 0 & 0 & -\Gamma_2^*\end{bmatrix}\begin{bmatrix}\delta\alpha_0\\ \delta\alpha_0^*\\ \delta\alpha_1\\ \delta\alpha_1^*\\ \delta\alpha_2\\ \delta\alpha_2^*\end{bmatrix}, \tag{C1}$$

则决定稳态解稳定性的特征值 λ 的方程为

$$\begin{vmatrix}-\Gamma_0-\lambda & 0 & -K\alpha_2^0 & 0 & -K\alpha_1^0 & 0\\ 0 & -\Gamma_0^*-\lambda & 0 & -K\alpha_2^{0*} & 0 & -K\alpha_1^{0*}\\ K\alpha_2^{0*} & 0 & -\Gamma_1-\lambda & 0 & 0 & K\alpha_0^0\\ 0 & K\alpha_2^0 & 0 & -\Gamma_1^*-\lambda & K\alpha_0^{0*} & 0\\ K\alpha_1^{0*} & 0 & 0 & K\alpha_0^0 & -\Gamma_2-\lambda & 0\\ 0 & K\alpha_1^0 & K\alpha_0^{0*} & 0 & 0 & -\Gamma_2^*-\lambda\end{vmatrix}=0, \tag{C2}$$

以上特征方程化为

$$
\begin{aligned}
&(\Gamma_0+\lambda)(\Gamma_0^*+\lambda)(\Gamma_1+\lambda)(\Gamma_1^*+\lambda)(\Gamma_2+\lambda)(\Gamma_2^*+\lambda)-4|\alpha_0^0|^2|\alpha_1^0|^2|\alpha_2^0|^2+(\Gamma_0+\lambda)(\Gamma_0^*+\lambda)|\alpha_0^0|^4+\\
&(\Gamma_1+\lambda)(\Gamma_1^*+\lambda)|\alpha_1^0|^4+(\Gamma_2+\lambda)(\Gamma_2^*+\lambda)|\alpha_2^0|^4-(\Gamma_0+\lambda)(\Gamma_0^*+\lambda)(\Gamma_1^*+\lambda)(\Gamma_2+\lambda)|\alpha_0^0|^2-\\
&(\Gamma_0+\lambda)(\Gamma_0^*+\lambda)(\Gamma_1+\lambda)(\Gamma_2^*+\lambda)|\alpha_0^0|^2+(\Gamma_0+\lambda)(\Gamma_1+\lambda)(\Gamma_1^*+\lambda)(\Gamma_2+\lambda)|\alpha_1^0|^2+\\
&(\Gamma_0^*+\lambda)(\Gamma_1+\lambda)(\Gamma_1^*+\lambda)(\Gamma_2^*+\lambda)|\alpha_1^0|^2+(\Gamma_0+\lambda)(\Gamma_1+\lambda)(\Gamma_2+\lambda)(\Gamma_2^*+\lambda)|\alpha_2^0|^2+\\
&(\Gamma_0^*+\lambda)(\Gamma_1^*+\lambda)(\Gamma_2+\lambda)(\Gamma_2^*+\lambda)|\alpha_2^0|^2-(\Gamma_0+\lambda)(\Gamma_1+\lambda)|\alpha_0^0|^2|\alpha_1^0|^2-\\
&(\Gamma_0^*+\lambda)(\Gamma_1^*+\lambda)|\alpha_0^0|^2|\alpha_1^0|^2-(\Gamma_0+\lambda)(\Gamma_2+\lambda)|\alpha_0^0|^2|\alpha_2^0|^2-(\Gamma_0^*+\lambda)(\Gamma_2^*+\lambda)|\alpha_0^0|^2|\alpha_2^0|^2+\\
&(\Gamma_1^*+\lambda)(\Gamma_2+\lambda)|\alpha_1^0|^2|\alpha_2^0|^2+(\Gamma_1+\lambda)(\Gamma_2^*+\lambda)|\alpha_1^0|^2|\alpha_2^0|^2=0.
\end{aligned}
$$

将 $\Gamma_0=\gamma_0+\mathrm{i}\Delta_0$, $\Gamma_1=\gamma_1+\mathrm{i}\Delta_1$, $\Gamma_2=\gamma_2+\mathrm{i}\Delta_2$ 代入上式，得

$$
\begin{aligned}
&[(\gamma_0+\lambda)^2+\Delta_0^2][(\gamma_1+\lambda)^2+\Delta_1^2][(\gamma_2+\lambda)^2+\Delta_2^2]-4|\alpha_0^0|^2|\alpha_1^0|^2|\alpha_2^0|^2+[(\gamma_0+\lambda)^2+\Delta_0^2]|\alpha_0^0|^4+\\
&[(\gamma_1+\lambda)^2+\Delta_1^2]|\alpha_1^0|^4+[(\gamma_2+\lambda)^2+\Delta_2^2]|\alpha_2^0|^4-2[(\gamma_0+\lambda)^2+\Delta_0^2][(\gamma_1+\lambda)(\gamma_2+\lambda)+\Delta_1\Delta_2]|\alpha_0^0|^2+\\
&2[(\gamma_1+\lambda)^2+\Delta_1^2][(\gamma_0+\lambda)(\gamma_2+\lambda)-\Delta_0\Delta_2]|\alpha_1^0|^2+2[(\gamma_2+\lambda)^2+\Delta_2^2][(\gamma_0+\lambda)(\gamma_1+\lambda)-\Delta_0\Delta_1]|\alpha_2^0|^2-\\
&2[(\gamma_0+\lambda)(\gamma_1+\lambda)-\Delta_0\Delta_1]|\alpha_0^0|^2|\alpha_1^0|^2-2[(\gamma_0+\lambda)(\gamma_2+\lambda)-\Delta_0\Delta_2]|\alpha_0^0|^2|\alpha_2^0|^2+\\
&2[(\gamma_1+\lambda)(\gamma_2+\lambda)+\Delta_1\Delta_2]|\alpha_1^0|^2|\alpha_2^0|^2=0. \qquad \text{(C3)}
\end{aligned}
$$