文章编号：0253-2239(2009)06-1459-08

带有完美匹配层的光波导中模式的渐近解

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摘要 对带有完美匹配层(PML)的光波导，通过 W. K. B. 方法，得到了关于波的模式的近似非线性方程。通过对该非线性方程进行渐近分析，得到了完美匹配层对波的各种模式的影响，推导出泄漏模的渐近解和 Berenger 模的渐近解。理论结果表明，对于光波导的横电(TE)模，泄漏模的渐近解和开放的波导是相同的，而对于横磁(TM)模，泄漏模的渐近解和开波导会有所不同；当上下包层的折射率不同时，Berenger 模的轨迹会有两条，而折射率相同时 Berenger 模的轨迹只有一条。数值模拟表明，泄漏模及 Berenger 模的渐近解在其精确解附近，且随着模的增大，误差越来越小。

关键词 导波光学；光波导计算；渐近解；W. K. B. 方法；完美匹配；泄漏模

中图分类号 O436 **文献标识码** A **doi:** 10.3788/AOS20092906.1459

Aysmptotic Solutions of Modes in Optical Waveguide with Perfectly Matched Layer

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Abstract Approximate nonlinear equation about modes is derived by W. K. B. method for waveguide with perfectly matched layer (PML). Asymptotic solutions for the leaky modes and Berenger modes are derived and the effect of PML is analyzed. Theoretical analysis shows that for leaky modes of transverse electric (TE) case, asymptotic solutions of waveguide are the same as that of open waveguide, but for transverse magnetic (TM) case, asymptotic solutions of waveguide are different from that of open waveguide. For both TE and TM cases of three-layer slab waveguide, if refractive index of top cladding is different form that of bottom cladding, there will be two traces of Berenger modes, otherwise there is only one trace of Berenger modes. Numerical simulations show that these asymptotic solutions of leaky modes and Berenger modes are very close to exact solutions and as the norms of modes become larger, the error will become smaller.

Key words guided-wave optics; computing in optical waveguide; asymptotic solutions; W. K. B method; perfectly matched layer (PML); leaky mode

1 引 言

在光通信、光传感和光信息处理中，光波导是光集成器件中最基本的器件。在光波导中传播计算的方法有有限元、时域有限差分方法以及子域合成法[1]、模式匹配法[2]等。在波的传播计算中，泄漏模非常重要，它代表消逝模和辐射模的连续谱。对于开放的光波导，现在比较有效的方法是采用完美匹配层(PML)[3,4]把开放的波导区域截断为有界的区域，在采用了完美匹配层之后，除了传播模和泄漏模之外，还会产生一种模，称之为 Berenger 模；同时完美匹配层的采用还会对泄漏模产生一定的影响。对于横变量方向一边无界的两层波导，Rogier He 等[5~7]推导出了关于泄漏模及 PML 模的近似解，Gaal Szabolcs B 等[8]给出了二维分层介质 PML 模的解。对于两边均无界的三层平板光波导，在折射率为分段常量时，可通过一个非线性方程推导出泄

收稿日期： 2008-08-28；**收到修改稿日期：** 2008-09-23

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漏模的渐近解[9]；然而在很多情况下折射率会变化的[10]，有时还可能是复数[11]。在折射率随横变量变化时，通过应用W. K. B方法得到泄漏模的渐近解[12]；这些渐近解可作为瑞利(Rayleigh)商迭代的初值求解更精确的泄漏模。

本文推导了采用完美匹配层之后横电(TE)波和横磁(TM)波泄漏模的渐近解，给出了 Berenger 模的渐近公式。

2 基本方程

对于二维折射率可变的光波导，中间为芯层，上下为包层，折射率为

$$n(x)=\begin{cases}n_1(x), & x<0\\ n_2(x), & x>d\\ n_0(x), & 0<x<d\end{cases} \tag{1}$$

式中 d 为芯层的厚度，$n_0(x)$ 为芯层折射率，$n_2(x)$ 为上包层折射率，$n_1(x)$ 下包层折射率。记

$$\begin{aligned}&n_0(0)=n_0(0^+), && n_0(d)=n_0(d^-),\\ &n_1(0)=n_1(0^-), && n_2(d)=n_2(d^+),\\ &n_0'(0)=n_0'(0^+), && n_0'(d)=n_0'(d^-),\\ &n_1'(0)=n_1'(0^-), && n_2'(d)=n_2'(d^+).\end{aligned}$$

要数值求解光波导上下无界的亥姆霍兹(Helmholtz)方程，必须在 x 方向进行截断，需应用 PML 技术，在数学上它相当于作一个坐标变换：$\hat{x}=\int_0^x f(t)\mathrm{d}t$。在三层波导中，

$$f(t)=1+\mathrm{i}\sigma(t),$$

$$\sigma(t)=\begin{cases}c_1\dfrac{t_1^3}{1+t_1^2},\\ c_2\dfrac{t_2^3}{1+t_2^2},\\ 0,\end{cases} \tag{2}$$

式中

$$t_1=\frac{x-d_1}{d_2-d_1},\quad d_1\leqslant x\leqslant d_2$$

$$t_2=\frac{x-d_4}{d_3-d_4},\quad d_4\leqslant x\leqslant d_3$$

$$d_3\leqslant x\leqslant d_1$$

d_1 为上包层厚度，d_2 为上 PML 层厚度，d_3 为下包层厚度，d_4 为下 PML 层厚度。

对于三层波导，在加了 PML 之后，原 Helmholtz 方程为

$$\begin{cases}u_{zz}+\rho_0\dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho_0}\dfrac{\partial u}{\partial x}\right)+\kappa_0^2n_0^2(z,x)u=0, & 0\leqslant x\leqslant d\\ u_{zz}+\rho_1\dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho_1}\dfrac{\partial u}{\partial x}\right)+\kappa_0^2n_1^2(z,x)u=0, & d_3\leqslant x\leqslant 0\\ u_{zz}+\rho_2\dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho_2}\dfrac{\partial u}{\partial x}\right)+\kappa_0^2n_2^2(z,x)u=0, & d\leqslant x\leqslant d_1\\ u_{zz}+\dfrac{\rho_1}{1+\mathrm{i}\sigma(x)}\dfrac{\partial}{\partial x}\left\{\dfrac{1}{\rho_1[1+\mathrm{i}\sigma(x)]}\dfrac{\partial u}{\partial x}\right\}+\kappa_0^2n_1^2(z,x)u=0, & d_4\leqslant x\leqslant d_3\\ u_{zz}+\dfrac{\rho_2}{1+\mathrm{i}\sigma(x)}\dfrac{\partial}{\partial x}\left\{\dfrac{1}{\rho_2[1+\mathrm{i}\sigma(x)]}\dfrac{\partial u}{\partial x}\right\}+\kappa_0^2n_2^2(z,x)u=0, & d_1\leqslant x\leqslant d_2\\ u(z,d^-)=u(z,d^+),\\ \dfrac{1}{\rho_0}\dfrac{\partial u}{\partial x}(z,d^-)=\dfrac{1}{\rho_2}\dfrac{\partial u}{\partial x}(z,d^+),\\ u(z,0^-)=u(z,0^+),\\ \dfrac{1}{\rho_0}\dfrac{\partial u}{\partial x}(z,0^+)=\dfrac{1}{\rho_1}\dfrac{\partial u}{\partial x}(z,0^-),\\ u(z,d_2)=0,\\ u(z,d_4)=0,\end{cases} \tag{3}$$

当 $\rho_i=1(i=1,2,3)$ 时为 TE 波，当 $\rho_i=n_i^2(x)(i=1,2,3)$ 时为 TM 波。

令 $u=\phi(x)\exp[\mathrm{i}(\sqrt{k}z-\omega t)]$，这里 k 是传播常数，ω 是角频率，并且设 $n(z,x)=n(x)$，得到对应的 算子的特征问题变为

$$
\begin{cases}
\rho_0 \dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho_0}\dfrac{\partial \phi}{\partial x}\right)+\kappa_0^2 n_0^2(x)\phi=\lambda\phi, & 0\leqslant x\leqslant d \\
\rho_1 \dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho_1}\dfrac{\partial \phi}{\partial x}\right)+\kappa_0^2 n_1^2(x)\phi=\lambda\phi, & d_3\leqslant x\leqslant 0 \\
\rho_2 \dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho_2}\dfrac{\partial \phi}{\partial x}\right)+\kappa_0^2 n_2^2(x)\phi=\lambda\phi, & d\leqslant x\leqslant d_1 \\
\dfrac{\rho_1}{1+\mathrm{i}\sigma(x)}\dfrac{\partial}{\partial x}\left[\dfrac{1}{\rho_1[1+\mathrm{i}\sigma(x)]}\dfrac{\partial \phi}{\partial x}\right]+\kappa_0^2 n_1^2(x)\phi=\lambda\phi, & d_4\leqslant x\leqslant d_3 \\
\dfrac{\rho_2}{1+\mathrm{i}\sigma(x)}\dfrac{\partial}{\partial x}\left[\dfrac{1}{\rho_2[1+i\sigma(x)]}\dfrac{\partial \phi}{\partial x}\right\}+\kappa_0^2 n_2^2(x)\phi=\lambda\phi, & d_1\leqslant x\leqslant d_2 \\
\phi(d^-)=\phi(d^+), & \\
\dfrac{1}{\rho_0}\dfrac{\partial \phi}{\partial x}(d^-)=\dfrac{1}{\rho_2}\dfrac{\partial \phi}{\partial x}(d^+), & \\
\phi(0^-)=\phi(0^+), & \\
\dfrac{1}{\rho_0}\dfrac{\partial \phi}{\partial x}(0^+)=\dfrac{1}{\rho_1}\dfrac{\partial \phi}{\partial x}(0^-), & \\
\phi(d_2)=0, & \\
\phi(d_4)=0, &
\end{cases}
\tag{4}
$$

3　带有 PML 的光波导 TE 波渐近解的研究

TE 波算子的特征 ϕ 的通解可通过 W. K. B. 方法得出，在应用边界条件及界面条件后，可以得到

$$
\exp\left[2\mathrm{i}\int_0^d \sqrt{\kappa_0^2 n_0^2(t)-k}\,\mathrm{d}t\right]=\frac{\gamma_{0d}+\gamma_{2d}+(\gamma_{2d}-\gamma_{0d})\exp\left\{2\mathrm{i}\int_d^{d_2}\sqrt{\kappa_0^2 n_2^2(t)-k}[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\}}{\gamma_{0d}-\gamma_{2d}-(\gamma_{2d}+\gamma_{0d})\exp\left\{2\mathrm{i}\int_d^{d_2}\sqrt{\kappa_0^2 n_2^2(t)-k}[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\}}\times
$$

$$
\frac{\gamma_{00}-\gamma_{10}-(\gamma_{00}+\gamma_{10})\exp\left\{2\mathrm{i}\int_0^{d_4}\sqrt{\kappa_0^2 n_1^2(t)-k}[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\}}{\gamma_{00}+\gamma_{10}+(\gamma_{10}-\gamma_{00})\exp\left\{2\mathrm{i}\int_0^{d_4}\sqrt{\kappa_0^2 n_2^2(t)-k}[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\}},
\tag{5}
$$

式中 $\hat{d}_2=d_2+\mathrm{i}\int_0^{d_2}\sigma(t)\mathrm{d}t$，$\hat{d}_4=d_4+\mathrm{i}\int_0^{d_4}\sigma(t)\mathrm{d}t$，$\gamma_{00}=\sqrt{\kappa_0^2 n_0^2(0)-k}$，$\gamma_{10}=\sqrt{\kappa_0^2 n_1^2(0)-k}$，$\gamma_{2d}=\sqrt{\kappa_0^2 n_2^2(d)-k}$，$\gamma_{0d}=\sqrt{\kappa_0^2 n_0^2(d)-k}$。

(5) 式可变为

$$
\frac{\gamma_{0d}+\mathrm{i}\cot[\gamma_{2d}(\hat{d}_2-d)]\gamma_{2d}+\kappa_0^2 n_2(d)n'_2(d)/(2\gamma_{2d}^2)-\kappa_0^2 n_0(d)n'_0(d)/(2\gamma_{0d}^2)}{\gamma_{0d}-\mathrm{i}\cot[\gamma_{2d}(\hat{d}_2-d))\gamma_{2d}-\kappa_0^2 n_2(d)n'_2(d)/(2\gamma_{2d}^2)+\kappa_0^2 n_0(d)n'_0(d)/(2\gamma_{0d}^2)}=
$$

$$
\exp\left[2\mathrm{i}\int_0^d \sqrt{\kappa_0^2 n_0^2(t)-k}\,\mathrm{d}t\right]\frac{\gamma_{00}+\mathrm{i}\cot[\gamma_{10}(\hat{d}_4)]\gamma_{10}+\kappa_0^2 n_1(0)n'_1(0)/(2\gamma_{10}^2)-\kappa_0^2 n_0(0)n'_0(0)/(2\gamma_{00}^2)}{\gamma_{00}-\mathrm{i}\cot[\gamma_{10}(\hat{d}_4)]\gamma_{10}-\kappa_0^2 n_1(0)n'_1(0)/(2\gamma_{10}^2)+\kappa_0^2 n_0(0)n'_0(0)/(2\gamma_{00}^2)},
\tag{6}
$$

更进一步得到

$$
\cot(\gamma_{0d}d)=\frac{\gamma_{00}\gamma_{0d}+\gamma_{2d}\gamma_{10}\cot[\gamma_{2d}(\hat{d}_2-d)]\cot(\gamma_{10}\hat{d}_4)}{-\gamma_{0d}\gamma_{10}\cot(\gamma_{10}\hat{d}_4)+\gamma_{2d}\gamma_{00}\cot[\gamma_{2d}(\hat{d}_2-d)]},
\tag{7}
$$

即

$$
\begin{aligned}
&\gamma_{0d}\gamma_{10}\cot(-\gamma_{10}\hat{d}_4)\cot(\gamma_{0d}d)+\gamma_{00}\gamma_{2d}\cot[\gamma_{2d}(\hat{d}_2-d)]\cot(\gamma_{0d}d)= \\
&\gamma_{00}\gamma_{0d}+\gamma_{2d}\gamma_{10}\cot[\gamma_{2d}(\hat{d}_2-d)]\cot(\gamma_{10}\hat{d}_4),
\end{aligned}
\tag{8}
$$

令 $k = k_n = |k_n|\exp(\mathrm{i}\theta_n)$，因此

$$\gamma_{10} = |k_n|^{1/2}\exp[\mathrm{i}(\theta_n - \pi)/2]\sqrt{1-\kappa_0^2 n_1^2(0)/k_n},$$
$$-\hat{d}_4 = \hat{d}_2 - d = |\hat{d}_4|\exp(\mathrm{i}\theta),$$
$$-\gamma_{10}\hat{d}_4 = |\hat{d}_4||k_n|^{1/2}\exp\{\mathrm{i}[(\theta_n-\pi)/2+\phi]\}\sqrt{1-\kappa_0^2 n_1^2(0)/k_n},$$
$$\gamma_{2d}(\hat{d}_2 - d) = |\hat{d}_2 - d||k_n|^{1/2}\exp\{\mathrm{i}[(\theta_n-\pi)/2+\phi]\}\sqrt{1-\kappa_0^2 n_2^2(d)/k_n},$$

如果 $-\gamma_{10}\hat{d}_4$，$\gamma_{2d}(\hat{d}_2-d)$ 和 $\gamma_{0d}d$ 虚部都趋向于无穷，由(8)式会矛盾，因此这三个量的虚部不能同时都趋向于无穷，这样就有以下两种情况：

1) $\theta_n \to \theta_* = \pi$，得到和开波导一致的结果[12]。

2) $\theta_n \to \theta_* = \pi - 2\phi$ 时得到

$$[\mathrm{i}\gamma_{00} + \gamma_{10}\cot(-\gamma_{10}\hat{d}_4)]\{\mathrm{i}\gamma_{0d} + \gamma_{2d}\cot[\gamma_{2d}(\hat{d}_2 - d)]\} = 0,$$

由此得出

$$\cot(-\gamma_{10}\hat{d}_4) = -\mathrm{i}(\gamma_{00}/\gamma_{10}), \tag{9}$$
$$\cot[\gamma_{2d}(\hat{d}_2 - d)] = -\mathrm{i}(\gamma_{0d}/\gamma_{2d}), \tag{10}$$

由(9)式可以得到 $we^w = \pm\mathrm{i}(a_0/2)\sqrt{\delta_{10}}$ [13]，$w = a_0\gamma_{10} + a_2/\gamma_{10}^2 + a_3/\gamma_{10}^3 + \cdots$，$a_0 = -\mathrm{i}\hat{d}_4$，$b_2 = \delta_{10}/4$，$b_4 = \delta_{10}^2/32$，$a_2 = b_2$，$a_3 = -a_2/a_0$，$a_4 = -(a_0 b_4 - a_3)/a_0$，$\delta_{10} = \kappa_0^2[n_0^2(0) - n_1^2(0)]$。

由于 δ_{10} 已知，可求得 w，由 $w = a_0\gamma_{10} + a_2/\gamma_{10}^2 + a_3/\gamma_{10}^3 + \cdots$，当只取第一项时得到主要渐近公式为

$$\gamma_{10} = w/a_0, \tag{11}$$

从而求出 γ_{10}，由 $\gamma_{10} = \sqrt{\kappa_0^2 n_1^2(0) - k}$ 进而可求得 k 的渐近解。

由(10)式可以得到 $w\mathrm{e}^w = \pm\mathrm{i}(\hat{a}_0/2)\sqrt{\delta_{2d}}$，这里 $w = \hat{a}_0\gamma_{2d} + \hat{a}_2/\gamma_{2d}^2 + \hat{a}_3/\gamma_{2d}^3 + \cdots$，$\hat{a}_0 = \mathrm{i}(\hat{d}_2 - d)$，$\hat{b}_2 = \delta_{2d}/4$，$\hat{b}_4 = -\delta_{2d}^2/32$，$\hat{a}_2 = \hat{b}_2$，$\hat{a}_3 = -\hat{a}_2/\hat{a}_0$，$\hat{a}_4 = -(\hat{a}_0\hat{b}_4 - \hat{a}_3)/\hat{a}_0$，$\delta_{2d} = \kappa_0^2[n_0^2(0) - n_2^2(d)]$。由于 δ_{2d} 已知，可求得 w，由 $w = \hat{a}_0\gamma_{2d} + \hat{a}_2/\gamma_{2d}^2 + \hat{a}_3/\gamma_{2d}^3 + \cdots$，当只取其第一项时，得到主要的近似项：

$$\gamma_{2d} = w/\hat{a}_0, \tag{12}$$

从而求出 γ_{2d}，由三 $\gamma_{2d} = \sqrt{\kappa_0^2 n_2^2(d) - k}$ 进而可求得 k 的渐近解。

4 带有 PML 的光波导 TM 波渐近解

TM 波对应的算子的特征问题中，令 $n(x) = \phi(x)p(x)$，应用 W. K. B 方法及上述条件并经过一系列变化后可得到

$$\exp\left[2\mathrm{i}\int_0^d s_0(t)\mathrm{d}t\right] = \frac{\mathrm{i}\dfrac{s_0(d)}{n_0^2(d)} - \dfrac{n_0'(d)}{n_0^3(d)} + \dfrac{n_2'(d)}{n_2^3(d)} + \mathrm{i}\dfrac{s_2(d)}{n_2^2(d)}\mathrm{i}\cot\left\{\displaystyle\int_d^{d_2} s_2(t)[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\}}{\mathrm{i}\dfrac{s_0(d)}{n_0^2(d)} + \dfrac{n_0'(d)}{n_0^3(d)} - \dfrac{n_2'(d)}{n_2^3(d)} - \mathrm{i}\dfrac{s_2(d)}{n_2^2(d)}\mathrm{i}\cot\left\{\displaystyle\int_d^{d_2} s_2(t)[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\}} \times$$
$$\frac{\mathrm{i}\dfrac{s_0(0)}{n_0^2(0)} + \dfrac{n_0'(0)}{n_0^3(0)} - \dfrac{n_1'(0)}{n_1^3(0)} - \mathrm{i}\dfrac{s_1(0)}{n_1^2(0)}\mathrm{i}\cot\left\{\displaystyle\int_0^{d_4} s_1(t)[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\}}{\mathrm{i}\dfrac{s_0(0)}{n_0^2(0)} - \dfrac{n_0'(0)}{n_0^3(0)} + \dfrac{n_1'(0)}{n_1^3(0)} + \mathrm{i}\dfrac{s_1(0)}{n_1^2(0)}\mathrm{i}\cot\left\{\displaystyle\int_0^{d_4} s_1(t)[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\}}, \tag{13}$$

式中

$$s_0(t) = \sqrt{\kappa_0^2 n_0^2(t) + \gamma_0''(t) - [\gamma_0'(t)]^2 - k}, \quad s_1(t) = \sqrt{\kappa_0^2 n_1^2(t) + \gamma_1''(t) - [\gamma_1'(t)]^2 - k},$$
$$s_2(t) = \sqrt{\kappa_0^2 n_2^2(t) + \gamma_2''(t) - [\gamma_2'(t)]^2 - k},$$
$$\gamma_0(x) = \ln[n_0(x)], \quad \gamma_1(x) = \ln[n_1(x)], \quad \gamma_2(x) = \ln[n_2(x)].$$

令

$$p_1 = \mathrm{i}\frac{s_0(d)}{n_0^2(d)}, \quad q_1 = \frac{n_0'(d)}{n_0^3(d)} - \frac{n_2'(d)}{n_2^3(d)} - \mathrm{i}\frac{s_2(d)}{n_2^2(d)}\mathrm{i}\cot\left\{\int_d^{d_2} s_2(t)[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\},$$

$$p_2 = \mathrm{i}\,\frac{s_0(0)}{n_0^2(0)},\quad q_2 = \frac{n'_0(0)}{n_0^3(0)} - \frac{n'_1(0)}{n_1^3(0)} - \mathrm{i}\,\frac{s_1(0)}{n_1^2(0)}\mathrm{i}\cot\left\{\int_0^{d_4} s_1(t)[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\},$$

可得到

$$\cot\left[\int_0^d s_0(t)\mathrm{d}t\right] = \mathrm{i}\,\frac{p_1 p_2 - q_1 q_2}{p_1 q_2 - p_2 q_1}, \tag{14}$$

和 TE 波推导情况相同，可以得到以下三种情况：

1) $$\exp\left[2\mathrm{i}\int_0^d s_0(t)\mathrm{d}t\right] = \frac{\mathrm{i}\,\frac{s_0(d)}{n_0^2(d)} - \left[\frac{n'_0(d)}{n_0^3(d)} - \frac{n'_2(d)}{n_2^3(d)} - \mathrm{i}\,\frac{s_2(d)}{n_2^2(d)}\right]}{\mathrm{i}\,\frac{s_0(d)}{n_0^2(d)} + \left[\frac{n'_0(d)}{n_0^3(d)} - \frac{n'_2(d)}{n_2^3(d)} - \mathrm{i}\,\frac{s_2(d)}{n_2^2(d)}\right]} \times \frac{\mathrm{i}\,\frac{s_0(0)}{n_0^2(0)} + \left[\frac{n'_0(0)}{n_0^3(0)} - \frac{n'_1(0)}{n_1^3(0)} + \mathrm{i}\,\frac{s_1(0)}{n_1^2(0)}\right]}{\mathrm{i}\,\frac{s_0(0)}{n_0^2(0)} - \left[\frac{n'_0(0)}{n_0^3(0)} - \frac{n'_1(0)}{n_1^3(0)} + i\,\frac{s_1(0)}{n_1^2(0)}\right]}, \tag{15}$$

2) $$\cot\left\{\int_d^{d_2} s_2(t)[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\} = \mathrm{i}\left[-\frac{n_2^2(d)}{n_0^2(d)}\frac{s_0(d)}{s_2(d)} + \mathrm{i}\,\frac{n'_0(d)}{n_0^3(d)}\frac{n_2^2(d)}{s_2(d)} - \mathrm{i}\,\frac{n'_2(d)}{n_2(d)s_2(d)}\right] \triangleq \mathrm{i}M_1, \tag{16}$$

3) $$\cot\left\{-\int_0^{d_4} s_1(t)[1+\mathrm{i}\sigma(t)]\mathrm{d}t\right\} = -\mathrm{i}\left[\frac{n_1^2(0)}{n_0^2(0)}\frac{s_0(0)}{s_1(0)} + \mathrm{i}\,\frac{n'_0(0)}{n_0^3(0)}\frac{n_1^2(0)}{s_1(0)} - \mathrm{i}\,\frac{n'_1(0)}{n_1(0)s_1(0)}\right] \triangleq \mathrm{i}M_2, \tag{17}$$

可见，由于使用了 PML，第 1)种情况的结果和开波导时稍有不同[12]；同时又产生了第 2)和第 3)种情况。

由第 1)种情况得到：

$$\exp\left[2\mathrm{i}\int_0^d s_0(t)\mathrm{d}t\right] \approx \left[c_0 + \frac{c_1}{s_0(0)} + \cdots\right]\left[e_0 + \frac{e_1}{s_0(0)} + \cdots\right] \approx c_0 e_0 + \frac{1}{s_0(0)}(c_1 e_0 + c_0 e_1) + \cdots \approx A_0 + A_1/s_0(0) + \cdots, \tag{18}$$

其中

$$c_0 = \frac{n_1^2(0) + n_0^2(0)}{n_1^2(0) - n_0^2(0)},\quad c_1 = \frac{-2in_0^2(0)n_1^4(0)}{[n_1^2(0) - n_0^2(0)]^2}\left[\frac{n'_0(0)}{n_0^3(0)} - \frac{n'_1(0)}{n_1^3(0)}\right],$$

$$e_0 = \frac{n_2^2(d) + n_0^2(d)}{n_2^2(d) - n_0^2(d)},\quad e_1 = \frac{2\mathrm{i}n_0^2(d)n_2^4(d)}{(n_2^2(d) - n_0^2(d))^2}\left[\frac{n'_0(d)}{n_0^3(d)} - \frac{n'_2(d)}{n_2^3(d)}\right],$$

$$A_0 = c_0 e_0,\qquad A_1 = c_1 e_0 + c_0 e_1.$$

推导同文献[9,12]，(18)式可变为

$$\exp[2\mathrm{i}ds_0(0)] \approx A_0 + \frac{A_1}{s_0(0)} + \cdots, \tag{19}$$

从而有

$$2\mathrm{i}ds_0(0) \approx 2\mathrm{i}\pi m + \ln A_0 + \frac{t_1}{s_0(0)} + \cdots,$$

$$t_1 = A_1 . A_0.$$

可以得到主要近似项为

$$s_0(0) \approx \frac{\pi m}{d} - \frac{\mathrm{i}\ln A_0}{2d} = K_m,$$
$$k = \kappa_0^2 n_0^2(0) + \gamma''_0(0) + [\gamma'_0(0)]^2 - K_m^2, \tag{20}$$

二阶近似项为

$$s_0(0) \approx K_m - \frac{\mathrm{i}t_1}{2dK_m},$$
$$k = \kappa_0^2 n_0^2(0) + \gamma''_0(0) + [\gamma'_0(0)]^2 - K_{m1}^2, \tag{21}$$

由第 2) 种情况得到

$$2\mathrm{i}s_2(d)(\hat{d}_2 - d) \approx \mathrm{i}(2n-1)\pi + \ln\left[\frac{n_0^2(d) - n_2^2(d)}{n_0^2(d) + n_2^2(d)}\right] \triangleq \hat{A}_0,$$

所以

$$k = \frac{\hat{A}_0^2}{4(\hat{d}_2 - d)^2} + \kappa_0^2 n_2^2(d) + \gamma''_2(d) - [\gamma'_2(d)]^2, \tag{22}$$

由第 3)种情况得到

$$-2\mathrm{i}s_1(0)(\hat{d}_4) \approx \mathrm{i}(2n-1)\pi + \ln\left[\frac{n_0^2(0) - n_1^2(0)}{n_0^2(0) + n_1^2(0)}\right] \triangleq \hat{B}_0,$$

所以

$$k = \frac{\hat{B}_0^2}{4(\hat{d}_4)^2} + \kappa_0^2 n_1^2(0) + \gamma''_1(0) - [\gamma'_1(0)]^2. \tag{23}$$

5　带有 PML 模的两层波导 TE 波渐近解

文献[7]给出了两层波导的 PML 模的渐近公式。同样地，本文也推导出了两层波导 TE 模的渐近公式。为了比较的方便，这里取与文献[7]一样的参量，芯层的相对介电常量和磁导率分别为 ε_r，μ_r，

空气中的介电常量和磁导率为 ε_0，μ_0；*PML* 层选取同文献[7]，本文也得到两种结果：

1) $\cot(\gamma_{0d}d)=\mathrm{i}\dfrac{r_{2d}\mu_r}{r_{0d}}$，对应于泄漏模的传播常量，在 $n(x)$ 为常数且 $\mu_r=1$ 时该结果和文献[7]相同。

2) $\cot(\gamma_{2d}[\hat{d}_2-d)]=-\mathrm{i}\dfrac{r_{0d}}{r_{2d}\mu_r}$，当 $\mu_r=1$ 时，

$$r_{2d}=\frac{(2m+1)\pi\mathrm{i}}{A_0}+\ln\left(\frac{\mu_r-1}{\mu_r+1}\right)\Big/A_0=K_m,$$

$$r_{2d}=K_m-\frac{a_0}{A_0K_m^2},\tag{24}$$

式中 $A_0=2\mathrm{i}(\hat{d}_2-d)$，$a_0=\dfrac{\delta_{2d}\mu_r}{\mu_r^2-1}$；

当 $\mu_r\neq1$ 时，

$$\begin{aligned}r_{2d}&=w/a_0,\\ r_{2d}&=w/a_0-a_0^2a_2/w^2,\end{aligned}\tag{25}$$

这里 $a_0=\mathrm{i}(\hat{d}_2-d)$，$a_2=b_2=\dfrac{\delta_{2d}}{4}$，$w\mathrm{e}^w=-\dfrac{a_0}{2}\sqrt{\delta_{2d}}$，进而可得到 Berenger 模的渐近解。

6 数值模拟

1) 当 $n(x)$ 是常数时，当波导是两层时，把本文的结果和文献[7]中的结果进行比较，如表 1 和表 2 所示。表 1 和表 2 中第 1 列表示精确的解，第 2 列表示由文献[7]中得到渐近解的相对误差，第 3 列表示本文中得到的渐近解的相对误差。

例 1

$d=9$ mm，$d_1=14$ mm，$d_2=17.5$ mm，$f=12$ GHz，PML 参量的选取同文献[7]，即 $\hat{d}_2=d_1+(d_2-d_1)(10-8\mathrm{i})$，$\mu_r=1$，$\varepsilon_r=3$。

例 2

$d=9$ mm，$d_1=14$ mm，$d_2=17.5$ mm，$f=12$ GHz，PML 参数的选取同文献[7]，即 $\hat{d}_2=d_1+(d_2-d_1)(10-8\mathrm{i})$，$\mu_r=2$，$\varepsilon_r=3$。

表 1 两层波导 TE 波中几个 Berenger 模的渐近解的相对误差

Table 1 The relative errors of asymptotic solution of some Berenger modes for TE case of two-layer slab waveguide

Exact solutions ($\sqrt{k}/\kappa_0$)	Relative error of Ref. [7]	Relative errors of formula (24)
2.8830−4.2447i	0.0189	0.0005
5.7481−8.5132i	0.0047	0.0005
8.6488−12.741i	0.0021	0.0005
11.562−16.956i	0.0012	0.0005
14.480−21.165i	0.0008	0.0005

表 2 两层波导 TE 波中几个 Berenger 模的渐近解的相对误差

Table 2 The relative errors of asymptotic solution of some Berenger modes for TE case of two-layer slab waveguide

Exact solutions ($\sqrt{k}/\kappa_0$)	Relative error of Ref. [7]	Relative errors of formula (24)
3.0237−4.5232i	0.0096	0.0149
5.9357−8.5198i	0.0025	0.0035
8.8635−12.687i	0.0021	0.0005
11.795−16.889i	0.0012	0.0005
14.729−21.088i	0.0008	0.0005

可见，本文方法得到的结果比文献[7]得到的结果更精确。

例 3

波导为三层波导时，$d=18$ mm，$d_1=23$ mm，$d_2=26.5$ mm，$f=10$ GHz，PML 参量的选取同文献[7]，即 $\hat{d}_2=d_1+(d_2-d_1)(30-10\mathrm{i})$，$\mu_r=1$，$\varepsilon_r=3$。这里只比较 PML 模，结果见图 1。

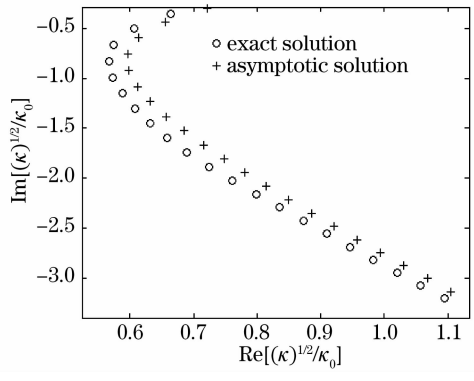

图 1 当 $n(x)$ 为常量时，TM 波中前几个 Berenger 模的比较

Fig. 1 Comparison of some Berenger modes for TM case when $n(x)$ is constant

2) 当 $n(x)$ 不是常量时

例 4

对于 TE 模，令 $n_0(x)=\sqrt{x+\kappa_0^2+20}/\kappa_0$，$n_1(x)=n_2(x)=1$，$\kappa_0=2\pi/\lambda$，$\lambda=6$ μm，$d=2$，$d_1=3$，$d_2=3.5$，$d_3=-1$，$d_4=-1.5$。该例子精确解可由艾里(Airy)函数给出，结果见图 2。

3) 无法给出精确解的例子，取 $c=100$，离散点的数目是 1000，PML 层的厚度为 0.5 时得到的解作为精确解。

例 5

令 $n_0(x)=3.3\{1-0.01[(x-1)/2.5]^2\}$，$n_1(x)=2.1$，$n_2(x)=3.17$，$\kappa_0=2\pi/\lambda$，$\lambda=1.55$ μm，$d=2$，$d_1=3$，$d_2=3.5$，$d_3=-1$，$d_4=-1.5$。由于折射率 $n_1(x)$ 和 $n_2(x)$ 取值的不同，会出现两条

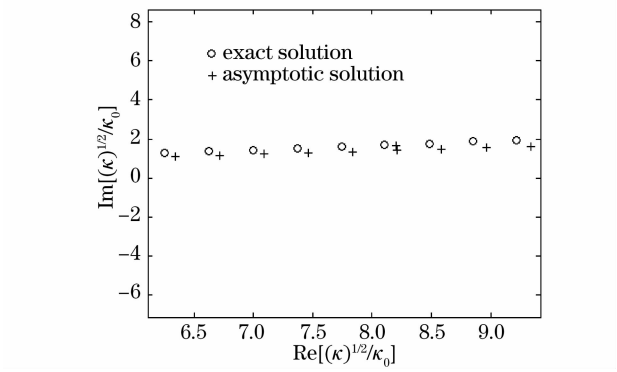

图 2　当 $n(x)$ 不是常量时，TM 模中前几个 Berenger 模的比较

Fig. 2　Comparison of some Berenger modes for TM case when $n(x)$ isn't constant

Berenger 模的轨迹。

表 3～表 6 中第 1 列也是表示精确解，第 2 列表示由本文得到的渐近解，第 3 列表示该渐近解的相对误差。

首先计算 TE 模，表 3、表 4 列举前几个 Berenger 模的渐近解及其相对误差。

表 3　TE 波中几个 Berenger 模的渐近解渐近解及其相对误差

Table 3　The relative errors of some Berenger modes for TE case

Exact solutions ($\sqrt{k}$)	Asymptotic solutions ($\sqrt{k}$) from formula (11)	Relative eeeor of asymptotic solutions from formula (11)
8.5216+0.0036i	8.5087+0.0112i	0.0018
8.5481+0.0143i	8.5265+0.0270i	0.0029
8.5920+0.0319i	8.5615+0.0464i	0.0038
8.6531+0.0560i	8.6143+0.0702i	0.0048
8.7309+0.0860i	8.6846+0.0990i	0.0055
8.8246+0.1213i	8.7720+0.1327i	0.0061

表 4　TE 波中几个 Berenger 模的渐近解及其相对误差

Table 4　Asymptotic solutions and the relative errors of some Berenger modes for TE case

Exact solutions ($\sqrt{k}$)	Asymptotic solutions ($\sqrt{k}$) from formula (12)	Relative eeeor of asymptotic solutions from formula (12)
12.854+0.0144i	12.862+0.0120i	0.0007
12.853+0.0046i	12.850+0.0052i	0.0002
12.872+0.0184i	12.923+0.0331i	0.0041
12.903+0.0307i	12.887+0.0210i	0.0015
12.946+0.0468i	13.031+0.0679i	0.0067
13.003+0.0665i	12.971+0.0487i	0.0028

图 3 是由离散矩阵方法得到 的 Berenger 模和渐近解方法得到的 Berenger 模。

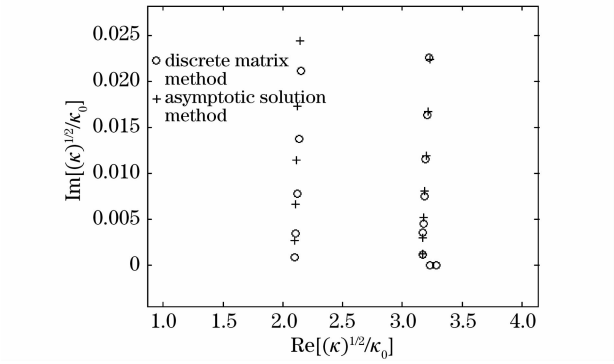

图 3　当无法给出精确解时，TE 波中前几个 Berenger 模的比较

Fig. 3　Comparison of some Berenger modes for TE case when exact solution cannot be given

其次计算 TM 模，表 5、表 6 列举前几个 Berenger 模渐近解及其相对误差。

表 5　TM 波中几个 Berenger 模的渐近解及其相对误差

Table 5　Asymptotic solutions and the relative errors of some Berenger modes for TM case

Exact solutions ($\sqrt{k}$)	Asymptotic solutions ($\sqrt{k}$) from formula (22)	Relative error of asymptotic solutions from formula (22)
6.4849+3.9354i	6.6712+4.0863i	0.0316
6.5033+6.1154i	6.6748+6.1540i	0.0197
6.8676+8.2501i	7.0137+8.2285i	0.0138
7.4197+10.292i	7.5425+10.231i	0.0108
8.0669+12.256i	8.1752+12.164i	0.0097
8.7950+14.160i	8.8685+14.042i	0.0084

图 4 是由离散矩阵方法得到的 Berenger 模和渐近解方法得到的 Berenger 模。

可见，对于光波导的 TE 模和 TM 模，渐近的 Berenger 模比较接近于精确的 Berenger 模，且随着模的增大，误差有变小趋势。

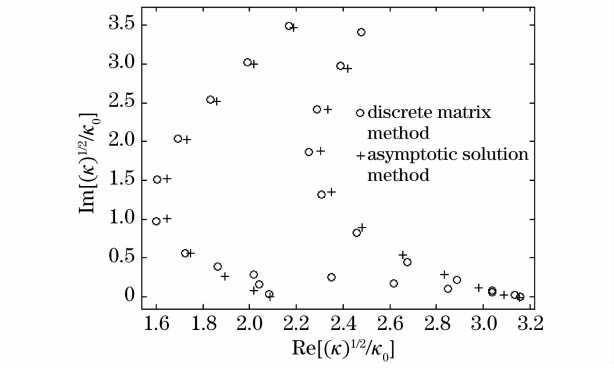

图 4　当无法给出精确解时，TM 波中前几个 Berenger 模的比较

Fig. 4　Comparison of some Berenger modes for TM case when exact solutions cannot be given

表 6 TM 波中几个 Berenger 模的渐近解及其相对误差

Table 6 Asymptotic solution and the relative errors of some Berenger modes for TM case

Exact solutions ($\sqrt{k}$)	Asymptotic solutions ($\sqrt{k}$) from formula (23)	Relative error of asymptotic solutions from formula (23)
10.850+1.8359i	10.7713+2.1748i	0.0316
9.9639+3.3463i	10.0579+3.6257i	0.0281
9.3537+5.3273i	9.5300+5.4908i	0.0223
9.1414+7.5515i	9.3341+7.6072i	0.0169
9.2710+9.7950i	9.4574+9.7812i	0.0139
9.6801+12.040i	9.8057+11.914i	0.0115

5 结 论

分析了在使用完美匹配层后光波导泄漏模的变化和 Berenger 模的渐近解的情况。理论结果表明，对于光波导的 TE 模，泄漏模的结果与开波导中得到结果是一致的，而光波导的 TM 模泄漏模会稍有不同。光波导在上下包层的折射率不同时会产生两类 Berenger 模的轨迹，但当折射率相同时，两条 Berenger 模的轨迹会重合。数值结果表明，Berenger 模的渐近解和精确解比较接近，并且随着模的增大误差有减小趋势。

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