第 14 卷　第 6 期　　光　学　学　报　　Vol. 14, No. 6
1994 年 6 月　　ACTA OPTICA SINICA　　June, 1994

矩孔光栅的矢量模式理论

严　瑗　陈　晖　朱文勇
（上海交通大学应用物理系，上海 200030）

摘　要　本文引入满足均匀矢量亥姆霍兹(Helmholtz)方程的矢量波函数作为基矢，对矩孔光栅的孔内外光场分别进行矢量模式展开和矢量平面波展开，并由耦合条件导出了求解展开系数的方程组，从方程组中求解出相应的振幅系数，可研究光栅的衍射场分布．该方法可研究入射场方向和偏振任意时的衍射效率和偏振特性等问题．

关键词　矩孔光栅，　矢量模式理论．

1　引　　言

矩孔光栅可用作滤波器[1]，法布里-珀罗干涉仪的反射单元[1]，在微波工程领域中也存在广泛的应用价值[2]．以往对矩孔光栅的衍射问题的研究，往往将场分解为 TE 场和 TM 场进行计算[3]，但这样不能研究衍射场的偏振特性问题．本文在矩形槽光栅矢量模式理论的基础上，建立了矩孔光栅的矢量模式理论，与以往理论相比，可以方便地得到任意入射方向和偏振场的衍射场分布和偏振态分布．

本文采用 Hansen 的方法[4]求解均匀矢量亥姆霍兹方程，即以相应的标量 Helmholtz 方程的解 ψ 作为生成函数，选择适当的常矢量 $\boldsymbol{P}$ 作为领示矢量，就可构成一组相互正交矢量波函数 $\boldsymbol{L}$, $\boldsymbol{M}$, $\boldsymbol{N}$，其中 $\boldsymbol{L}=\nabla\psi$, $\boldsymbol{M}=\nabla\times(\boldsymbol{P}\psi)$, $\boldsymbol{N}=k^{-1}\nabla\times\boldsymbol{M}$. 如根据标量亥姆霍兹方程及具体边界条件，得到一组完备归一的标量基 $\{\psi_n\}$．由它们生成的集 $\{\boldsymbol{L}_n, \boldsymbol{M}_n, \boldsymbol{N}_n\}$ 就是一组完备的正交归一矢量基，矢量亥姆霍兹方程的解就可用该组矢量基函数展开．对于无源问题，仅以 $\{\boldsymbol{M}_n, \boldsymbol{N}_n\}$ 作为基函数．该方法已被用于处理矩形槽光栅[5]和对称型闪耀光栅[6]的衍射问题等．

2　矢量基函数

对于如图 1 所示的光栅结构，考虑理想导体情况，x-y 平面为光栅平面，x, y 方向均为无限周期结构．x 方向周期为 d_1，孔宽 a，y 方向周期 d_2，孔宽 b，孔深为 h. 将整个光场分成两个场区，（Ⅰ）区为光栅平面以上区域，

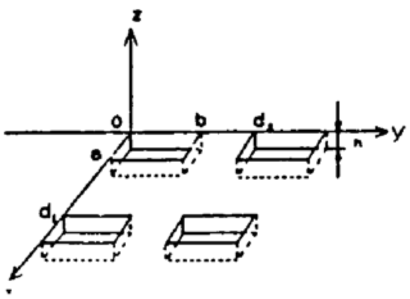

Fig. 1 The model of the grating

---

* 国家 863 高科技基金资助项目．
收稿日期：1992 年 5 月 25 日

(Ⅱ)区为光栅孔内区域. 由于是无源问题，$L=0$.

2.1 (Ⅰ)**区生成函数为平面波函数**：

$$\psi^{(1)}=\exp\left[i(k_x+k_y+k_z)\right],\tag{1}$$

取 z 方向单位矢量 $\boldsymbol{a}_z$ 为领示矢量，可得该区域的矢量基表达式：

$$\begin{cases}\boldsymbol{M}_{mn}^{(1)}=\dfrac{i}{\sqrt{k_{xm}^2+k_{yn}^2}}\left(k_{yn}\boldsymbol{a}_x-k_{xm}\boldsymbol{a}_y\right)\exp\left[i(k_{xm}x+k_{yn}y+k_{zmn}z)\right],\\ \boldsymbol{N}_{mn}^{(1)}=\dfrac{1}{k\sqrt{k_{xm}^2+k_{yn}^2}}\left[-k_{xm}k_{zmn}\boldsymbol{a}_x-k_{yn}k_{zmn}\boldsymbol{a}_y+(k_{xm}^2+k_{yn}^2)\boldsymbol{a}_z\right]\\ \qquad\exp\left[i(k_{xm}x+k_{yn}y+k_{zmn}z)\right],\end{cases}\tag{2}$$

式中 $k_{xm}=k_x+2m\pi/d_1$，$k_{yn}=k_y+2n\pi/d_2$，$k_{zmn}^2=k^2-k_{xm}^2-k_{yn}^2$，$(m,n=0,\pm1,\pm2,\cdots)$ 各基矢前的系数 $(k_{xm}^2+k_{yn}^2)^{-1/2}$ 是为了保持完备和归一.

2.2 (Ⅱ)**区的电场的矢量基在边界上应满足** Dirichlet **条件**：

$$\boldsymbol{n}\times\begin{Bmatrix}\boldsymbol{M}^{(\mathrm{II})}\\ \boldsymbol{N}^{(\mathrm{II})}\end{Bmatrix}\Bigg|_{\substack{x=0,\,a\\ y=0,\,b\\ z=-h}}=0\tag{3}$$

磁场的矢量基在边界上应满足 Neumann 条件：

$$\boldsymbol{n}\times\nabla\times\begin{Bmatrix}\boldsymbol{M}^{(\mathrm{II})'}\\ \boldsymbol{N}^{(\mathrm{II})'}\end{Bmatrix}\Bigg|_{\substack{x=0,\,a\\ y=0,\,b\\ z=-h}}=0,\tag{4}$$

$\boldsymbol{n}$ 为界面法向单位矢量. 相应生成函数组为：

$$\begin{cases}\psi_{eomn}=\cos\dfrac{m\pi x}{a}\cos\dfrac{n\pi y}{b}\dfrac{\sin\left[\mu_{mn}(z+h)\right]}{\mu_{mn}}, & m,n=0,1,2,\cdots\\ \psi_{oemn}=\sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}\cos\left[\mu_{mn}(z+h)\right], & m,n=1,2,3,\cdots\end{cases}\tag{5}$$

式中前两个下标分别表示函数对 x,y 和对 z 的奇偶性，$\mu_{mn}^2=k^2-(m\pi/a)^2-(n\pi/b)^2$，仍取 $\boldsymbol{a}_z$ 为领示矢量，由生成函数可得满足以上边界条件的矢量基函数分别为：

$$\begin{cases}\boldsymbol{M}_{eomn}^{(\mathrm{II})}=\dfrac{1}{\sqrt{(m\pi/a)^2+(n\pi/b)^2}}\dfrac{\sin\left[\mu_{mn}(z+h)\right]}{\mu_{mn}}\left(-\dfrac{n\pi}{b}\cos\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}\boldsymbol{a}_x\right.\\ \qquad\left.+\dfrac{m\pi}{a}\sin\dfrac{m\pi x}{a}\cos\dfrac{n\pi y}{b}\boldsymbol{a}_y\right),\\ \boldsymbol{N}_{eomn}^{(\mathrm{II})}=\dfrac{1}{k\sqrt{(m\pi/a)^2+(n\pi/b)^2}}\left\{-\dfrac{m\pi}{a}\sin\dfrac{m\pi x}{a}\cos\dfrac{n\pi y}{b}\cos\left[\mu_{mn}(z+h)\right]\boldsymbol{a}_x\right.\\ \qquad-\dfrac{n\pi}{b}\cos\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}\cos\left[\mu_{mn}(z+h)\right]\boldsymbol{a}_y\\ \qquad\left.+\dfrac{\sin\left[\mu_{mn}(z+h)\right]}{\mu_{mn}}\cos\dfrac{m\pi x}{a}\cos\dfrac{n\pi y}{b}\left[\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right]\boldsymbol{a}_z\right\},\end{cases}\tag{6}$$

$$
\begin{cases}
\boldsymbol{M}^{(\mathrm{II})}_{oemn} = \dfrac{1}{\sqrt{(m\pi/a)^2 + (n\pi/b)^2}} \cos\left[\mu_{mn}(z+h)\right] \left(\dfrac{n\pi}{b} \sin \dfrac{m\pi x}{a} \cos \dfrac{n\pi y}{b}\, \boldsymbol{a}_x \right. \\
\qquad \left. - \dfrac{m\pi}{a} \cos \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b}\, \boldsymbol{a}_y \right), \\
\boldsymbol{N}^{(\mathrm{II})}_{oemn} = \dfrac{1}{k\sqrt{(m\pi/a)^2 + (n\pi/b)^2}} \left\{ -\mu_{mn} \dfrac{m\pi}{a} \cos \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b} \sin\left[\mu_{mn}(z+h)\right] \boldsymbol{a}_x \right. \\
\qquad - \dfrac{n\pi}{b} \mu_{mn} \sin \dfrac{m\pi x}{a} \cos \dfrac{n\pi y}{b} \sin\left[\mu_{mn}(z+h)\right] \boldsymbol{a}_y \\
\qquad \left. + \cos\left[\mu_{mn}(z+h)\right] \sin \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b} \left[\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2 \right] \boldsymbol{a}_z \right\},
\end{cases}
\tag{7}
$$

3　场的矢量展开

设入射波是矢量平面波，其电场矢量为：

$$
\boldsymbol{E}_i = \boldsymbol{I} \exp\left[i(k_x x + k_y y - k_z z)\right] \tag{8}
$$

式中 k_z 为正实数，$\boldsymbol{I}$ 是单位偏振矢量. $\boldsymbol{E}_i$ 可用(Ⅰ)区的一对基矢 $\boldsymbol{M}_i$，$\boldsymbol{N}_i$ 展开：

$$
\boldsymbol{E}_i = A_i \boldsymbol{M}_i + iB_i \boldsymbol{N}_i = \left\{ \frac{iA_i}{\sqrt{k_x^2 + k_y^2}} (k_y \boldsymbol{a}_x - k_x \boldsymbol{a}_y) + \frac{iB_i}{k\sqrt{k_x^2 + k_y^2}} \left[k_x k_z \boldsymbol{a}_x + k_y k_z \boldsymbol{a}_y + (k_x^2 + k_y^2) \boldsymbol{a}_z \right] \right\} \exp\left[i(k_x x + k_y y - k_z z)\right] \tag{9}
$$

式中 $A_i = \boldsymbol{M}_i^* \cdot \boldsymbol{E}_i$，$B_i = -i\boldsymbol{N}_i^* \cdot \boldsymbol{E}_i$. 第$(m,n)$级衍射波可用$(\boldsymbol{M}^{(\mathrm{I})}_{mn}, \boldsymbol{N}^{(\mathrm{I})}_{mn})$展开. 于是(Ⅰ)区光场可表示为：

$$
\begin{bmatrix} \boldsymbol{E}^{(\mathrm{I})} \\ \boldsymbol{H}^{(\mathrm{I})} \end{bmatrix} = A_i \begin{bmatrix} \boldsymbol{M}_i \\ -i\sqrt{\varepsilon/\mu}\, \boldsymbol{N}_i \end{bmatrix} + B_i \begin{bmatrix} i\boldsymbol{N}_i \\ \sqrt{\varepsilon/\mu}\, \boldsymbol{M}_i \end{bmatrix} + \sum_{mn} \left\{ A_{mn} \begin{bmatrix} \boldsymbol{M}^{(\mathrm{I})}_{mn} \\ -i\sqrt{\varepsilon/\mu}\, \boldsymbol{N}^{(\mathrm{I})}_{mn} \end{bmatrix} + B_{mn} \begin{bmatrix} i\boldsymbol{N}^{(\mathrm{I})}_{mn} \\ \sqrt{\varepsilon/\mu}\, \boldsymbol{M}^{(\mathrm{I})}_{mn} \end{bmatrix} \right\} \tag{10}
$$

类似地，(Ⅱ)区光场可展开为：

$$
\begin{bmatrix} \boldsymbol{E}^{(\mathrm{II})} \\ \boldsymbol{H}^{(\mathrm{II})} \end{bmatrix} = \sum_{mn} \left\{ C_{mn} \begin{bmatrix} \boldsymbol{M}^{(\mathrm{II})}_{eomn} \\ -i\sqrt{\varepsilon/\mu}\, \boldsymbol{N}^{(\mathrm{II})}_{eomn} \end{bmatrix} + D_{mn} \begin{bmatrix} \boldsymbol{N}^{(\mathrm{II})}_{oemn} \\ -i\sqrt{\varepsilon/\mu}\, \boldsymbol{M}^{(\mathrm{II})}_{oemn} \end{bmatrix} \right\} \tag{11}
$$

式中 C_{mn} 的角标 m，n 不同时为零，D_{mn} 的角标 m，n 均从 1 开始.

4　场的耦合

在光栅平面 $(z=0)$ 上

$$
\begin{cases}
\boldsymbol{a}_z \times \boldsymbol{E}^{(\mathrm{I})}\big|_{z=0} = \begin{cases} \boldsymbol{a}_z \times \boldsymbol{E}^{(\mathrm{II})}\big|_{z=0} & (0 \leqslant x \leqslant a \text{ and } 0 \leqslant y \leqslant b) \\ 0 & (\text{otherwise}) \end{cases} \\
\boldsymbol{a}_z \times \boldsymbol{H}^{(\mathrm{I})}\big|_{z=0} = \boldsymbol{a}_z \times \boldsymbol{H}^{(\mathrm{II})}\big|_{z=0} \quad (0 \leqslant x \leqslant a \text{ and } 0 \leqslant y \leqslant b)
\end{cases}
\tag{12}
$$

将(10)、(11)式代入(12)式，利用(2)、(6)、(7)式，可以得到计算振幅 A_{mn}，B_{mn} 和 C_{mn}、D_{mn} 的线性方程组. 它们可用超矩阵形式表示：

$$\left\{\begin{aligned}&\begin{bmatrix}\underline{A}\\ \underline{B}\end{bmatrix}=\begin{bmatrix}u^{(1)} & v^{(1)}\\ u^{(2)} & v^{(2)}\end{bmatrix}\begin{bmatrix}\underline{C}\\ \underline{D}\end{bmatrix}+\begin{bmatrix}\delta(-A_i)\\ \delta(B_i)\end{bmatrix},\\ &\begin{bmatrix}\underline{C}\\ \underline{D}\end{bmatrix}=\begin{bmatrix}\alpha^{(1)}A_i+\beta^{(1)}B_i\\ \alpha^{(2)}A_i+\beta^{(2)}B_i\end{bmatrix}+\begin{bmatrix}\alpha^{(3)} & \beta^{(3)}\\ \alpha^{(4)} & \beta^{(4)}\end{bmatrix}\begin{bmatrix}\underline{A}\\ \underline{B}\end{bmatrix},\end{aligned}\right. \tag{13}$$

$$\underline{A}=(\cdots, A_{m,-n}, \cdots, A_{00}, A_{01}, \cdots, A_{m,n}, \cdots)^T,$$

$$\underline{B}=(\cdots, B_{m,n}, \cdots, B_{00}, B_{01}, \cdots, B_{m,n}, \cdots)^T,$$

$$\underline{C}=(C_{01}, C_{02}, \cdots, C_{10}, C_{11}, \cdots, C_{pq}, \cdots)^T,$$

$$\underline{D}=(D_{11}, D_{12}, \cdots, D_{pq}, :)^T,$$

$$\delta(x)=(x\,\delta_{m0}\delta_{n0}, x\,\delta_{m0}\delta_{n0}, \cdots)^T,$$

$$\left\{\begin{aligned}u^{(1)}_{mnpq}&=\frac{i}{d_1d_2}\frac{1}{\sqrt{k_{xm}^2+k_{yn}^2}}\frac{1}{\sqrt{(p\pi/a)^2+(q\pi/b)^2}}\frac{\sin(\mu_{pq}h)}{\mu_{pq}}\left(\frac{p\pi}{a}I^*_{pqmn}k_{xm}+\frac{q\pi}{b}J^*_{pqmn}k_{yn}\right)\\ u^{(2)}_{mnpq}&=\frac{-i}{d_1d_2}\frac{k}{k_{zmn}\sqrt{k_{xm}^2+k_{yn}^2}}\frac{1}{\sqrt{(p\pi/a)^2+(q\pi/b)^2}}\frac{\sin(\mu_{pq}h)}{\mu_{pq}}\left(\frac{q\pi}{b}J^*_{pqmn}k_{xm}-\frac{p\pi}{a}I^*_{pqmn}k_{yn}\right)\\ v^{(1)}_{mnpq}&=\frac{-i}{d_1d_2}\frac{1}{k\sqrt{k_{xm}^2+k_{yn}^2}}\frac{\mu_{pq}}{\sqrt{(p\pi/a)^2+(q\pi/b)^2}}\sin(\mu_{pq}h)\left(\frac{q\pi}{b}I^*_{pqmn}k_{xm}-\frac{p\pi}{a}J^*_{pqmn}k_{yn}\right)\\ v^{(2)}_{mnpq}&=\frac{-i}{d_1d_2}\frac{1}{k_{zmn}\sqrt{k_{xm}^2+k_{yn}^2}}\frac{\mu_{pq}}{\sqrt{(p\pi/a)^2+(q\pi/b)^2}}\sin(\mu_{pq}h)\left(\frac{p\pi}{a}J^*_{pqmn}k_{xm}+\frac{q\pi}{b}I^*_{pqmn}k_{yn}\right)\end{aligned}\right.$$

$$\left\{\begin{aligned}\alpha^{(3)}_{pqmn}&=\frac{k_{zmn}\sqrt{(p\pi/a)^2+(q\pi/b)^2}}{\sqrt{k_{xm}^2+k_{yn}^2}}\frac{[(p/a)k_{xm}I_{pqmn}+(q/b)k_{yn}J_{pqmn}]}{(\pi/4)\cos(\mu_{pq}h)[(b/a)p^2(1+\delta_{q0})+(a/b)q^2(1+\delta_{p0})]}\\ \alpha^{(4)}_{pqmn}&=\frac{k_{zmn}\sqrt{(p\pi/a)^2+(q\pi/b)^2}}{k\sqrt{k_{xm}^2+k_{yn}^2}}\frac{[(p/a)k_{yn}J_{pqmn}(1+\delta_{q0})-(q/b)k_{xm}I_{pqmn}(1+\delta_{p0})]}{(\pi/2)\cos(\mu_{pq}h)[(b/a)p^2(1+\delta_{q0})+(a/b)q^2(1+\delta_{p0})]}\\ \beta^{(3)}_{pqmn}&=\frac{k\sqrt{(p\pi/a)^2+(q\pi/b)^2}}{\sqrt{k_{xm}^2+k_{yn}^2}}\frac{[(p/a)k_{yn}I_{pqmn}-(q/b)k_{xm}J_{pqmn}]}{(\pi/4)\cos(\mu_{pq}h)[(b/a)p^2(1+\delta_{q0})+(a/b)q^2(1+\delta_{p0})]}\\ \beta^{(4)}_{pqmn}&=-\frac{\sqrt{(p\pi/a)^2+(q\pi/b)^2}}{\sqrt{k_{xm}^2+k_{yn}^2}}\frac{[(p/a)k_{xm}J_{pqmn}(1+\delta_{q0})+(q/b)k_{yn}I_{pqmn}(1+\delta_{p0})]}{(\pi/4)\cos(\mu_{pq}h)[(b/a)p^2(1+\delta_{q0})+(a/b)q^2(1+\delta_{p0})]}\end{aligned}\right.$$

$$\alpha^{(1)}_{pq}=-\alpha^{(3)}_{pq00},\quad \alpha^{(2)}_{pq}=-\alpha^{(4)}_{pq00},\quad \beta^{(1)}_{pq}=\beta^{(3)}_{pq00},\quad \beta^{(2)}_{pq}=\beta^{(4)}_{pq00}$$

$$\left\{\begin{aligned}I_{pqmn}&=\int_0^b\int_0^a\sin\frac{p\pi x}{a}\cos\frac{q\pi y}{b}\exp[i(k_{xm}x+k_{yn}y)]\,dxdy\\ J_{pqmn}&=\int_0^b\int_0^a\cos\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\exp[i(k_{xm}x+k_{yn}y)]\,dxdy\end{aligned}\right.$$

对(13)式进行数值计算，并取适当阶次截断，就可求得各级衍射波的振幅 $\{A_{mn}, B_{mn}\}$，A_{mn} 和 B_{mn} 一般情况下是复数，$A_{mn}=|A_{mn}|\exp(i\theta_{mn})$，$B_{mn}=|B_{mn}|\exp(i\phi_{mn})$，$\delta_{mn}=\phi_{mn}-\theta_{mn}$ 决定衍射场的偏振态，当 k_{zmn} 是实数时，第(m,n)级衍射波是真实传播的衍射光，其衍射效率为

$$E(m,n)=\frac{(|A_{mn}|^2+|B_{mn}|^2)}{(|A_i|^2+|B_i|^2)}\frac{k_{zmn}}{k_z}$$

本文建立了矩孔光栅的矢量模式理论，可研究入射方向和偏振任意的平面波的衍射效率、偏振特性等问题. 采用该方法进行数值计算，收敛状况较好，已得到部分结果，与实验结果较好相符.

参 考 文 献

[1] R. Ulrich, Interference filters for the far infrared. *Appl. Opt.*, 1968, **7**(10): 1987~96

[2] Chao-Chun Chen, Transmission of microwave through perforated flat plates of finite thickness. *IEEE Trans. Microwave Theory & Tech.*, 1973, MTT-**21**(1): 1~6

[3] R. Petit, *Electromagnetic Theory of Grating*. Springer-Verlag Berlin Heidelberg New York, 1980, 227~239

[4] W. W. Hansen, New type of expansion in radiation problems. *Phys. Rev.*, 1935, **47**(1): 139~143

[5] 杨宝成,庄松林,周学松, 矩形槽光栅的矢量模式理论. 光学学报, 1989, **9**(3): 270~277

[6] 林维德,庄松林,周学松, 对称型闪耀光栅的矢量模式理论. 光学学报, 1991, **11**(7): 624~629

A Vector Modal Theory for Perfectly-Conducting Rectangular-Aperture Grating

Yan Yuan Chen Hui Zhu Wenyong
(*Department of Applied Physics, Shanghai Jiaotong University, Shanghai* 200030)
(Received 25 May 1993)

Abstract The vector wave functions satisfying vector homogeneous Helmholtz equation are introduced as the basic vectors, and the fields both outside and inside the apertures of the grating are expanded by vector plane wave functions and vector modes respectively. Applying the field coupling conditions, the diffraction equations for the amplitudes are derived. The method can be used to study the diffraction of optical plane waves with arbitrary incident direction and polarization.

Key words rectangular-aperture grating, vector modal theory.