Freezing quantum coherence with weak measurement

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We show how to optimally protect quantum states and freeze coherence under incoherent channels using a quantum weak measurement and quantum measurement reversal. In particular, we present explicit formulas for the conditions for freezing quantum coherence in a given quantum state.

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Quantum coherence is a fundamental feature of quantum mechanics. It has been widely used as a resource and root concept in quantum information processing[1], quantum metrology[2–5], entanglement creation[6,7], thermo dynamics[8–12], and quantum biology[13–16]. Recently, a rigorous theory of coherence as a physical resource has been developed[17–19], and necessary constraints have been put forward to assess valid quantifiers of coherence[17]. Some coherence measures based on various physical contexts, such as the l_1 norm of coherence, the relative entropy of coherence[17], and the skew information[20,21], have been put forward.

Quantum coherence is a useful physical resource, but the coherence of a quantum state is often destroyed by the noise of the environment. A challenge in exploiting the resource is to protect the coherence from decoherence caused by noise. Studies on this topic were started in Ref. [22], where the authors found that the coherence is frozen for some particular initial states only when a quantum system undergoes the local identical bit flip and bit phase flip channels. Then, the question we ask is whether there exist methods to freeze coherence when a quantum system undergoes other channels.

Both coherence and entanglement capture the quantumness of a physical system, and it is well known that entanglement also stems from the superposition principle, which is the essence of coherence. In practice, quantum entanglement is fragile with respect to environmental noises[23]. Some ideas are proposed to protect quantum states and quantum entanglements from decoherence using a quantum weak measurement and quantum measurement reversal[24–28]. So, we can optimize quantum states and freeze coherence from decoherence using a weak measurement and quantum measurement reversal.

First, we need to review some notions, such as incoherent states, incoherent operations, and coherence measures. For an N-qubit system associated with a Hilbert space C^{2^N}, the computational basis, $\{|0\rangle,|1\rangle\}^{\otimes N}$, is fixed as the reference basis, and the incoherent states are those whose density matrix, δ, is diagonal in the reference basis:

$$\delta=\sum_{i_1,\ldots,i_N} d_{i_1,\ldots,i_N}|i_1,\ldots,i_N\rangle\langle i_1,\ldots,i_N|. \tag{1}$$

A quantum channel is described by a completely positive and trace-preserving (CPTP) map, Λ, whose action on the state ρ of the system can be characterized by a set of Kraus operators, $\{K_j\}$, such that $\Lambda(\rho)=\sum_j K_j\rho K_j^+$, where $\sum_j K_j^+K_j=I$, and I is the identity operator. Incoherent quantum channels [incoherent CPTP (ICPTP) maps] constitute a subset of quantum channels that satisfy the additional constraint $K_j\tau K_j^+\subset\tau$ for all j, where τ is the set of incoherent states[17].

We consider paradigmatic instances of incoherent channels that embody typical noise sources in quantum information processing and whose action on a single qubit is described as follows[1]. The bit flip, bit phase flip, and phase flip channels are represented in Kraus form by $K_0^{F_i}=\sqrt{1-q/2}I$, $K_{j,k\neq i}^{F_i}=0$, and $K_i^{F_i}=\sqrt{q/2}\sigma^i$, with $i=1$, $i=2$, and $i=3$, respectively; σ^i is the ith Pauli matrix, and $q\in[0,1]$ encodes the strength of the noise and depends on time t. The amplitude-damping (AD) channel is represented by

$$K_0^A=\begin{bmatrix}1 & 0\\ 0 & \sqrt{1-q}\end{bmatrix},\qquad K_1^A=\begin{bmatrix}0 & \sqrt{q}\\ 0 & 0\end{bmatrix}. \tag{2}$$

The action of N independent and identical local noisy channels on each qubit of an N-qubit system maps the system state, ρ, into the evolved state,

$$\Lambda_q^{\otimes N}(\rho)=\sum_{j_1,\ldots,j_N}(K_{j_1}\otimes\cdots\otimes K_{j_N})\rho(K_{j_1}^+\otimes\cdots\otimes K_{j_N}^+). \tag{3}$$

We recall the well-known measures of coherence[17]. The l_1 norm of coherence, C_{l_1}, measures coherence in an intuitive way via the off-diagonal elements of a density matrix, ρ, in the reference basis, $C_{l_1}(\rho)=\sum_{i\neq j}|\rho_{ij}|$. The relative entropy of coherence is given by $C_r(\rho)=S(\rho_d)-S(\rho)$ for any state ρ, where ρ_d is the matrix containing only the diagonal elements of ρ in the reference basis, and $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is the von Neumann entropy. Reference [29] has proven that the coherence is frozen for all coherence measures when the relative entropy of coherence is frozen.

A weak measurement is a type of quantum measurement that would not cause the quantum system to collapse fully[30], thus the weak-measured system may be recovered through some reversal operations. Usually a weak measurement[24,26] can be parametrized as $M = \mathrm{diag}\{1, \sqrt{1-p}\}$, and the measurement reversal operator is written as $N = \mathrm{diag}\{\sqrt{1-p_r}, 1\}$. For convenience and generality, the weak measurements are given by $M = \mathrm{diag}\{1, m\}$ and $N = \mathrm{diag}\{n, 1\}$, with $m, n \in [0, +\infty)$. M is the projective measurement when $m = 0$. When $m \in (0, 1)$, then M is a measurement partially collapsing on the ground state, and when $m \in (1, +\infty)$, then M can be written as $M = m \cdot \mathrm{diag}\{1/m, 1\}$, where m is an overall factor and $1/m < 1$; thus, M is a weak measurement partially collapsing on the excited state[25]. For the normalization of the final state, one should multiply by the factor $\min\{1, 1/m^2\}$. A similar analysis is valid for N.

In order to freeze the coherence, we should perform two weak measurements, M and N, before and after the qubit is put into the incoherent channel, respectively. With these weak measurements implemented, the final state is

$$\rho_f^w = N\Lambda_q(M\rho M^+)N^+, \tag{4}$$

where Λ_q is defined by Eq. (3).

We now analyze the conditions when the coherence is frozen during the evolution of a quantum system under the AD channels. For a pure qubit state, $|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, with $\theta \in [0, \pi]$. According to Eq. (4), the final density matrix of the qubit in the reference basis, $\{|0\rangle, |1\rangle\}$, is

$$\rho_f^w = \frac{1}{T}\begin{bmatrix} n^2(\cos^2\theta + m^2 q \sin^2\theta) & mn\sqrt{1-q}\cos\theta\sin\theta \\ mn\sqrt{1-q}\cos\theta\sin\theta & m^2(1-q)\sin^2\theta \end{bmatrix}, \tag{5}$$

where $T = n^2\cos^2\theta + m^2(1-q+n^2 q)\sin^2\theta$ is the normalization factor. The overall success probability is

$$P_s = T \cdot \min\{1, 1/m^2\} \cdot \min\{1, 1/n^2\}. \tag{6}$$

The elements ρ_{12} and ρ_{21} of the final density matrix, ρ_f^w, are

$$\rho_{12} = \rho_{21} = \frac{mn\sqrt{1-q}\cos\theta\sin\theta}{n^2\cos^2\theta + m^2(1-q+n^2 q)\sin^2\theta}. \tag{7}$$

Using the inequality $x + y \geq 2\sqrt{xy}$ (equality is obtained if, and only if, $x = y$) and ensuring that the elements ρ_{12} and ρ_{21} are not related to the strength, q, of the noise, we can show that ρ_{12} and ρ_{21} reach freezing values when the following conditions are met:

$$m = f_1(\theta)/\sqrt{q}, \qquad n = \sqrt{(1-q)/q}\, f_1(\theta)\tan\theta, \tag{8}$$

where $f_1(\theta)$ is an arbitrary meaningful function about θ. The corresponding elements of the final density matrix are $\rho_{11} = (1 + f_1^2(\theta)\tan^2\theta)/\overline{T}$ and $\rho_{12} = \rho_{21} = \rho_{22} = 1/\overline{T}$, with $\overline{T} = 2 + f_1^2(\theta)\tan^2(\theta)$. The l_1 norm and relative entropy of coherence are frozen under these conditions. We can see that if $\theta = 0$ or $\theta = \pi$, the success probability is zero. If $\theta = \pi/2$, then $\rho_{12} = \rho_{21} = 0$. That is to say, this method of freezing coherence cannot create coherence from the initial incoherent state.

Particularly, when $f_1(\theta) = \cot\theta$, then $m = \cot\theta/\sqrt{q}$ and $n = \sqrt{(1-q)/q}$. We obtain the freezing values: $\rho_{12} = \rho_{21} = 1/3$, $\rho_{11} = 2/3$, and $\rho_{22} = 1/3$. The l_1 norm and the relative entropy of coherence are $C_{l_1}(\rho) = 2/3$ and $C_r(\rho) \approx 0.37$.

So, we can conclude that the coherence is frozen under these conditions of weak measurements when a one-qubit quantum system undergoes an AD channel. Interestingly, for these particular conditions, the frozen coherence is independent of the initial state.

We will show that the coherence always manifests as frozen in the case of two qubits undergoing identical AD channels by using prior weak measurements and post weak measurements.

Consider the two-qubit Bell-like state $|\varphi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, with $\theta \in [0, \pi]$. The weak measurement is a nonunitary operation, which can be written as

$$\begin{aligned} M_1 &= \begin{bmatrix} 1 & 0 \\ 0 & m_1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & m_2 \end{bmatrix}, \\ N_1 &= \begin{bmatrix} n_1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} n_2 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{9}$$

In terms of Eq. (4), in the reference basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the final density matrix of the two qubit turns out to be

$$\rho_{MN} = \frac{1}{T_1}\begin{bmatrix} a_1 & 0 & 0 & e_1 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ e_1 & 0 & 0 & d_1 \end{bmatrix}, \tag{10}$$

with

$$\begin{aligned} a_1 &= n_1^2 n_2^2(\cos^2\theta + m_1^2 m_2^2 q^2 \sin^2\theta), \\ b_1 &= n_1^2 m_1^2 m_2^2 q(1-q)\sin^2\theta, \\ c_1 &= n_2^2 m_1^2 m_2^2 q(1-q)\sin^2\theta, \\ d_1 &= m_1^2 m_2^2(1-q)^2\sin^2\theta, \\ e_1 &= n_1 n_2 m_1 m_2(1-q)\sin\theta\cos\theta, \\ T_1 &= n_1^2 n_2^2\cos^2\theta + [n_1^2 n_2^2 q^2 + (n_1^2 + n_2^2)q(1-q) \\ &\quad + (1-q)^2]m_1^2 m_2^2\sin^2\theta, \end{aligned}$$

where T_1 is the normalization factor. The overall success probability is

$$P_{s1} = T_1 \prod_{x=\{m_1,m_2,n_1,n_2\}} \min\{1, 1/x^2\}. \tag{11}$$

Using similar methods as the single-qubit case above, we analyze the elements $\rho_{14} = \rho_{41} = e_1/T_1$ of the final density matrix, ρ_{MN}. After calculation, we get the conditions for freezing the quantum coherence:

$$m_1 m_2 = f_2(\theta)/q, \quad n_1 = n_2 = \sqrt{[(1-q)f_2(\theta)\tan\theta]/q}, \tag{12}$$

where $f_2(\theta)$ is an arbitrary meaningful function about θ. The corresponding elements of the final density matrix, ρ_{MN}, are $\rho_{11} = (1 + f_2^2(\theta)\tan^2\theta)/\overline{T}_1$, $\rho_{22} = \rho_{33} = (f_2(\theta)\tan\theta)/\overline{T}_1$, and $\rho_{44} = \rho_{14} = \rho_{41} = 1/\overline{T}_1$, with $\overline{T}_1 = 1 + (1 + f_2(\theta)\tan\theta)^2$. Under these conditions, we can see that the l_1 norm and the relative entropy of coherence are frozen. Similar to the single qubit, if $\theta = 0$ or $\theta = \pi$, the success probability is zero; if $\theta = \pi/2$, then $\rho_{14} = \rho_{41} = 0$.

Particularly, when $f_2(\theta) = \cot\theta$, then $m_1 m_2 = \cot\theta/q$ and $n_1 = n_2 = \sqrt{(1-q)/q}$, and we obtain the corresponding elements of the final density matrix: $\rho_{14} = \rho_{41} = 0.2$, $\rho_{11} = 0.4$, and $\rho_{22} = \rho_{33} = \rho_{44} = 0.2$. The l_1 norm and the relative entropy of coherence are $C_{l_1}(\rho) = 0.4$ and $C_r(\rho) \approx 0.22$. Under the conditions given, the l_1 norm and relative entropy of coherence are frozen with fixed values.

One can see that one of the conditions is $m_1 m_2 = f_2(\theta)/q$; we can set $m_2 = 1$, i.e., the prior weak measurement on the second qubit is not necessary, and the coherence can be frozen by adjusting m_1. If we set $m_1 = m_2 = \sqrt{f_2(\theta)/q}$, then the prior weak measurement is implemented on the two qubits.

So, we can conclude that the coherence of a two-qubit Bell-like state is frozen under these conditions of weak measurements when the quantum system undergoes identical AD channels.

Extending the above conclusions into multi-qubit systems, we can freeze the coherence of multi-qubit systems by using similar methods when each qubit undergoes identical AD channels. For N-qubit states, $|\phi\rangle = \cos\theta|0^{\otimes X}\rangle + \sin\theta|1^{\otimes X}\rangle$, with $\theta \in [0, \pi]$ and $X \geq 3$. We perform prior weak measurements $M^{\otimes X}$ on each qubit of the system, then let the qubits enter the AD channels; after obtaining the qubits, we do post weak measurement $N^{\otimes X}$ on each qubit. When $m = f_3(\theta)/\sqrt{q}$ and $n = \sqrt{(1-q)/q}f_4(\theta)$, where $f_{3(4)}(\theta)$ is an arbitrary meaningful function about θ, the coherence of the multi-qubit will be frozen when the qubits undergo identical AD channels.

We now analyze and discuss the conditions for freezing coherence from decoherence at a finite temperature using a weak measurement. At a nonzero temperature, the channel is more complicated. The AD channel under a finite temperature can be modeled by the following generalized amplitude damping (GAD) channel[1]:

$$E_0 = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{bmatrix}, \qquad E_1 = \sqrt{p}\begin{bmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{bmatrix},$$
$$E_2 = \sqrt{1-p}\begin{bmatrix} \sqrt{1-q} & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_3 = \sqrt{1-p}\begin{bmatrix} 0 & 0 \\ \sqrt{q} & 0 \end{bmatrix}, \tag{13}$$

where $q \in [0, 1]$ encodes the strength of the noise and depends on time t, and p is a function of temperature. The GAD channel describes a situation where the system can both lose and gain excitations by interacting with the environment. We perform two weak measurements, M and N, respectively, before and after the qubit is put into the GAD channel.

For a pure qubit state, $|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, with $\theta \in [0, \pi]$. According to Eq. (4), the final density matrix of the qubit in the reference basis is

$$\rho^w_{\mathrm{GAD}} = \frac{1}{T_2}\begin{bmatrix} a_2 & c_2 \\ d_2 & b_2 \end{bmatrix}, \tag{14}$$

with $a_2 = n^2[(1-p+pq)\cos^2\theta + m^2 pq\sin^2\theta]$,

$$\begin{aligned} b_2 &= m^2(1-pq)\sin^2\theta + (1-p)q\cos^2\theta, \\ c_2 &= d_2 = mn\sqrt{1-q}\cos\theta\sin\theta, \\ T_2 &= [n^2(1-q+pq) + (1-p)q]\cos^2\theta \\ &\quad + m^2[1-pq+n^2pq]\sin^2\theta, \end{aligned}$$

where T_2 is the normalization factor. The overall success probability is

$$P_{s2} = T_2 \cdot \min\{1, 1/m^2\} \cdot \min\{1, 1/n^2\}. \tag{15}$$

The elements ρ_{12} and ρ_{21} of the final density matrix, ρ^w_{GAD}, are $\rho_{12} = \rho_{21} = c_2/T_2$. We analyze the elements ρ_{12} and ρ_{21} by using the inequality $x + y \geq 2\sqrt{xy}$ (equality is obtained if, and only if, $x = y$). After calculation, we can see that if $p(1-p) = x \cdot [(1-q)/q^2]$, i.e., $p = (1 \pm \sqrt{1 - 4x(1-q)/q^2})/2$, with $x > 0$, is an arbitrary number, then the elements of the final density matrix are fixed values when the following conditions are met:

$$\begin{aligned} m &= \sqrt[4]{\frac{(1-q+pq)(1-p)}{(1-pq)p}}\cot\theta, \\ n &= \sqrt[4]{\frac{(1-pq)(1-p)}{p(1-q+pq)}}. \end{aligned} \tag{16}$$

The corresponding elements of the final density matrix are $\rho_{12} = \rho_{21} = 1/[2(\sqrt{x} + \sqrt{1+x})]$ and $\rho_{11} = \rho_{22} = 0.5$.

We can see that the coherence is frozen under these conditions of weak measurements when a one-qubit quantum system undergoes a GAD channel at a certain temperature. Interestingly, the frozen coherence is independent of the initial state.

We calculated the conditions for freezing coherence of multi-qubit systems with weak measurements when each qubit undergoes an identical GAD channel; the result was similar to that for one qubit. So, we can freeze the coherence of multi-qubit systems by using similar methods; but, when the initial state is an incoherent state, the methods are invalid.

As can be seen from above, the coherence of a quantum system can be frozen when it undergoes AD or GAD channels by performing prior weak measurements and post weak measurements. Is this method is valid for other incoherent channels? We tested that for a single qubit subject to Markovian bit flip, bit phase flip, phase flip, depolarizing, and phase damping channels. We found that only the bit flip and bit phase flip channels allowed for nonzero frozen coherence, while all the other considered incoherent channels were invalid for this method.

In conclusion, we determine the conditions for which the coherence of a quantum system is dynamically varied and frozen: this occurs for an arbitrary number of qubits, initialized in a coherent state, using prior weak measurements and post weak measurements on each qubit of the quantum system before and after undergoing local independent and identical incoherent channels. But, the incoherent channels only include the bit flip, bit phase flip, and AD channel. The conditions of the weak measurements are determined by the initial state and the parameters of the channel. We show that there are general agreements on freezing conditions both for the l_1 norm and the relative entropy of coherence. This method of freezing coherence to ensure a durable physical exploitation of coherence is feasible in theory, thus it will be interesting to explore practical realizations of such dynamical conditions.

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