# Superimposed orbital angular momentum mode of multiple Hankel–Bessel beam propagation in anisotropic non-Kolmogorov turbulence

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Received April 12, 2016; accepted June 14, 2016; posted online July 25, 2016

Mathematical models for the superimposed orbital angular momentum (OAM) mode of multiple Hankel–Bessel (HB) beams in anisotropic non-Kolmogorov turbulence are developed. The effects of anisotropic turbulence and source parameters on the mode detection spectrum of the superimposed OAM mode are analyzed. Anisotropic characteristics of the turbulence in the free atmosphere can enhance the performance of OAM-based communication. The HB beam is a good source for mitigating the turbulence effects due to its nondiffraction and self-focusing properties. Turbulence effects on the superimposed OAM mode can be effectively reduced by the appropriate allocation of OAM modes at the transmitter based on the reciprocal features of the mode cross talk.

*OCIS codes: 010.0010, 010.1300, 010.1330, 010.3310, 270.5585.*
*doi: 10.3788/COL201614.080102.*

Light can carry both spin angular momentum (SAM) and orbital angular momentum (OAM), where SAM is associated with polarization and OAM is related to the spatial profile of the intensity and phase[1]. The SAM with two orthogonal modes is well known and widely exploited in present free-space optical (FSO) systems to double the channel capacity[2]. Recently, there has been a growing interest in utilizing advanced OAM modes of light to enhance the channel capacity and spectral efficiency in FSO links[3]. OAM-based FSO systems have been employed to demonstrate high-capacity information transmission with a link distance of ~1 m at the lab scale[4]. OAM modes are inevitably sensitive to turbulence interference in the atmosphere because of their natural spatial structure[5]. The ability to transmit OAM modes over long-distance free space is crucial in order to take such lab-scale experiments into the real world[6]. Several investigations[5–10] have revealed that the effective range of the transmitted OAM modes with a small ratio of the mode cross talk in the atmosphere were restricted to a relatively narrow range because of turbulence effects. Turbulence-induced scintillation significantly caused the cross talk among OAM modes[7], reduced information capacity[8], and induced the spread of the OAM mode detection spectrum[9,10]. Superimposed OAM modes of multiple vortex beams possess more peculiar distribution structures and more controllable parameters for their potential applications in microparticle manipulation[11] and optical information encoding[12] compared with the conventional single-OAM mode. The superimposed OAM mode of multiple Laguerre–Gaussian (LG) beams have been studied in depth because of their simplicity of generation[12,13]. Classical information encoded by the superimposed OAM mode has recently started to be tested in real scenarios over a 3 km-long distance with significant turbulence, and showed that the phase of the superimposed OAM mode could be well conserved during the transmission in the turbulent atmosphere[12]. A low-density parity check-coded modulation architecture using coherent superimposed OAM mode for an optimal signal constellation was proposed in Ref. [13]. Recently, nondiffraction vortex beams[14–17] were widely adopted as the beam sources to mitigate the turbulence effect and proved to be superior to classical LG beams. A hypergeometric (HyG) beam is an important member of nondiffraction vortex beams and suffers lower diffraction than Bessel–Gauss beams in free space[18]. A Hankel–Bessel (HB) beam is a particular case of HyG beams with a simpler representation[19]. The probability density[16] and detection probability[17] of the single-OAM mode of HB beams in the turbulence were investigated. However, all of these research still stayed in isotropic turbulent atmosphere.

As stated in the literature[20–27], optical turbulence appears to be largely anisotropic and non-Kolmogorov in the free atmosphere above the boundary layer. Under these circumstances, the conventional isotropic Kolmogorov power spectrum may not properly describe the real turbulence behavior in the free atmosphere. Recently, several anisotropic non-Kolmogorov turbulence power spectra[23–25] have been proposed to theoretically investigate the effect of anisotropic turbulence in the free atmosphere. An anisotropic non-Kolmogorov spectrum with a circular symmetric assumption of turbulence eddies over planes orthogonal to the propagation path was introduced in Ref. [23]. Andrews *et al.*[24] proposed a general anisotropic turbulence spectrum by introducing two anisotropic parameters to describe the asymmetric property of turbulence eddies in the horizontal and vertical directions. However, these two turbulence spectrums are only valid in the inertial subrange. In 2014, Toselli[25] introduced the finite turbulence inner and outer scales into the anisotropic turbulence spectrum by combining the von Karman spectrum[26] and the anisotropic non-Kolmogorov spectrum[23]. Propagation behavior of partially coherent electromagnetic beams in anisotropic turbulence was investigated in Ref. [27] based on the spectrum proposed in Ref. [25]. To the best of our knowledge, there are almost no discussions with respect to anisotropic turbulence effects on the transmission properties of the superimposed OAM mode of multiple HB beams.

Here, we were interested in the transmission properties of the superimposed OAM mode of multiple HB beams in anisotropic turbulence. Rytov approximation theory was employed to establish the transmission model for the superimposed OAM mode in a weak turbulence channel. The influences of anisotropic turbulence and source parameters on the normalization modal power and OAM mode detection spectrum for the superimposed OAM mode carried by HB beams were discussed in detail. Our research might be useful for practical OAM-based FSO communication between airplanes at a high altitude.

The complex amplitude of the superimposed OAM mode of the multiple HB beams, each with the same beam parameters but with a different OAM mode $m_0$ space without turbulence interference can be written as

$$U_{\text{free}}(\rho,\varphi,z)=\sum_{m_0} t_{m_0} E_{\text{free}}^{m_0}(\rho,\varphi,z), \tag{1}$$

where $\boldsymbol{\rho}=(\rho,\varphi)$ is the two-dimensional position vector and $\varphi$ is the azimuthal angle; the complex amplitude $E_{\text{free}}^{m_0}(\rho,\varphi,z)$ of HB beams can be described as[19]

$$E_{\text{free}}^{m_0}(\rho,\varphi,z)=i^{3|m_0|+1}|m_0|!A_0\sqrt{\frac{\pi}{2kz}}J_{|m_0|/2}\left(\frac{k\rho^2}{4z}\right) \times \exp[i(kz-\pi|m_0|/4-\pi/4)]\exp(im_0\varphi), \tag{2}$$

where $A_0$ is a constant, $k=2\pi/\lambda$ is the wave number of the incident wavelength $\lambda$, and $J_m(x)$ is the Bessel function of first kind with $m$ order; $t_{m_0}$ donates the modal coefficients for an OAM mode $m_0$

$$t_{m_0}(z)=(2\pi)^{-1/2}\int_0^{2\pi} U_{\text{free}}(\rho,\varphi,z)\exp(-im_0\varphi)\rho\mathrm{d}\varphi. \tag{3}$$

We define $\gamma_{m_0}(z)=|t_{m_0}^2(z)|^2$, the modal weighting for mode $m_0$; the normalization of OAM modes guarantees that $\sum_{m_0}\gamma_{m_0}(z)=1$.

For two HB beams with the same beam parameter, but carrying different OAM modes $m_0$ and $l_0$, respectively, the superposition of two OAM modes is simply[13]

$$U_{\text{free}}^{m_0,l_0}(\rho,\varphi,z)=t_{m_0}E_{\text{free}}^{m_0}(\rho,\varphi,z)+t_{l_0}E_{\text{free}}^{l_0}(\rho,\varphi,z). \tag{4}$$

In the weak fluctuation region[24], the Rytov approximation can be employed to obtain the complex amplitude of the superimposed OAM mode carried by HB beams,

$$U_{\text{at}}^{m_0,l_0}(\rho,\varphi,z)=U_{\text{free}}^{m_0,l_0}(\rho,\varphi,z)\exp[\psi(\rho,\varphi,z)], \tag{5}$$

where $\Psi(\rho,\varphi,z)$ is the complex phase perturbation of spherical waves in the turbulence.

Using the superposition theory of spiral plane modes, $U_{\text{at}}^{m_0,l_0}(\rho,\varphi,z)$ in the turbulence can be written as[9]

$$U_{\text{at}}^{m_0,l_0}(\rho,\varphi,z)=\sum_{m=-\infty}^{\infty} t'_m(z)\exp(im\varphi). \tag{6}$$

The new modal weighting $\gamma'_m(z)=|t'^2_m(z)|^2$ for the spiral plane mode $m$ after the energy redistribution by the turbulence can be expressed as

$$\gamma'_m(z)=(2\pi)^{-1}\iint \exp[-im(\varphi-\varphi')]\,U_{\text{at}}^{m_0,l_0}(\rho,\varphi,z) \times U_{\text{at}}^{m_0,l_0,*}(\rho',\varphi',z)\mathrm{d}\varphi\mathrm{d}\varphi'. \tag{7}$$

By substituting Eq. (5) into Eq. (7), and averaging over turbulence ensembles, we have the ensembles average modal weighting for the superimposed OAM mode of multiple HB beams in anisotropic turbulence,

$$\langle\gamma'_m(z)\rangle=(2\pi)^{-1}\iint \exp[-im(\varphi-\varphi')] \times U_{\text{free}}^{m_0,l_0}(\rho,\varphi,z)\,U_{\text{free}}^{m_0,l_0,*}(\rho',\varphi',z) \times \langle\exp[\psi_1(\rho,\varphi,z)+\psi_1^*(\rho',\varphi',z)]\rangle_{\text{at}}\mathrm{d}\varphi\mathrm{d}\varphi', \tag{8}$$

where $*$ denotes the complex conjugate and $\langle\rangle_{\text{at}}$ denotes the average over the ensemble of the turbulence.

By using paraxial approximation of the wave structure function, average term $\langle\rangle_{\text{at}}$ in Eq. (8) can be obtained as[28]

$$\langle\exp[\psi_1(\rho,\varphi,z)+\psi_1^*(\rho',\varphi',z)]\rangle_{\text{at}} \approx \exp[-(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi'))/\rho_0^2], \tag{9}$$

where $\rho_0$ is the spatial coherence length of the spherical waves in anisotropic non-Kolmogorov turbulence,

$$\rho_0=\left[\pi^2k^2z/3\int_0^{\infty}\kappa^3\phi_n(\kappa)\mathrm{d}\kappa\right]^{-1/2}, \tag{10}$$

where $\phi_n(\kappa)$ denotes the spatial power spectrum of the atmospheric turbulence. Here, we assume that the turbulence anisotropy exists only along the propagation direction of the beam, and employ the anisotropic non-Kolmogorov von Karman spectrum proposed in Ref. [25] with the turbulence inner scale and outer scale effects considered,

$$\phi_n(\kappa)=A(\alpha)C_n^2\mu^2\frac{\exp\{-[\mu^2(\kappa_x^2+\kappa_y^2)+\kappa_z^2]/\kappa_l^2\}}{[\mu^2(\kappa_x^2+\kappa_y^2)+\kappa_z^2+\kappa_0^2]^{\alpha/2}}, \tag{11}$$

where $\kappa=\sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}$, $\alpha$ is the non-Kolmogorov power law with $3<\alpha<4$; $\mu$ is the effective anisotropy coefficient,

$C_n^2$ is a generalized refractive index structure parameter with units $\mathrm{m}^{3-\alpha}$ characterizing the strength of the turbulence[29], $\kappa_l = c(\alpha)/l_s$, $\kappa_0 = 2\pi/L_0$, $l_s$ is the inner scale, and $L_0$ is the outer scale, and $A(\alpha)$ and $c(\alpha)$ are given by

$$A(\alpha) = \Gamma(\alpha - 1)\cos(\pi\alpha/2)/4\pi^2, \tag{12}$$

$$c(\alpha) = \{\pi A(\alpha)\Gamma(3/2 - \alpha/2)[(3-\alpha)/3]\}^{1/(\alpha-5)}, \tag{13}$$

with symbol $\Gamma(x)$ being the Gamma function.

Utilizing the Markov approximation, ignoring the spatial wave number component $\kappa_z$ in Eq. (11), and substituting Eq. (11) into Eq. (10) yields

$$\rho_0 = \left\{\mu^{2-\alpha}\frac{\pi^2 k^2 z A(\alpha)}{6(\alpha-2)} C_n^2\left[\tilde{\kappa}_l^{2-\alpha}\gamma \exp\left(\frac{\kappa_0^2}{\kappa_l^2}\right)\Gamma\left(2-\frac{\alpha}{2},\frac{\kappa_0^2}{\kappa_l^2}\right) - 2\tilde{\kappa}_0^{4-\alpha}\right]\right\}^{-1/2}, \tag{14}$$

where $\gamma = 2\tilde{\kappa}_0^2 - 2\tilde{\kappa}_l^2 + \alpha\tilde{\kappa}_l^2$, $\tilde{\kappa}_0^2 = \kappa_0^2/\mu^2$, $\tilde{\kappa}_l^2 = \kappa_l^2/\mu^2$, and $\Gamma(x, y)$ is the incomplete Gamma function.

We need to define a modal power $M_m(z)$ for the superimposed OAM mode of the multiple HB beams[17],

$$M_m(z) = \iint \langle \gamma'_m(z)\rangle \rho' \rho \mathrm{d}\rho' \mathrm{d}\rho. \tag{15}$$

Based on the integral expression[30],

$$\int_0^{2\pi} \exp[-in\varphi_1 + \eta\cos(\varphi_1 - \varphi_2)]\mathrm{d}\varphi_1 = 2\pi \exp(-in\varphi_2)I_n(\eta), \tag{16}$$

where $I_n(x)$ is the Bessel function of the second kind with $n$ order. Making use of Eqs. (2), (8), (9), (15), and the Bessel function orthogonal completeness,

$$\int_0^\infty J_n(\rho,z)J_m(\rho',z)\rho'\mathrm{d}\rho' = \begin{cases}|J_n(\rho,z)|^2, & \text{if } m=n\\ 0, & \text{if } m\neq n\end{cases}, \tag{17}$$

and we can obtain the modal power for the mode $m$,

$$M_m(z) = \int \frac{\pi}{2kz}A_0^2 \exp\left(-\frac{2\rho^2}{\rho_0^2}\right) \times \left[\gamma_{m_0}(z)(|m_0|!)^2\left|J_{|m_0|/2}\left(\frac{k\rho^2}{4z}\right)\right|^2 \times I_{m-m_0}\left(\frac{2\rho^2}{\rho_0^2}\right) + \gamma_{l_0}(z)(|l_0|!)^2\left|J_{|l_0|/2}\left(\frac{k\rho^2}{4z}\right)\right|^2 \times I_{m-l_0}\left(\frac{2\rho^2}{\rho_0^2}\right)\right]\rho\mathrm{d}\rho. \tag{18}$$

The normalization modal power for the mode $m$ of the superimposed OAM mode can be expressed as[16]

$$p_m(z) = M_m(z)/I, \tag{19}$$

where $I$ is the total power,

$$I = \int_0^\infty \int_0^{2\pi} |U_{\text{free}}^{m_0,l_0}(\rho,\varphi,z)|^2 \rho\mathrm{d}\rho\mathrm{d}\phi = \sum_m M_m(z). \tag{20}$$

Analogously, we can obtain the normalization modal power and the OAM mode detection spectrum for an arbitrary superimposed OAM mode of the multiple HB beams in anisotropic turbulence.

In this section, we discuss the numerical results of the normalization modal power and the OAM mode detection spectrum for the superimposed OAM mode in anisotropic turbulence. The following parameters are assumed unless otherwise specified: $\lambda = 1550$ nm; $\mu = 30$; $\alpha = 3.37$; $L_0 = 50$ m; $l_s = 1$ mm; $C_n^2 = 10^{-15}\ \mathrm{m}^{3-\alpha}$; $z = 2$ km. Due to the lack of sufficient experiments on the anisotropic turbulence in the free atmosphere, the parameters adopted here are set as examples for theoretical analyses and other values can also be adopted.

The normalization modal power for the transmitted single-OAM mode $m_0$ of HB beams in the free atmosphere (anisotropic turbulence, $\mu = 30$) and atmospheric boundary layer (isotropic turbulence, $\mu = 1$) are evaluated in Fig. 1, where the mode quantum numbers $m_0$ range from 1 to 5. Optical turbulence in the free atmosphere produces a much weaker effect on the OAM modes than that in the atmospheric boundary layer with the same turbulence strength $C_n^2$. As a matter of fact, $C_n^2$ usually varies as a function of altitude and has a smaller value at a higher altitude[26]. Thus, the free atmosphere may be very suitable as an application scenario for OAM-based FSO communication.

In Fig. 2, we illustrate the impact of the beam source on the single-OAM mode of HB beams in the free atmosphere. To this end, we change the transmitted OAM mode $m_0$ of the HB beams and the LG beams ($p = 0$) from 1 to 5. Atmospheric turbulence effects on the normalization modal power for the transmitted OAM modes of both the HB beams and LG beams increase as the transmitted OAM mode quantum number increases. This is because vortex beams with a larger quantum number have wider radii that result in an increase in the negative effects of turbulence. However, the decline rate of the normalization modal power for the transmitted OAM mode of the HB beams is much gentler than that of the LG beams with the increasing $m_0$. The HB beams

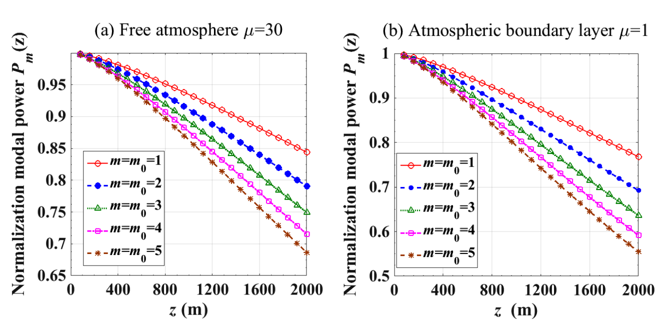


Fig. 1. Normalization modal power for the transmitted single-OAM mode of HB beams in the free atmosphere (anisotropic) and atmospheric boundary layer (isotropic) against $z$ for $m_0$.

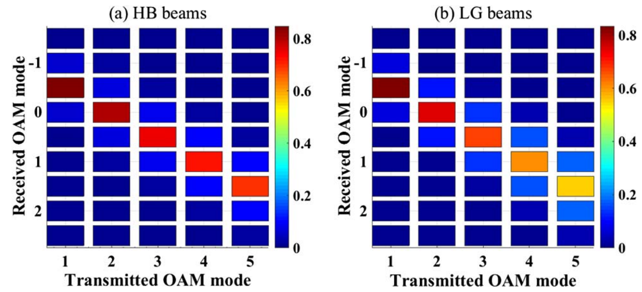


Fig. 2. Normalization modal power for the transmitted single-OAM mode of BG beams and LG beams in anisotropic non-Kolmogorov turbulence against $z$ for $m_0$.

provide excellent performance improvement of the LG beams because of their nondiffraction and self-focusing characteristics[19]. Nondiffraction vortex beams may be more efficient in practical OAM-based FSO communication in the atmospheric turbulence.

Without turbulence, we obtain a perfect OAM mode detection spectrum with no mode cross talk on transmitted single-OAM mode $m_0$, as observed in Fig. 3(a). Figure 3 presents the effect of turbulent anisotropy on the OAM mode detection spectrum for the transmitted OAM mode $m_0 = 4$ of HB beams by changing $\mu = 30, 3, 1.5, 1.1$, and 1 (isotropic turbulence). Atmospheric turbulence obviously causes the mode cross talk among the transmitted OAM modes. For instance, the modal power for the transmitted OAM mode $m_0 = 4$ will be detected by the receiver on modes $m = \{1, 2, 3, 4, 5, 6, \text{and } 7\}$ with OAM mode spreading. Compared with the atmospheric boundary layer ($\mu = 1$), anisotropic characteristics of the turbulence in the free atmosphere could further enhance the effective range of the transmitted single-OAM mode. As the anisotropy coefficient $\mu$ decreases, optical turbulence becomes stronger, the spectrum becomes wider, which induces higher modes of cross talk values. This is because the anisotropy turbulence introduces a rescaling of the spatial coherence length in the atmospheric turbulence, which is given by the anisotropy factor $\mu^{2-\alpha}$. This phenomenon can also be physically explained by the change in the curvature of the anisotropic turbulence eddies with respect to the isotropic case. The increase of $\mu$ will lead to anisotropic turbulent cell acting as a lens with a longer radius of curvature, thus a beam that passes through an anisotropic turbulent cell will be less deviated from the direction of propagation. HB beams with smaller beam radii have a higher normalization modal power of the transmitted OAM mode in the turbulence.

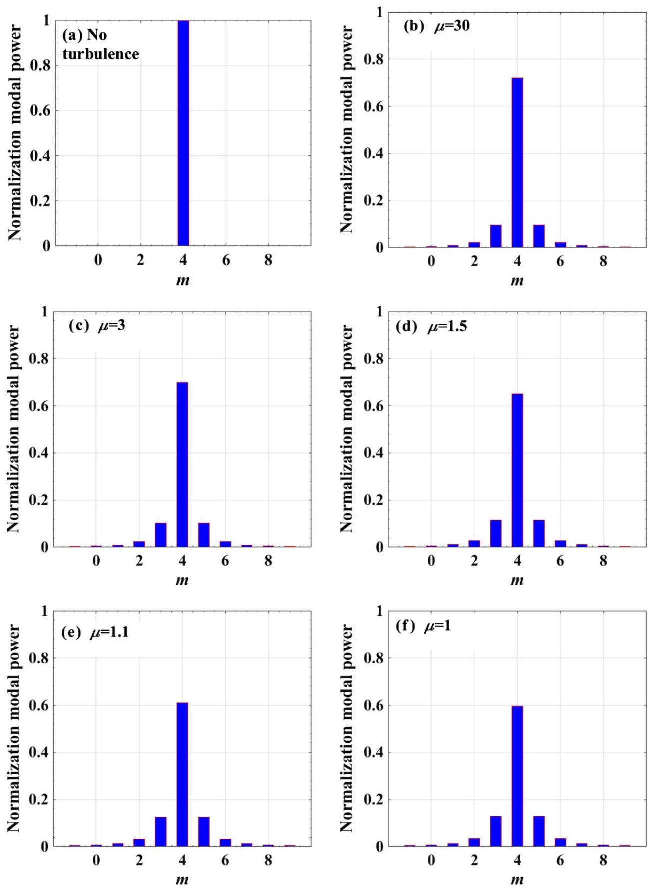


Fig. 3. OAM mode detection spectrum for single-OAM modes of HB beams propagating in anisotropic turbulence with different $\mu$.

Next, we focused on the OAM mode detection spectrum for the superimposed OAM mode carried by multiple HB beam propagation in anisotropic non-Kolmogorov turbulence. We first created a superimposed OAM mode of two HB beams having different OAMs $l_0$ and $m_0$ with equal modal weighting 1/2. The influence of the non-Kolmogorov power index $\alpha$ on this superimposed OAM mode in anisotropic non-Kolmogorov turbulence is depicted in Fig. 5 by changing $\alpha = 3.37$, 3.67, and 3.97. It is shown that, with the increase of $\alpha$, the anisotropic non-Kolmogorov turbulence produces more effects on the OAM mode detection spectrum of the superimposed OAM mode of multiple HB beams. This conclusion is consistent with that in isotropic turbulence[17]. We can also see that the superimposed OAM mode of two closer modes has a better performance than that of two farther modes in the atmospheric turbulence. For instance, the superimposed OAM mode of two OAM modes $l_0 = 3$ and $m_0 = 4$ [Fig. 4(e)] is easier to recognize than that of $l_0 = 2$ and $m_0 = 4$ [Fig. 4(f)], after propagation in anisotropic non-Kolmogorov turbulence with $\alpha = 3.97$. It is reasonable that, since the OAM mode cross talk is mutual, the cross talk of other transmitted OAM modes can also make a good supplement for its own power leakage. The majority of the cross talk of the transmitted OAM mode for HB beams happens with the immediate neighboring modes. Thus, atmospheric turbulence effects on the superimposed OAM mode can be effectively reduced by the appropriate allocation of OAM modes at the transmitter based on the reciprocal features of the mode cross talk.

In Fig. 5, we evaluate the influence of anisotropic turbulence on the OAM mode detection spectrum for the superimposed four OAM modes $m_1 = -8$, $m_2 = -4$,

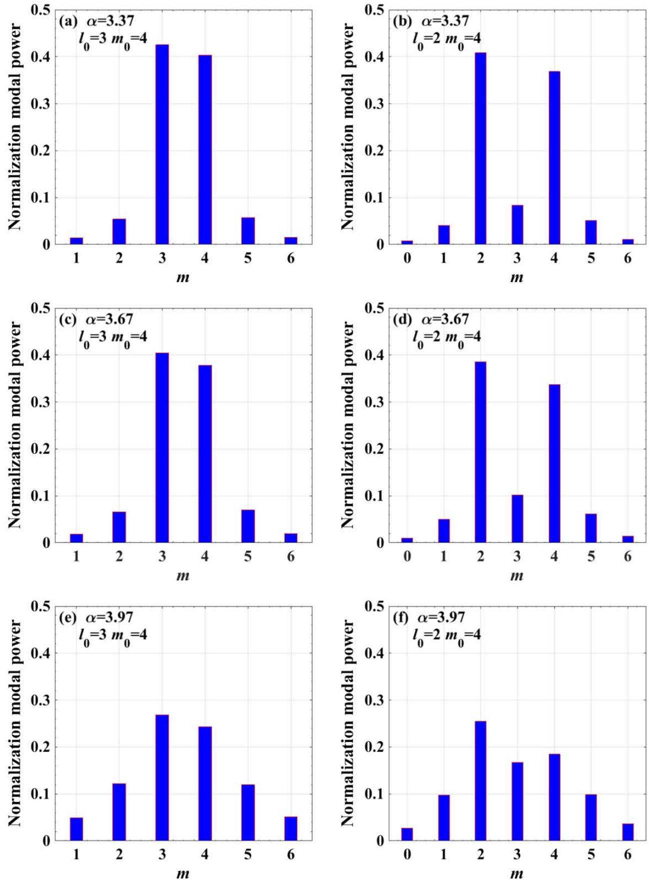


Fig. 4. OAM mode detection spectrum for the superimposed OAM mode of multiple HB beams in anisotropic turbulence with different $\alpha$.

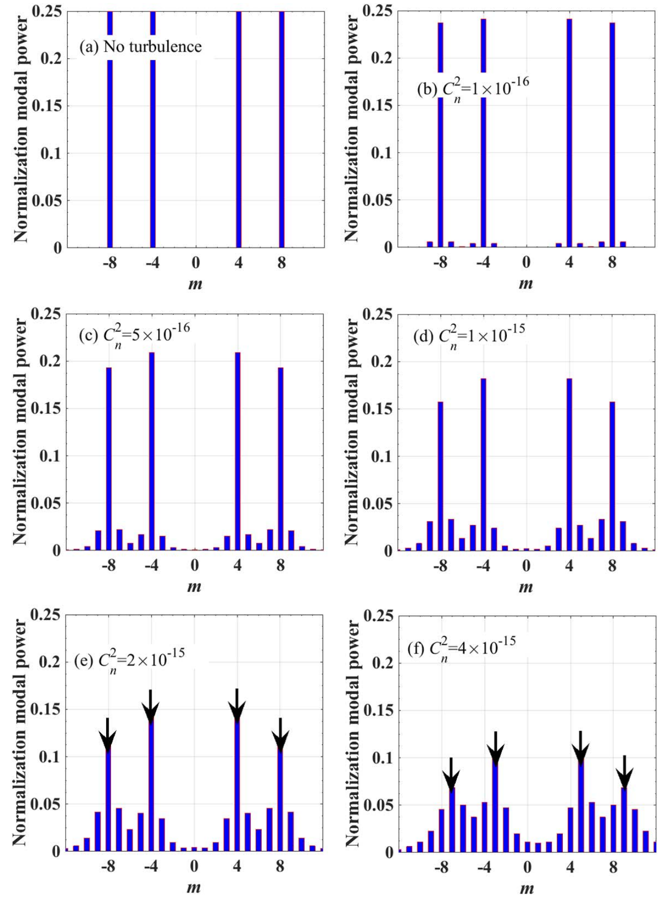


Fig. 5. OAM mode detection spectrum for the superimposed OAM mode of multiple HB beams in anisotropic turbulence with different $C_n^2$.

$m_3 = 4$, and $m_4 = 8$ (with the equal modal weighting 1/4) carried by HB beam propagation in the free atmosphere. Five different turbulence strengths (corresponding to the refractive index structure parameter values of $C_n^2 = 10^{-16}\ \mathrm{m}^{3-\alpha}$, $C_n^2 = 5\times10^{-16}\ \mathrm{m}^{3-\alpha}$, $C_n^2 = 10^{-15}\ \mathrm{m}^{3-\alpha}$, $C_n^2 = 2\times10^{-15}\ \mathrm{m}^{3-\alpha}$, and $C_n^2 = 4\times10^{-15}\ \mathrm{m}^{3-\alpha}$) are used. At a weaker turbulence level ($C_n^2 = 10^{-16}\ \mathrm{m}^{3-\alpha}$), the OAM mode detection spectra spread into neighboring OAM modes, but $m_0 = -8$, $m_0 = -4$, $m_0 = 4$, and $m_0 = 8$ still have a higher detection probability [Figs. 5(b) and 5(c)]. As the turbulence becomes stronger, the spectrum broadens, which induces higher cross talk values. The results show the high sensitivity of the OAM modes of light beams to atmospheric turbulence. However, the superimposed OAM mode still can be effectively distinguished due to the reciprocal features of the mode cross talk in the atmospheric turbulence.

In conclusion, we quantitatively describe the effects of anisotropic turbulence and source parameters on the normalization modal power of the transmitted OAM modes and the OAM mode detection spectrum for the superimposed OAM mode carried by nondiffraction HB beams in the free atmosphere. The combined influence of the mode quantum number, anisotropy coefficient, propagation distance, the non-Kolmogorov power index, and the refractive index structure parameter are considered. Turbulent aberrations evidently cause the mode cross talk among the transmitted OAM modes and induce the spread of the OAM mode detection spectrum in the atmospheric boundary layer within hundreds of meters of propagation. The anisotropic characteristics of the optical turbulence in the free atmosphere can further enhance the effective range of OAM-based FSO communication. The free atmosphere might be very suitable as an application scenario for OAM-based FSO communication. The HB beams provide an excellent performance improvement on LG beams because of their nondiffraction and self-focusing characteristics. The majority of the cross talk of the transmitted OAM modes for the HB beams happens with the immediate neighboring modes. The superimposed OAM mode of multiple closer modes will have a better performance in the atmospheric turbulence due to the reciprocal features of the mode cross talk. Atmospheric turbulence effects on the superimposed OAM mode can be effectively reduced by appropriate allocation of OAM modes at the transmitter. The negative effect of the turbulence aberrations on the transmitted OAM modes for HB beams increases with the increase of propagation distance, mode quantum number, and refractive index structure parameter, and with the decrease of the anisotropy coefficient, and the non-Kolmogorov power index.

This work was supported by the National Natural Science Foundation of China under Grant No. 61372033.

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