Uncertainty, certainty, and relativity

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One of the most fascinating principles in quantum mechanics must be Heisenberg's uncertainty principle, which can be briefly stated as follows: every physical observation cannot be precisely determined without some degree of error or uncertainty. And it is by no means can one use the principle within the limit of certainty region, as will be shown in this Letter. Two of the most important pillars in modern physics must be Einstein's relativity theory and Schrödinger's contribution to quantum mechanics. Yet, there is a profound connection between these discoveries by means of the uncertainty relationship, in which we shown that the observation of a high-speed object is conceivable if the speed of the observer keeps up with object's speed.

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Two of the most important discoveries in modern physics must be Einstein's relativity theory[1] and Schrödinger's equations of quantum theory[2]. These discoveries remain at the core of our understanding of fundamental physics. Nonetheless, these two branches of modern physics are profoundly connected by Heisenberg's uncertainty principle[3], as I will show in this Letter. The fact is, Heisenberg's uncertainty principle is one of the major accepted physical limitations in observation or detection in quantum mechanics. It states that every physical observation or measurement cannot be precisely determined due to the consequence of perturbation. This is the major constraint imposed upon on every observation and detection in practice.

Let us start the analysis of the uncertainty principle using a finite bandwidth approach[4], as follows: strictly speaking, all physical systems are finite bandwidth systems. A low-pass system is defined as a system that possesses a nonzero transfer characteristic from zero frequency to a definite frequency. Since the analysis of a bandpass system can be easily reduced to the case of an equivalent low-pass system, we restrict our discussion to only the low-pass analysis. Let us now discuss the low-pass system shown in Fig. 1.

Its transfer function is written as

$$H(\nu) = \begin{cases} 1 & |\nu| \leq \Delta\nu/2 \\ 0 & |\nu| > \Delta\nu/2 \end{cases}. \tag{1}$$

If the input signal to this low-pass system has a finite time-duration of Δt, then to have a good output reproduction of the input signal, it is required that the system's bandwidth $\Delta\nu$ be greater than or at least equal to $1/\Delta t$, that is:

$$\Delta\nu \geq 1/\Delta t, \tag{2}$$

where $1/\Delta t$ is known as the input signal bandwidth. Alternatively, we can write the following relationship:

$$\Delta\nu \, \cdot \, \Delta t \geq 1. \tag{3}$$

This is, in fact, a well-known relationship to Heisenberg's uncertainty principle in quantum mechanics as given by

$$\Delta x \, \cdot \, \Delta p \geq h, \tag{4}$$

where Δx and Δp are the position and momentum errors, respectively, and h is Planck's constant. In other words, the position and momentum observations cannot be simultaneously determined, as they are limited by Planck's constant.

Furthermore, the Heisenberg uncertainty relation can also be written in the form of energy and time variables, as follows:

$$\Delta E \, \cdot \, \Delta t \geq h. \tag{5}$$

The significance of this relationship implies that every bit of information takes time and energy to transmit, to process, to record, to retrieve, to learn, to assemble, to observe, and to detect, and it is not free.

Nonetheless, the essence of these uncertainty formulations is that they are the constraint of the region (i.e., either 1 or h), but not the shapes. In other words, as long they are limited by the constraint (i.e., either 1 or h), spectral and time resolutions can be traded, momentum and position errors can be exchanged, and the energy and time variables can be compensated.

The uncertainty relationship implies that in practice the accuracy of observation (or detection) cannot be obtained within the limits that are imposed on those
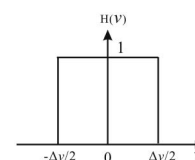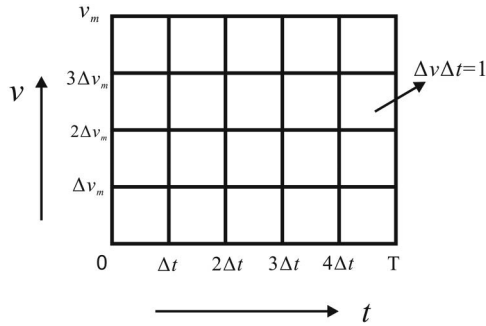

Fig. 1. Ideal low-pass filter.

Fig. 2. Gabor's information cell.

constraints, either 1 or h, as stated. This is in fact an impossibility, that one should not be violating the uncertainty relationship in any practical observation. Yet, the uncertainty relationship did not actually impose to being used within the region that is smaller than 1 or h, as will be seen later.

In 1946, Gabor[5] published a paper entitled "Theory of Communication" in the *Journal of the Institute of Electrical Engineers*, about two years before Shannon's[6] classical article, "A Mathematical Theory of Communication," appeared in the *Bell System Technical Journal*. Several of Gabor's concepts on information aspects were quite consistent with Shannon's theory of information. Here, we briefly illustrate one of his concepts of information as related to the Heisenberg uncertainty principle. Let us look at the frequency and time plot shown in Fig. 2.

Here, ν_m and T are the frequency and time limits of a time signal. Notice that this frequency-time plot can be subdivided into elementary information elements or cells (Gabor called them logons), as given by

$$\Delta\nu \cdot \Delta t = 1. \qquad (6)$$

Notice that this is essentially the lower bound of the uncertainty relation. Referring to Fig. 2, the plot contains

$$N' = (\nu_m/\Delta\nu)(T/\Delta t) \qquad (7)$$

numbers of information cells. Nonetheless, the signal within each of the cells can accommodate two possible elementary signals, symmetrical and anti-symmetrical signals (i.e., orthogonal signals). Thus, we see that the total number of information cells would be twice the numbers, as given by

$$N = 2N'. \qquad (8)$$

Notice that the shapes of the information cells are not particularly critical. The unit area is given by

$$\Delta\nu \cdot \Delta t = 1. \qquad (9)$$

As for the elementary signals, Gabor proposed using the two orthogonal Gaussian cosine and sine wavelets shown Fig. 3.

We further note that each information cell is in fact the lower bound of the Heisenberg uncertainty principle in
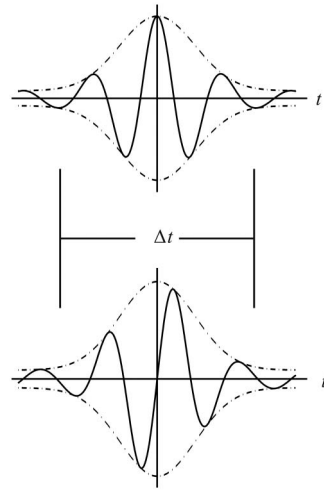

Fig. 3. Gaussian envelope; cosine and sine elementary signals.

quantum mechanics, and the band-limited signal must be of a very special type, in which the function has to be well behaved. In other words, the function contains no discontinuity, no sharp angles, and has only rounded features. These types of signals must be the analytic functions.

In view of the two orthogonal Gaussian wavelets illustrated in Fig. 3, the lower bounds of the uncertainty principle, strictly speaking, can be written in the following forms:

$$\Delta\nu \cdot \Delta t \geq 1/2, \qquad (10)$$

$$\Delta E \cdot \Delta t \geq h/2, \qquad (11)$$

$$\Delta x \cdot \Delta p \geq h/2, \qquad (12)$$

where the regions are bounded by either $1/2$ or $h/2$.

It is interesting to use an example to show that uncertainty principle indeed holds in practical observation. For this example, we provide the set of sound spectrogram analyses shown in Fig. 4, where we see that the spectral and time resolutions cannot be observed (or determined) simultaneously.

On the left-hand side, we see a wide-band sound spectrogram in which the time resolution Δt (i.e., time striation) can be easily identified, but at the expense of the finer frequency resolution, $\Delta\nu$. On the other hand,
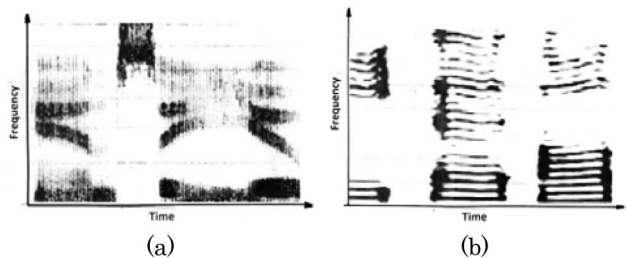

Fig. 4. Sound spectrogram analyses. (a) Broadband sound spectrogram. (b) Narrow-band sound spectrogram.

as we view the narrow-band analysis on the right-hand side, we see that the finer spectral resolution $\Delta \nu$ can be seen, but at the expense of the time striation, Δt. In view of these results, we note that the observations are quite consistent with the uncertainty principle's prediction: one cannot resolve (or observe) the frequency resolution $\Delta \nu$ and the time resolution Δt simultaneously[7].

Although the Heisenberg uncertainty principle seems not to be violated, it does not mean that one cannot work within the limited region. Let us show that, if one reverses the inequality of the uncertainty principle using

$$\Delta \nu \, \cdot \, \Delta t \leq 1, \tag{13}$$

then it is reasonable to name the preceding inequality the "certainty principle," in contrast with the uncertainty principle[3]. This means that when the light beam (e.g., the signal) propagates within the time resolution Δt, the complex light field preserves a high degree of certainty. Thus, as the bandwidth Δv of the light beam becomes narrower, the signal property is self-preserved (i.e., unchanged) within a longer time window Δt, or vise versa. This is in fact precisely the temporal coherence limit of the light beam (or signal). If one multiplies the preceding certainty inequality by the velocity of light $\boldsymbol{c}$, we have

$$c\Delta t \leq c/\Delta v, \tag{14}$$

which is essentially the coherence length (or certainty distance) of a signal beam (or light source), as written by

$$\Delta d \leq c/\Delta v. \tag{15}$$

This means that within the coherence length or certainty distance Δd, the transmitted signal is highly correlated with the original signal within a time window, as expressed by the mutual coherence function[8] (or certainty function):

$$\Gamma_{12}(\Delta t) = \lim_{T \to \infty} \frac{1}{T} \int_0^T u_1(t)u_2^*(t + \Delta t)\mathrm{d}t, \tag{16}$$

where $\Delta t \leq 1/\Delta \nu$, $*$ denotes the complex conjugate, u_1 is the received signal, and u_2^* is the signal before Δt.

The degree of certainty (i.e., mutual coherence) can be determined by the following equation:

$$\gamma_{12}(\Delta t) = \frac{\Gamma_{12}(\Delta t)}{[\Gamma_{11}(0)\Gamma_{22}(0)]^{1/2}}. \tag{17}$$

As discussed earlier, the shape of the information cell (i.e., $\Delta \nu \, \cdot \, \Delta t$) is not a critical issue, as long it is within a unit cell. Notice that it is this unit region that has not been fully exploited yet, as applied to signal transmission, information processing, measurement, and imaging.

Let us show with a couple of practical examples that images can be obtained within the certainty region, $\Delta \nu \, \cdot \, \Delta t \leq 1$.

Successful applications within the certainty regime (i.e., the certainty principle) must be due to the wavefront reconstruction (i.e., holography) of Gabor[9]. We all know

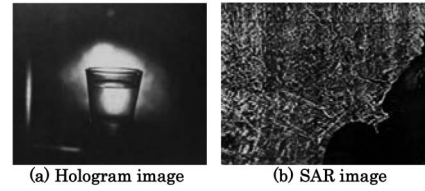(a) Hologram image   (b) SAR image

Fig. 5. (a) Hologram image. (b) SAR image.

that a successful holographic construction depends on the coherence length (i.e., temporal coherence) of the light source. This light source provides the constraint that the object beam and the reference beam are mutually coherent. Otherwise, the complex wavefront would not be properly recorded on a photographic film. On the left-hand side of Fig. 5, we can see a virtual image obtained from a hologram.

Another example: the application of synthetic aperture radar (SAR) imaging, in which the returned radar signal is required to be combining with a highly coherent local signal so that the complex distribution of the returned radar wavefront can be synthesized on a square-law medium. The right-hand side of Fig. 5 shows a SAR image, which was obtained using a SAR-recorded format[10]. We further note that some microwave radar has a very narrow bandwidth, and its coherence length (or certainty distance) Δd could be over hundreds of thousands of feet.

Since every bit of information is limited by the Heisenberg uncertainty principle,

$$\Delta \nu \, \cdot \, \Delta t \geq 1, \tag{18}$$

in which we notice that the spectral resolution and time resolution can be simply traded (or exchanged) and it is the unit cell, but not the shape of the cell, that sets the limit. In practice however, it is usually the constraint of the time window Δt that imposes the limit on improving the spectral resolution $\Delta \nu$. In other words, if one can elongate the time window, then a finer, detectable spectral resolution (by the constraint of the unity region) can, in principle, be observed (or detected). This means that if one can actually elongate the observation time window, then one should be able to push the spectral resolution to a lower limit without violating the constraints of the uncertainty principle.

Let us now take the time-dilation equation of Einstein's special theory of relativity[1], as given by

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}, \tag{19}$$

where $\Delta t'$ is the dilated time window of the observer (or detector), in which we assume the observer is traveling at a velocity ν. Δt is original time window of the observer at a standstill (i.e., $\nu = 0$), and c is the light velocity.

Thus, we see that when the observer is traveling at a velocity ν and it is observing an object that is standing still (i.e., at zero velocity), then it is the time window $\Delta t'$ for observing the object, instead of Δt, that must be used in the uncertainty principle. Therefore, we use

$$\Delta\nu \cdot \Delta t' \geq 1, \tag{20}$$

and since the dilated time window $\Delta t'$ is larger than Δt,

$$\Delta t' \geq \Delta t. \tag{21}$$

Thus, we see that a finer (or narrower) spectral resolution $\Delta\nu$ can conceivably be observed by the observer. The gain in spectral resolution is apparently due to the compensation of a wider observation time window $\Delta t'$.

It is interesting to note that, as the velocity of the observer ν approaches to the speed of light (i.e., $v \to c$), then the observation time window becomes infinitely wide (i.e., $\Delta t' \to \infty$), which provides a huge time window for the observer to observe (or detect). This means that when the observer is traveling at speed of light, the observer would have an infinitesimal spectral resolution (i.e., $\Delta\nu \to 0$), under the assumption that Einstein's relativity theory and the Heisenberg uncertainty principle are correct.

Another scenario is that, by interchanging the positions of the observer and the object to be observed, that is, the observer stands still and the object is traveling at a velocity ν instead, the time window as given by

$$\Delta t = \Delta t' \sqrt{1 - v^2/c^2}. \tag{22}$$

One should use this observation time window Δt for the uncertainty evaluation. Then, we see that a broader (i.e., poorer) spectral resolution $\Delta\nu$ is expected under the constraint of the uncertainty principle, since $\Delta t \leq \Delta t'$.

Now, if the observed object is traveling at the speed of light (i.e., $v \to c$), then the observer (or the detector) would have no time to observe the object, since $\Delta t \to 0$. In this case, the spectral resolution would be infinitely broad (i.e., $\Delta\nu \to \infty$).

The Heisenberg uncertainty principle can also be written in the following forms:

$$\Delta E \cdot \Delta t \geq h, \tag{23}$$

$$\Delta p \cdot \Delta x \geq h. \tag{24}$$

Notice that the constraints of these expressions are not created by the shapes, but by the limitation of Planck's constant, h. Again, we see that the energy resolution ΔE and the time resolution Δt in principle can be traded, as well for the position Δx and momentum errors Δp in principle, but not within the limit of Planck's constant. The applications of the time dilations $\Delta t'$ and Δt are dependent upon on the positions of the observer and the object to be observed, as described earlier.

We would further emphasize that the usage of the time dilation in the uncertainty principle may be one of the crucial connections between quantum mechanics and the relativity theory, in which we see that a particle (or substance) travels at a very high speed (e.g., close to the speed of light or a very short life) and can be observed (or detected) in principle.

In the proceeding, we have provided two applications within the certainty region (i.e., within the unit information cell) as applied to holography and to SAR imaging. For these reasons, it is conceivable that the application of the relativity theory within the certainty region is possible. We further emphasize that the application is not just limited to spectral and time resolutions (i.e., $\Delta\nu \cdot \Delta t \leq 1$): it can also be extended to the energy and time variables (i.e., $\Delta E \cdot \Delta t \leq h$), and as well to the position and momentum errors (i.e., $\Delta p \cdot \Delta x \leq h$).

Within the current physical constraints, observation is limited by the light's speed. If we assume the Schrödinger's quantum mechanics and Einstein's special theory of relativity are correct, then there exists a profound relationship between these two pillars of modern physics, which are connected by Heisenberg's uncertainty relationship.

As we all know, the basic constraint, as advocated by the uncertainty principle, is that one cannot observe time and spectral resolutions simultaneously (i.e., $\Delta\nu \cdot \Delta t \geq 1$), or position and momentum precisely. Then, there is the dilemma of observing objects that are traveling at a very high speed or close to the speed of light. We have shown in the preceding that the constraint for any physical observation can be alleviated by using a wider observation time window, but with a price tag for a higher observation speed.

In order to mitigate this constraint, we let the observer is travel at a speed of $\nu 1$, and the object travel at a velocity of $\nu 2$. For simplicity, we assume that the observer and the object are traveling in the same direction, and the observer travels faster than the object (i.e., $\nu 1 > \nu 2$). Then, the observation time dilation can be written as

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - (v1 - v2)^2/c^2}}. \tag{25}$$

Thus, we see that as the velocity of the observer increases, a longer observation time window can be obtained, since the observer's time is running slower than the object's time. Thus, a finer spectral resolution can, in principle, be observed or detected.

In the other scenario, if the object runs faster than the observer (i.e., $\nu 1 < \nu 2$), the observer would have a narrower observation time window, as given by

$$\Delta t = \Delta t' \sqrt{1 - (v1 - v2)^2/c^2}. \tag{26}$$

Then, observer would have a broader (i.e., poorer) spectral resolution.

In view of the preceding illustrations, we see that observing (or detecting) an object that travels at a very high speed (or near the speed of light) in principle is conceivable, as long the observer can keep up with the object's speed.

The essence of the Heisenberg uncertainty principle can be stated as follows: every physical observation cannot be precisely determined without some degree of error or uncertainty. The principle defines the fundamental limits

that time and spectra, energy and time, and position and momentum cannot be observed (or detected) simultaneously. As Dennis Gabor showed, the uncertainty relationship is in fact related to an information cell called the logon, and he showed that time and spectra, energy and time, and position and momentum can in principle be traded. In this context of the uncertainty relation, we have shown that every bit of information takes time and energy to transmit, to process, to record, to retrieve, to learn, to observe, and to detect, and it is not free. Although the uncertainty principle intrinsically cannot be violated in terms of observation or detection, we have shown that applications such as holography and SAR imagery can take place within the certainty limit. As we know, two of the most distinguished pillars in modern physics must be Einstein's special theory of relativity and Schrödinger's quantum mechanics. Yet, we have shown that these two pillars are profoundly connected by means of the uncertainty principle; we have shown that

the observation (or detection) of a high-speed object is conceivable in principle, if the observer can keep up with the object's speed.

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