Theoretical bounds on Fresnel compressive holography performance
(Invited Paper)

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During the last years the theory of compressive sensing has found significant utility in the digital holography realm. In this letter we summarize and extend our previous theoretical results which determine the relation between the number of Fresnel samples required on the object illumination type, illumination wavelength, imaging geometry and sensor's size and resolution.

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1. Introduction

Digital holography (DH) provides an indirect framework to capture the complex field amplitude of a wavefront propagated from an object. Digital holography is used in many areas including digital holographic microscopy, 3D macroscopic imaging, aberration correction, holographic interferometry, quantitative phase imaging of cells, object surface and tomographic imaging. The incoming object wavefront is captured using a semiconductor based device (such as CCD) and reconstructed using numerical means on a computer[1].

In this work, we overview our previously published results which treat DH as a compressive sensing (CS)[2−4] mechanism. Specifically, we overview and generalize our previous results in Ref. [5] that relate the CS performance with the DH setup and its components specifications. Here we generalize our theoretical results in Ref. [5] to include different types of object illumination and summarize the practical findings in an accessible form for computational imaging and holography practitioners.

Compressive sensing theory asserts that one can recover sparse signals from far fewer samples or measurements than traditional methods require. The "compressive" part relays on the signal's sparsity assumption, which expresses the idea that a signal can be represented using a low-dimensional mathematical model. Indeed, typical holographic images are often composed from small objects sparsely spread in the field-of-view, or they can be mathematically transformed into a domain in which their coefficients are sparsely distributed. The "sensing" part relates to the physical mechanism which projects the information from the object space to the measurement space. In this paper, we consider the simplest such mechanism, namely that of free space propagation. The propagated field can be simply recorded using DH. The simplicity and yet effectiveness of this sensing mechanism, makes it an attractive compressive sensing choice[5], which can be used in scenarios of limited number of available sensor pixels, reconstruction of highly distorted objects (including occluded objects) and inference of 3D object tomography from its 2D hologram[6].

In the compressive sensing framework, the signal is recovered using algorithms from the family of ℓ_1-norm minimization solvers[2−4]. Several conditions which relate to the signal's sparsity and the sensing operator must apply in order to guarantee accurate and unique solution. These sampling and recovering conditions are formulated in CS literature via various mathematical constructs, such as the restricted isometry principle[2] and the null space condition[3]. Another practical recovery condition applicable for general sensing problems is via what is known as the *coherence parameter*[2−4,7]. The coherence parameter can be used to link between the number of sparse signal elements, S, and the number of measurements, M, where the ambient dimension of the signal to be recovered, N, is such that $S < M \leqslant N$. In this work we evaluate the performance of three compressive Fresnel DH setups by calculating the coherence parameter of their sensing model.

2. Reconstruction guarantees for randomly subsampled Fresnel fields

Let us consider the case where one wishes to design a sensing system that samples the object's diffraction field at some distance away from it using only a small number of available detectors, placed uniformly at random in the detector plane. Mathematically this is described as the process of uniformly picking M out of N rows of $\mathbf{\Phi}$ at random, where $\mathbf{\Phi}$ is an $N \times N$ matrix describing the optical sensing operator in nominal sampling conditions. The coherence parameter is given by[2−4,7]

$$\mu = \max_{i,j} |\langle \varphi_i, \psi_j \rangle| , \tag{1}$$

where ϕ_i is a row vector of the sensing operator $\mathbf{\Phi}$, ψ_j is a column vector of $\mathbf{\Psi}$, which is the sparsifying operator and $\langle \cdot, \cdot \rangle$ denotes inner product operation. The coherence parameter, μ, measures the incoherence, or dissimilarity, between the sensing and the sparsifying operators. In the common case that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are orthonormal bases it can be shown that $1/\sqrt{N} \leqslant \mu \leqslant 1$[2]. We note that in some of the CS literature the coherence parameter is defined by the expression in Eq. (1) multiplied with a factor of $\sqrt{N}$, yielding $1 \leqslant \mu \leqslant \sqrt{N}$. In this paper, we follow the definition in Eq. (1). For a given mutual coherence parameter, the CS theory asserts that the signal can be reconstructed from M uniformly at random projections, provided that[2]

$$M \geqslant C\mu^2 NS \log N. \tag{2}$$

In the following, we shall consider the case $\boldsymbol{\Psi} = \boldsymbol{I}$, where $\mathbf{I}$ is the canonical basis, i.e., the signal we wish to reconstruct is sparse in the space domain. More general conditions will be discussed at the end of this section.

The input object, $f(x, y)$, is illuminated by a plane wave of wavelength λ, and the resultant object wavefield propagates in free space till reaches the CCD, positioned at a distance z away from the input plane, as illustrated in Fig. 1. This object complex field amplitude interferes with a reference wave, U_R, which is required to allow the extraction of the object's complex field amplitude from the captured intensity. Several methods may apply for this holographic recording process, such as off-axis holography and phase shifting holography[1]. After the complex field amplitude extraction, our sensing operator, $\boldsymbol{\Phi}$, accounts only for the free space propagation of the object's wavefield from the object to the sensor plane.

By applying the Fresnel transform as an approximation to the free space propagation, we get

$$\begin{aligned} g(x,y) &= f(x,y) * \exp\left\{\frac{\mathrm{j}\pi}{\lambda z}\left(x^2+y^2\right)\right\} \\ &= \exp\left\{\frac{\mathrm{j}\pi}{\lambda z}\left(x^2+y^2\right)\right\} \iint f(\xi,\eta) \\ &\quad \cdot \exp\left\{\frac{\mathrm{j}\pi}{\lambda z}\left(\xi^2+\eta^2\right)\right\} \exp\left\{\frac{-\mathrm{j}2\pi}{\lambda z}\left(x\xi+y\eta\right)\right\} \mathrm{d}\xi \mathrm{d}\eta. \end{aligned} \tag{3}$$

where "$*$" denotes convolution operation. Since in CS framework the signal is numerically reconstructed, the dependence of compressive digital holographic sensing on the system parameters should be analyzed by inspecting the numerical version of the Fresnel wave propagation. Usually the Fresnel numerical approximation is divided to *near* and *far* field numerical approximations[5,8]. The numerical *near field* approximation is given by

$$\begin{aligned} g\left(p\Delta x_o, q\Delta x_o\right) =& \mathcal{F}_{\mathrm{2D}}^{-1} \exp\left\{-\mathrm{j}\pi\lambda z\left(\frac{m^2}{N\Delta x_0^2}+\frac{n^2}{N\Delta y_0^2}\right)\right\} \\ & \cdot \mathcal{F}_{\mathrm{2D}}\left\{f\left(l\Delta x_0, k\Delta y_0\right)\right\}, \end{aligned} \tag{4}$$

where Δx_0, Δy_0 are object and CCD resolution pixel size, with $0 \leqslant p, q, k, l \leqslant \sqrt{N}-1$ and $\mathcal{F}_{2D}$ denotes the 2D Fourier transform. We assume that the sizes of the object and of the sensor are $\sqrt{N}\Delta x_0 \times \sqrt{N}\Delta y_0$. The near field model is valid for the regime where $z \leqslant z_0 = \max(\sqrt{N}\Delta x_0^2/\lambda, \sqrt{N}\Delta y_0^2/\lambda)$[6]. For the working regime of $z \geqslant z_0 = \max(\sqrt{N}\Delta x_0^2/\lambda, \sqrt{N}\Delta y_0^2/\lambda)$ the far field numerical approximation is given by

$$\begin{aligned} & g(p\Delta x_z, q\Delta y_z) = \exp\left\{\frac{\mathrm{j}\pi}{\lambda z}\left(p^2\Delta x_z^2+q^2\Delta y_z^2\right)\right\} \\ & \mathcal{F}_{\mathrm{2D}}\left[f(k\Delta x_0, l\Delta y_0)\exp\left\{\frac{\mathrm{j}\pi}{\lambda z}\left(k^2\Delta x_0^2+l^2\Delta y_0^2\right)\right\}\right], \end{aligned} \tag{5}$$

where $\Delta x_z = \lambda z/(\sqrt{N}\Delta x_0)$; $\Delta y_z = \lambda z/(\sqrt{N}\Delta y_0)$ is the output field's pixel size.

In this case, the coherence parameter is given by[6,9]

$$\mu_{\text{near field}} = \max_i |\phi_i| \approx \left[\Delta x_0 \Delta y_0/\left(\lambda z\right)\right], \tag{6}$$

where ϕ_i is a column vector (or PSF) of the matrix $\boldsymbol{\Phi}$, which represents Eq. (5) in a matrix-vector multiplication form; $\mathbf{g} = \boldsymbol{\Phi}\mathbf{f}$. Using Eq. (6) with Eq. (2), it can be shown[5] that the number of compressive measurements that are needed to accurately reconstruct the object is given by

$$M \geqslant C' N_F^2 \frac{S}{N} \log N, \tag{7}$$

where N_F denotes the recording device Fresnel number[10] given by $N_F = N\frac{\Delta x_0 \Delta y_0}{4\lambda z}$, and C' is a small constant factor[2,11]. Equation (7) determines that as the working distance gets larger, N_F decreases implying that fewer samples are needed for accurate reconstruction. For the case that $z > \max(\sqrt{N}\Delta x_0^2/\lambda, \sqrt{N}\Delta y_0^2/\lambda)$ the numerical near field approximation is not valid (see Ref. [8]). In this case, we need to use the far field numerical approximation [Eq. (5)], yielding the coherence parameter[6]

$$\mu_{\text{far field}} = 1/\sqrt{N}. \tag{8}$$

This is the smallest value that the mutual coherence can get, therefore according to Eq. (2), the number of required measurements requires for exact reconstruction is the smallest:

$$M \geqslant CS \log N. \tag{9}$$

This bound on the number of measurement remains constant through the far field regime. It is possible to get a physical intuition about this result by noticing that the object's diffraction pattern spatial spread is inversely proportional to its Fresnel number. Thus, as we move away from the object plane (and the Fresnel number decreases) each sample contains information about a larger portion of the object. This implies that even if we discard some of the samples, it is possible to reconstruct the object because the missing information can be extracted from other samples. Hence the signal can be accurately reconstructed from less than N detectors.

In the above analysis, we have assumed that object is sparse in the spatial domain. In the case where the

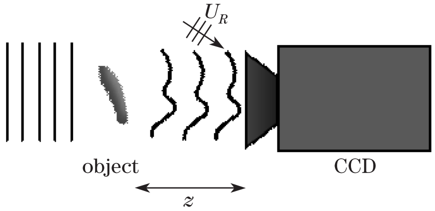

Fig. 1. Digital Holographic recording of the free space propagated field from an object illuminated by a plane wave.

object of interest is sparsely represented using a sparsifying operator, which is different from the canonical basis, e.g., it is sparse in wavelet transform or in its number of gradients, it is practically difficult to derive exact closed form expressions as those obtained in Eqs. (7) and (9). However, a numerical investigation which was described in Ref. [6] shows that the general behavior predicted from Eqs. (7) and (9) also holds for cases where common sparsifiers are applied.

3. Compressive digital holographic sensing of spherically illuminated objects

In many holographic applications, the object is illuminated by a spherical wavefront, especially in compact microscopy (lensless) systems. This illumination condition is illustrated in Fig. 2. The object is illuminated by a spherical wave originating form a point source at distance z_i from it.

In this case, the calculation of the coherence parameter needs to be changed accordingly. The Fresnel approximation of a diverging spherical wave in the free space is given by

$$\begin{aligned} g(x,y) &= \exp\Big(\mathrm{j}\pi\frac{x^2+y^2}{\lambda z_i}\Big) f(x,y) * \exp\Big(\mathrm{j}\pi\frac{x^2+y^2}{\lambda z}\Big) \\ &= \exp\left(\mathrm{j}\pi\frac{x^2+y^2}{\lambda z}\right)\int\int f(\xi,\eta)\exp\Big(\mathrm{j}\pi\frac{\xi^2+\eta^2}{\lambda}\Big(\frac{1}{z}+\frac{1}{z_i}\Big)\Big) \\ &\cdot \exp\Big\{\frac{-\mathrm{j}2\pi}{\lambda z}(x\xi+y\eta)\Big\}\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \quad (10)$$

Using similar arguments that we have used for the planar illumination case, we can define the Fresnel kernel sampling condition[8] (expressed for the 1D case, for simplicity):

$$\frac{\Delta x_0^2}{\lambda}\left(\frac{1}{z}+\frac{1}{z_i}\right) < \frac{1}{\sqrt{N}}, \quad (11)$$

Consequently, the working distance which defines the limit between the near and far field numerical approximation is given by

$$z = \frac{\sqrt{N}\Delta x_0^2}{\lambda - \sqrt{N}\Delta x_0^2/z_i}. \quad (12)$$

As a further elaboration of previous work in Ref. [6], using similar arguments used for the planar illumination case [Eq. (6)], the coherence parameter for the diverging

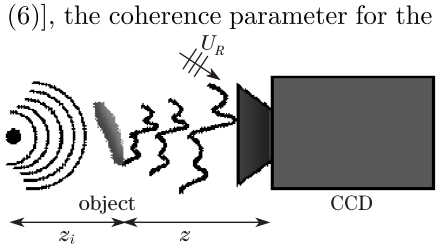

Fig. 2. Digital Holographic recording of the free space propagated field from an object illuminated by a diverging spherical wave.

spherical illumination is found:

$$\mu_{\mathrm{1D}}^{\mathrm{diverging}} = \max_i \left|\phi_i^{\mathrm{diverging}}\right| \approx \Delta x_0 \Big/ \sqrt{\lambda\left[\frac{z\times z_i}{z+z_i}\right]}, \quad (13)$$

where $\phi_i^{\mathrm{diverging}}$ is the column vector (or PSF) of the matrix $\mathbf{\Phi}^{\mathrm{diverging}}$, which is the 1D matrix-vector representation of Eq. (10); $\mathbf{g} = \mathbf{\Phi}^{\mathrm{diverging}}\mathbf{f}$. As in the previous case, in the far field approximation $\mu_{\text{far field}}^{\mathrm{diverging}} = 1/\sqrt{N}$. Extending our result in Eq. (13) to 2D yields[9]

$$\mu_{\mathrm{2D}}^{\mathrm{diverging}} \approx \frac{\Delta x_0^{\Delta} y_0}{\lambda\left[\frac{z\times z_i}{z+z_i}\right]}. \quad (14)$$

Note that by taking the limit $z_i \to \infty$, Eq. (13) reduces to the coherence parameter, μ, found in the planar field illumination case [Eq. (6)].

Combining Eqs. (14) and (2), we find that the required number of samples in the near field regime is given by

$$M \geqslant C\left\{\Delta x_0 \Delta y_0/\lambda\left[\frac{z\times z_i}{z+z_i}\right]\right\}^2 NS\log N. \quad (15)$$

This result can also be extended to converging spherical wave illumination. In this case, the coherence parameter is given by

$$\mu_{2D}^{\mathrm{converging}} \approx \frac{\Delta x_0 \Delta y_0}{\lambda\left[\frac{z\times z_i}{z-z_i}\right]}. \quad (16)$$

It can be seen that by using convergent object illumination the mutual coherence can be reduced, thus yielding higher compression [according to Eq. (2)]. This implies that for applications with limited number of detector pixels and that require proximity between the sample and the detector, illuminating the object with a converging wavefront should be preferred.

4. Summary and conclusion

We have overviewed the reconstruction guarantees for applying compressive Fresnel digital holography with different types of object illumination. The analysis reveals the dependence of the amount of subsampling permitted in the detector plane on the illumination type, wavelength, working distance, sensor size and resolution, and on the object's spatial size and its resolution. We have shown that for all types of object illumination, if the sensor is placed far enough from the object, a maximum compression is possible according to Eq. (9). However, in the near field the number of samples which are required for accurate object reconstruction is proportional to the coherence parameter. Table 1 summarizes the values of the coherence parameter for various object illumination conditions and the appropriate definition of the numerical near field regimes.

Table 1. Values of the Coherence Parameter for Various Object Illumination Conditions and the Appropriate Definition of the Numerical Near Field Regimes

	Numerical Near Field Approximation Condition	Coherence Parameter (μ_{2D}) for Object to Detector Distance, z
Planar Illumination	$z \leqslant \max(\sqrt{N}\Delta x_0^2/\lambda, \sqrt{N}\Delta y_0^2/\lambda)$	$\frac{\Delta x_0 \Delta y_0}{\lambda z}$
Diverging Spherical Wave Illumination	$z < \max(\frac{\sqrt{N}\Delta x_0^2}{\lambda-\sqrt{N}\Delta x_0^2/z_i}, \frac{\sqrt{N}\Delta y_0^2}{\lambda-\sqrt{N}\Delta y_0^2/z_i})$	$\frac{\Delta x_0 \Delta y_0}{\lambda\left[\frac{z\times z_i}{z+z_i}\right]}$
Converging Spherical Wave Illumination	$z < (\frac{\sqrt{N}\Delta x_0^2}{\lambda+\sqrt{N}\Delta x_0^2/z_i}, \frac{\sqrt{N}\Delta y_0^2}{\lambda+\sqrt{N}\Delta y_0^2/z_i})$	$\frac{\Delta x_0 \Delta y_0}{\lambda\left[\frac{z\times z_i}{z-z_i}\right]}$

These presented analytical guarantees can help to design "compressible sensors" where the number of pixels is much smaller than that dictated by the classical Nyquist criterion. The sensing mechanism is the free space propagation of the object's wavefield, which is natural, requires no special hardware to generate or store it and easy to acquire. Reconstruction of free space propagation of the object's wavefield can also be beneficial for subsampling mechanism which are not random, i.e., imposed by physical attributes of the system. Such cases were also described in Ref. [12] for reconstruction of partially occluded objects, in Ref. [13] for reconstruction of sub-pixel resolution movement estimation and in Refs. [14–20] for high resolution 3D object tomography from their single 2D hologram.

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