Temporal power spectral models of angle of arrival fluctuations for optical waves propagating through weak non-Kolmogorov turbulence

Linyan Cui (崔林艳)[1,2], Bingdang Xue (薛斌党)[2*], Wenyue Zhu (朱文越)[1], Xiangzhi Bai (白相志)[2], Xiaoguang Cao (曹晓光)[2], and Fugen Zhou (周付根)[2]

[1] *Key Laboratory of Atmospheric Composition and Optical Radiation, Chinese Academy of Sciences, Hefei 230031, China*

[2] *School of Astronautics, Beihang University, Beijing 100191, China*

[*] *Corresponding author: xuebd@buaa.edu.cn*

Received August 20, 2013; accepted October 17, 2013; posted online November 8, 2013

Analytical expressions of the temporal power spectral models of angle of arrival (AOA) fluctuations are derived using the generalized exponential spectral model for optical waves propagating through weak non-Kolmogorov turbulence. Compared with expressions of temporal power spectral models derived from the general non-Kolmogorov spectral model, the new expressions consider the influences of the inner and outer scales of finite turbulence. Numerical calculations show that large outer scales of turbulence increase the value of the temporal power spectrum of AOA fluctuations in low-frequency regions.

OCIS codes: 010.1290, 010.1300, 010.1330.

doi: 10.3788/COL201311.110102.

Atmospheric turbulence produces a series of effects, which include angle of arrival (AOA) fluctuations, irradiance scintillation, beam spread, and so on, on imaging or laser systems. AOA fluctuations of optical waves in the plane of a receiver aperture are related to image dancing in the focal plane of an imaging or laser system[1]. Several researchers have focused on the case of non-Kolmogorov turbulence, which covers a wide range of atmospheric layers[2−8]. The non-Kolmogorov atmospheric turbulence spectral model has been used to investigate the temporal power spectrum of AOA fluctuations for plane and spherical waves propagating through weak non-Kolmogorov turbulence[9]. However, no study is yet available on the influences of the inner and outer scales of turbulence. AOA fluctuations are mainly caused by large-scale turbulence cells[10]. As such, the outer scale of turbulence plays a very important role in the present analysis. Previous studies have shown that the generalized Von Karman spectrum[11] and the generalized exponential spectrum[12] may be used to analyze the influence of the outer scale of turbulence on AOA fluctuations[11−14]. In this letter, the generalized exponential spectral model[12] is adopted to derive theoretical expressions of the temporal power spectral models of AOA fluctuations for plane and spherical waves propagating through weak non-Kolmogorov turbulence.

The generalized exponential spectral model is expressed as[12]

$$\Phi_n(\kappa,\alpha,l_0,L_0) = A(\alpha)\cdot \hat{C}_n^2\cdot\kappa^{-\alpha}\cdot f(k,l_0,L_0,\alpha)\quad 0\leqslant\kappa<\infty, 3<\alpha<5, \tag{1}$$

$$f(\kappa,l_0,L_0,\alpha)=\left[1-\exp\left(-\frac{\kappa^2}{\kappa_0^2}\right)\right]\cdot\exp\left(-\frac{\kappa^2}{\kappa_l^2}\right), \tag{2}$$

where $\widehat{C}_n^2$ is the generalized refractive-index structure parameter, κ denotes the magnitude of the spatial-frequency vector and is related to the size of turbulence cells, and $\kappa_l = c(\alpha)/l_0$, $\kappa_0 = C_0/L_0$, l_0 and L_0 are the inner and outer scales of turbulence, respectively. C_0 depends on the specific application; in this study, it is set to 4π, similar to that in Ref. [10]. $A(\alpha)$ and $c(\alpha)$ are expressed as[12]

$$A(\alpha)=\frac{\Gamma(\alpha-1)}{4\pi^2}\sin\left[(\alpha-3)\frac{\pi}{2}\right],$$
$$c(\alpha)=\left\{\pi A(\alpha)\left[\Gamma\left(-\frac{\alpha}{2}+\frac{3}{2}\right)\left(\frac{3-\alpha}{3}\right)\right\}^{\frac{1}{\alpha-5}}. \tag{3}$$

Following Ref. [1], the temporal power spectrum of AOA fluctuations $W_\theta(\omega,\beta)$ is the Fourier transform of the temporal covariance function of AOA $C_\theta(t,\beta)$

$$W_\theta(\omega,\beta)=4\int_0^\infty C_\theta(t,\beta)\cos(\omega t)\,\mathrm{d}t. \tag{4}$$

Using the Taylor frozen turbulence hypothesis, $C_\theta(t,\beta)$ can be determined from the spatial covariance function of AOA $C_\theta(\rho,\beta)$[15]:

$$C_\theta(\rho,\beta)=\pi k^{-2}\int_0^\infty\kappa^3 W_\phi(\kappa)\,G_D(\kappa)\cdot[J_0(\rho\kappa)-\cos(2\beta)J_2(\rho\kappa)]\,\mathrm{d}\kappa, \tag{5}$$

where ρ represents the geometrical separation between points in the plane transverse to the direction of propagation, β is the angle between the baseline and the AOA observation axis, and $k=2\pi/\lambda$, where λ denotes the optical wavelength. $J_0(\rho\kappa)$ and $J_2(\rho\kappa)$ respectively denote the zero and second order Bessel functions. The Taylor frozen turbulence hypothesis satisfies the association of $\rho=v_\perp t$, where $\nu_\perp$ denotes the wind velocity perpendicular to the propagation path of optical waves. In this case, $C_\theta(t,\beta)$ can be written as

$$C_\theta(t,\beta) = \pi k^{-2}\int_0^\infty \kappa^3 W_\phi(\kappa)\, G_D(\kappa) \left[J_0(\nu_\perp t\kappa) - \cos(2\beta)\, J_2(\nu_\perp t\kappa)\right] \mathrm{d}\kappa, \tag{6}$$

where $W_\phi(\kappa)$ is the wave-front phase power spectrum. For plane and spherical waves, $W_\phi(\kappa)$ takes different expressions[16,17]:

$$W_{\phi(\mathrm{pl})}(\kappa) = 2\pi k^2 \int_0^L \Phi_n(\kappa)\cos^2\left(\frac{\kappa^2 z}{2k}\right) \mathrm{d}z, \tag{7}$$

$$W_{\phi(sp)}(\kappa) = 2\pi k^2 \int_0^L \Phi_n(\kappa)\left(\frac{z}{L}\right)^2 \cos^2\left[\frac{\kappa^2 z(L-z)}{2kL}\right] \mathrm{d}z, \tag{8}$$

where $W_{\phi(\mathrm{pl})}(\kappa)$ and $W_{\phi(\mathrm{sp})}(\kappa)$ are the wave-front phase power spectrum functions for plane and spherical waves, respectively.

$G_D(\kappa)$ denotes the point-spread function of the receiver aperture[18]

$$G_D(\kappa) = \exp\left[-\frac{c^2 D^2 \kappa^2}{4}\right], c = 0.52. \tag{9}$$

For weak non-Kolmogorov turbulence, temporal power spectral models of AOA fluctuations for plane and spherical waves can be expressed by substituting the generalized exponential spectral model (Eq. (1)) into Eqs. (7) and (8):

$$W_{\theta(\mathrm{pl})}(\alpha, l_0, L_0, \omega, \beta) = 8\pi^2 \int_0^\infty \kappa^3 \Phi_n(\kappa,\alpha,l_0,L_0) G_D(\kappa) \cdot \left[J_0(\nu_\perp t\kappa) - \cos(2\beta)\, J_2(\nu_\perp t\kappa)\right] \times \int_0^L \cos^2\left(\frac{\kappa^2 z}{2k}\right) \int_0^\infty \cos(\omega t)\, \mathrm{d}\kappa \mathrm{d}z \mathrm{d}t, \tag{10}$$

$$W_{\theta(\mathrm{sp})}(\alpha, l_0, L_0, \omega, \beta) = 8\pi^2 \int_0^\infty \kappa^3 \Phi_n(\kappa,\alpha,l_0,L_0) G_D(\kappa) \times \left[J_0(\nu_\perp t\kappa) - \cos(2\beta)\, J_2(\nu_\perp t\kappa)\right] \times \int_0^L \cos^2\left(\frac{\kappa^2 z(L-z)}{2kL}\right)\left(\frac{z}{L}\right)^2 \int_0^\infty \cos(\omega t)\, \mathrm{d}\kappa \mathrm{d}z \mathrm{d}t, \tag{11}$$

where $W_{\theta(\mathrm{pl})}(\alpha, l_0, L_0, \omega, \beta)$ and $W_{\theta(\mathrm{sp})}(\alpha, l_0, L_0, \omega, \beta)$ are the temporal power spectrum functions of AOA fluctuations for plane and spherical waves, respectively. These functions consider the influences of the inner and outer scales of finite turbulence as well as general spectral power law values.

Integrating[19]:

$$\int_0^\infty J_0(ax)\cos(bx)\,\mathrm{d}x = \begin{cases} \left(a^2-b^2\right)^{-1/2} & 0<b<a \\ 0 & b>a \end{cases}, \tag{12}$$

$$\int_0^\infty J_{2n}(ax)\cos(bx)\,\mathrm{d}x = \begin{cases} (-1)^n \left(a^2-b^2\right)^{-1/2} T_{2n}(b/a) & 0<b<a \\ 0 & b>a \end{cases}, \tag{13}$$

where $T_\mathrm{m}(z)$ is the Tchebichef polynomial,

$$T_\mathrm{m}(z) = \cos\left(m\cos^{-1} z\right), \tag{14}$$

the integrations of Eqs. (10) and (11) with respect to t are obtained. Then, assuming geometrical optics behaviors, $\cos^2\left(\kappa^2 z/2k\right) \approx 1$. Under the condition that the Fresnel zone $(L/k)^{1/2}$ is much smaller than the receiver aperture diameter $(L/k)^{1/2} \ll D$ and using the following integration function[19]:

$$\int_0^\infty (t+a)^{2\mu-1}(t-b)^{2\nu-1}\exp(-pt) = \begin{cases} 0, & 0<t<b \\ \Gamma(2\nu)(a+b)^{\mu+\nu-1} p^{-\mu-\nu}\exp\left[p(a-b)/2\right] W_{\mu-\nu,\mu+\nu-1/2}(bp+ap), & t>b \end{cases}, \tag{15}$$

the analytical expressions for $W_{\theta(\mathrm{pl})}(\alpha, l_0, L_0, \omega, \beta)$ and $W_{\theta(\mathrm{sp})}(\alpha, l_0, L_0, \omega, \beta)$ are finally derived as

$$W_{\theta(\mathrm{pl})}(\alpha, l_0, L_0, \omega, \beta) = 4\pi^2 \hat{A}(\alpha)\,\hat{C}_n^2 L \cdot \left[g_1(D,\alpha,l_0,L_0,\omega,\beta) - g_2(D,\alpha,l_0,L_0,\omega,\beta)\right], \tag{16}$$

$$W_{\theta(\mathrm{sp})}(\alpha, l_0, L_0, \omega, \beta) = \frac{4}{3}\pi^2 \hat{A}(\alpha)\,\hat{C}_n^2 L \cdot \left[g_1(D,\alpha,l_0,L_0,\omega,\beta) - g_2(D,\alpha,l_0,L_0,\omega,\beta)\right]. \tag{17}$$

$$\begin{aligned} g_1(D,\alpha,l_0,L_0,\omega,\beta) = &\frac{1}{\omega_0}\Gamma\left(\frac{1}{2}\right)\left(\frac{k}{L}\right)^{(3-\alpha)/4}\left(\frac{\omega}{\omega_0}\right)^{(1-\alpha)/2}\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}\right)^{(\alpha-5)/4} \\ &\times \exp\left[-\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}\right)\frac{k\omega^2}{2L\omega_0^2}\right]\left\{\left[1-\cos(2\beta)\right]\times W_{\frac{3-\alpha}{4},\frac{3-\alpha}{4}}\left[\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}\right)\frac{k\omega^2}{L\omega_0^2}\right]\right. \\ &\left.+ 2\frac{\omega}{\omega_0}\cos(2\beta)\times\left(\frac{k}{L}\right)^{1/2}\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}\right)^{1/2} W_{\frac{1-\alpha}{4},\frac{1-\alpha}{4}}\left[\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}\right)\frac{k\omega^2}{L\omega_0^2}\right]\right\}, \end{aligned} \tag{18}$$

$$g_2(D,\alpha,l_0,L_0,\omega,\beta)=\frac{1}{\omega_0}\Gamma\left(\frac{1}{2}\right)\left(\frac{k}{L}\right)^{(3-\alpha)/4}\left(\frac{\omega}{\omega_0}\right)^{(1-\alpha)/2}\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}+\frac{1}{\kappa_0^2}\right)^{(\alpha-5)/4}$$
$$\times\exp\left[-\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}+\frac{1}{\kappa_0^2}\right)\frac{k\omega^2}{2L\omega_0^2}\right]\left\{[1-\cos(2\beta)]\times W_{\frac{3-\alpha}{4},\frac{3-\alpha}{4}}\left[\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}+\frac{1}{\kappa_0^2}\right)\frac{k\omega^2}{L\omega_0^2}\right]\right.$$
$$\left.+2\frac{\omega}{\omega_0}\cos(2\beta)\left(\frac{k}{L}\right)^{1/2}\times\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}+\frac{1}{\kappa_0^2}\right)^{1/2}W_{\frac{1-\alpha}{4},\frac{1-\alpha}{4}}\left[\left(\frac{c^2D^2}{4}+\frac{1}{\kappa_l^2}+\frac{1}{\kappa_0^2}\right)\frac{k\omega^2}{L\omega_0^2}\right]\right\}, \tag{19}$$

where $W_{\mu,\nu}(z)$ is the Whittaker's confluent hypergeometric function. The quantity ω_0, sometimes called the Fresnel frequency, is uniquely defined by the Fresnel scale and the mean wind speed, $\omega_0=\nu_\perp(L/k)^{-1/2}$. This parameter physically represents the transition frequency at which the temporal power spectrum of irradiance fluctuations begins to decay under weak fluctuations.

Numerical calculations are performed to analyze the influences of the finite outer scale, general spectral power law, β, and D on the temporal power spectrum of AOA fluctuations. In the following calculations, parameters are set as $L = 1\,000$ m, $\lambda = 1.55$ μm, and $\nu_\perp = 4$ m/s; other values can also be chosen. The temporal power spectra of AOA fluctuations normalized by the variance of AOA fluctuations (the expressions of variance of AOA fluctuations have been derived in Ref. [20]) may be plotted as a function of ω/ω_0 for plane and spherical waves and take the same expressions for both types of waves.

To analyze the influence of the outer scale of turbulence on the temporal power spectrum of AOA fluctuations, the other parameters are fixed to $\alpha = 10/3$ (helical turbulence[21,22]), $\beta = 0$, and $D = 0.25$ m. Note that these values are chosen as an example, and other values may be selected if necessary. As AOA fluctuations are caused mainly by large-scale turbulence cells, the influence of the inner scale of turbulence on the final expressions may be ignored (Fig. 1). In this calculation, $l_0 = 5$ mm is chosen as an example. Figure 2 shows the numerical calculation results with different outer scales of turbulence. As the outer scale of turbulence increases, the temporal power spectrum of AOA fluctuations for plane/spherical wave increases, especially at low-frequency regions ($\omega < 0.1\omega_0$). This phenomenon results from phase fluctuations, which are contributed to mostly by the refractive effects of large-scale turbulence cells (sizes larger than the Fresnel scale $\sqrt{\lambda L}$[1]). For atmospheric turbulence, L_0 is usually expressed in units of meters; thus, the condition of $L_0 \gg \sqrt{\lambda L}$ is basically satisfied. According to the Richardson energy cascade theory of turbulence[1], the number of turbulence cells with sizes in the order of $\sqrt{\lambda L}$ increases as L_0 increases. Optical waves then meet a major number of large-scale turbulent cells along its propagation length, and these cells yield higher values of AOA fluctuations relative to the case of lower outer scale values, where more large scales are cut out. Therefore, the temporal power spectrum of AOA fluctuations increases with increasing L_0 values.

Figures 3, 4, and 5 plot the curves of the temporal power spectrum of AOA fluctuations normalized by the

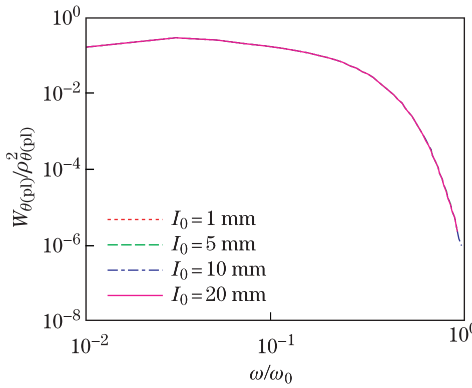

Fig. 1. Normalized temporal power spectrum of AOA fluctuations as a function of ω/ω_0 with different inner scales of turbulence.

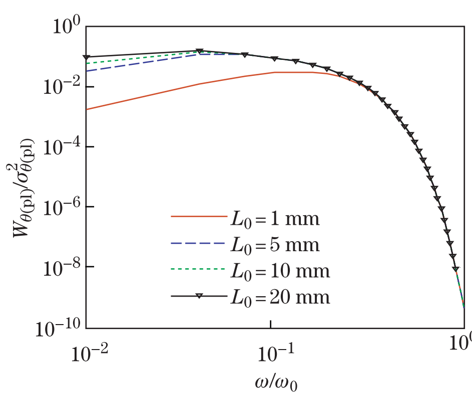

Fig. 2. Normalized temporal power spectrum of AOA fluctuations as a function of ω/ω_0 with different outer scales of turbulence.

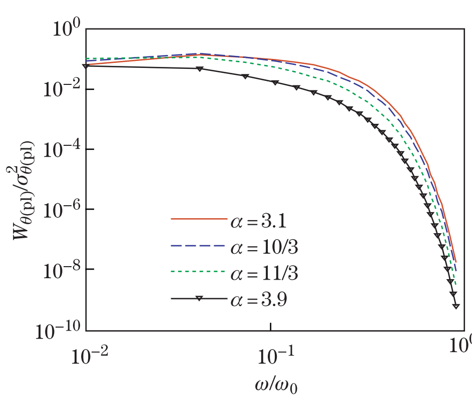

Fig. 3. Normalized temporal power spectrum of AOA fluctuations as a function of ω/ω_0 with different α values.

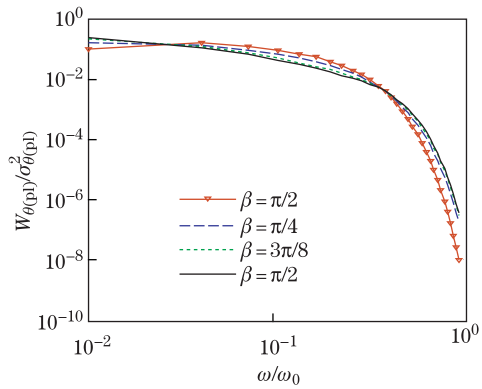

Fig. 4. Normalized temporal power spectrum of AOA fluctuations as a function of ω/ω_0 with different β values.

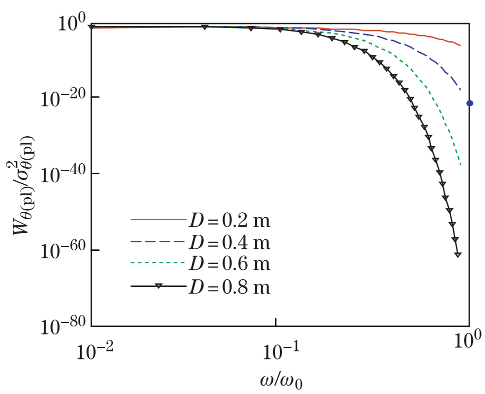

Fig. 5. Normalized temporal power spectrum of AOA fluctuations as a function of ω/ω_0 with different D values.

Table 1. Parameters Adopted in Figs. 3–5 (Other Values Can Also be Adopted)

	L_0 (m)	l_0 (mm)	α	β (rad)	D (m)
Fig. 3	10	5	3.1,10/3, 11/3,3.9	0	0.25
Fig. 4	10	5	10/3	0, $\pi/4$, $3\pi/8$,$\pi/2$	0.25
Fig. 5	10	5	10/3	0	0.2,0.4, 0.6,0.8

variance of AOA fluctuations with variable α, β, and D, respectively. The parameters are set and listed in Table 1. The parameters α, β, and D produce influences on the temporal power spectrum of AOA fluctuations identical to those described in Ref. [8]. The findings obtained indicate that introduction of inner and outer scales of turbulence to the expressions of the temporal power spectral models of AOA fluctuations will not change the influences of α, β, and D on the final expressions.

In conclusion, new analytical expressions of the temporal power spectrum of AOA fluctuations are derived for both plane and spherical waves propagating horizontally through weak non-Kolmogorov atmospheric turbulence. Calculations show that the temporal power spectrum of AOA fluctuations increases with increasing outer scale of turbulence, especially in low-frequency regions ($\omega < 0.1\omega_0$). In addition, the slopes of the curves of normalized temporal spectrum versus ω/ω_0 decrease with increasing power law α. As D increases, the average aperture effect introduced by D is exhibited. The results of this study will support future investigations on the effects of turbulence on plane and spherical optical waves propagating horizontally through non-Kolmogorov atmospheric turbulence.

This work was supported by the Beijing Natural Science Foundation (No. 4122047), the Innovation Foundation of AVIC (No. CXY2010BH02), the Fundamental Research Fund for the Central Universities, and the Open Research Fund of Key Laboratory of Atmospheric Composition and Optical Radiation, Chinese Academy of Sciences.

References

1 V. I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation* (trans. for NOAA by Israel Program for Scientific Translations, Jerusalem, 1971).
2 D. T. Kyrazis, J. B. Wissler, D. B. Keating, A. J. Preble, and K. P. Bishop, Proc. SPIE **2120,** 43 (1994).
3 M. S. Belen'kii, S. J. Karis, J. M. Brown, and R. Q. Fugate, Proc. SPIE **3126,** 113 (1997).
4 M. S. Belen'kii, E. Cuellar, K. A. Hughes, and V. A. Rye, Proc. SPIE **6304,** 63040U (2006).
5 A. Zilberman, E. Golbraikh, N. S. Kopeika, A. Virtser, I. Kupershmidt, and Y. Shtemler, Atmos. Res. **88,** 66 (2008).
6 A. S. Gurvich and M. S. Belen'kii, J. Opt. Soc. Am. A **12,** 2517 (1995).
7 M. S. Belen'kii, Opt. Lett. **20,** 1359 (1995).
8 X. Zhou, Y. Yang, Y. Shao, and J. Liu, Chin. Opt. Lett. **10,** 110603 (2012).
9 W. Du, L. Tan, J. Ma, and Y. Jiang, Opt. Express **18,** 5763 (2010).
10 L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media* (SPIE Optical Engineering Press, Bellingham, 2005).
11 I. Toselli, L. C. Andrews, R. L. Phillips, and V. Ferrero, Proc. SPIE **6551,** 65510E (2007).
12 L. Cui, B. Xue, X. Cao, J. Dong, and J. Wang, Opt. Express **18,** 21269 (2010).
13 L. Tan, W. Du, and J. Ma, Journal of Russian Laser Research **30,** 552 (2009).
14 B. Xue, L. Cui, W. Xue, X. Bai, and F. Zhou, Opt. Express **19,** 8433 (2011).
15 R. Conan, J. Borgnino, A. Ziad, and F. Martin, J. Opt. Soc. Am. A **17,** 1807 (2000).
16 D. M. Winker, J. Opt. Soc. Am. A **11,** 1568 (1991).
17 P. Hickson, J. Opt. Soc. Am. A **11,** 1667 (1994).
18 Y. Cheon and A. Muschinski, J. Opt. Soc. Am. A **24,** 415 (2007).
19 L. C. Andrews, *Special Functions of Mathematics for Engineers*, (2nd ed). (SPIE Optical Engineering Press, Bellingham, Wash., 1998).
20 W. Du, S. Yu, L. Tan, J. Ma, Y. Jiang, and W. Xie, Opt. Commun. **282,** 705 (2009).
21 A. Zilberman, E, Golbraikh, and N. S. Kopeika, Appl. Opt. **47,** 6385 (2008).
22 N. Kopeika, A. Zilberman, and E. Gobraikh, Proc. SPIE **7588,** 758808 (2010).