

# New approach for normalization and photon-number distributions of photon-added (-subtracted) squeezed thermal states

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Received December 9, 2011; accepted February 24, 2012; posted online May 16, 2012

Using the thermal field dynamics theory to convert the thermal state into a “pure” state in doubled Fock space, we find that the average value of  $e^{fa^\dagger a}$  under squeezed thermal state (STS) is just the generating function of Legendre polynomials. Based on this remarkable result, the normalization and photon-number distributions of  $m$ -photon added (or subtracted) STSs are conveniently obtained as the Legendre polynomials. This new concise method can be expanded to the entangled case.

OCIS codes: 270.0270, 270.5290.

doi: 10.3788/COL201210.082701.

The nonclassicality of optical fields is helpful in understanding the fundamentals of quantum optics and has many applications in quantum information processing<sup>[1]</sup>. The subtraction or addition of photons from/to traditional quantum states or Gaussian states has been proposed to generate and manipulate various nonclassical optical field<sup>[2–11]</sup>. For example, photon addition and subtraction have been experimentally demonstrated to probe quantum commutation rules<sup>[9]</sup>. Recently, photon-added (-subtracted) Gaussian states have received more attention from both experimentalists and theoreticians<sup>[12–21]</sup>, because these states exhibit numerous nonclassical properties and may provide access to a complete engineering of quantum states and fundamental quantum phenomena.

Theoretically, the normalization factors of such quantum states are essential for studying their nonclassical properties. Very recently, Fan *et al.*<sup>[22]</sup> presented a new concise approach for normalizing  $m$ -photon-added (-subtracted) squeezed vacuum state (pure state) by constructing a generating function. However, most systems are not isolated, are immersed in a thermal reservoir, and thus, we often have no enough information to specify completely the state of a system. In such situations, the system only can be described by mixed states, such as thermal states. In addition, the squeezed thermal states (STSs) can be considered as the generalized Gaussian states.

In this letter, we shall extend this case to the mixed state, i.e., by using the thermal field dynamics (TFD) theory to convert the thermal state into a “pure” state in doubled Fock space. We present a new concise method for normalizing photon-added (-subtracted) STSs (PASTSs, PSSTs) and deriving their photon-number distributions (PNDs), which have been a major topic of studies on quantum optics and statistics. The normalization factors and PNDs were found to be related to the Legendre polynomials in compact form.

We begin by briefly reviewing the properties of a thermal state.

For a single mode with frequency  $\omega$  in a thermal equilibrium state corresponding to absolute temperature  $T$ , the density operator is

$$\rho_{\text{th}} = \sum_{n=0}^{\infty} \frac{n_c^n}{(n_c + 1)^{n+1}} |n\rangle \langle n|, \quad (1)$$

where  $n_c = \{\exp[\hbar\omega/(kT)] - 1\}^{-1}$  is the average photon number of the thermal state  $\rho_{\text{th}}$  and  $k$  is the Boltzmann’s constant.  $|n\rangle = a^{\dagger n}/\sqrt{n!}|0\rangle$  and the normally ordering form of vacuum projector  $|0\rangle \langle 0| =: \exp(-a^\dagger a):$  (the symbol  $::$  denotes normal ordering). One can express Eq. (1) as

$$\rho_{\text{th}} =: \frac{1}{n_c + 1} e^{-\frac{1}{n_c + 1} a^\dagger a}: = \frac{1}{n_c + 1} e^{a^\dagger a \ln \frac{n_c}{n_c + 1}}, \quad (2)$$

where in the last step, the operator identity  $\exp(\lambda a^\dagger a) =: \exp[(e^\lambda - 1) a^\dagger a]:$  is used.

The elemental spirit of the TFD introduced by Takahashi *et al.*<sup>[23–25]</sup>, is to convert the calculations of ensemble averages for a mixed state  $\rho$ ,  $\langle A \rangle = \text{tr}(A\rho)/\text{tr}(\rho)$ , where  $\text{tr}$  denotes the trace operation over the system, to the equivalent expectation values with a pure state  $|0(\beta)\rangle$ , i.e.

$$\langle A \rangle = \langle 0(\beta) | A | 0(\beta) \rangle, \quad (3)$$

where  $\beta = 1/kT$  and  $k$  is the Boltzmann’s constant. Thus, for the density operator  $\rho_{\text{th}}$ , using the partial trace method in Ref. [26], i.e.,  $\rho_{\text{th}} = \tilde{\text{tr}}[|0(\beta)\rangle \langle 0(\beta)|]$ , where  $\tilde{\text{tr}}$  denotes the trace operation over the environment freedom (denoted as operator  $\tilde{a}^\dagger$ ), one can obtain the explicit expression of  $|0(\beta)\rangle$  in doubled Fock space

$$|0(\beta)\rangle = \text{sech}\theta \exp(a^\dagger \tilde{a}^\dagger \tanh\theta) |0\tilde{0}\rangle = S(\theta) |0\tilde{0}\rangle, \quad (4)$$

where  $|0\tilde{0}\rangle$  is annihilated by  $\tilde{a}$  and  $a$ ,  $[\tilde{a}, \tilde{a}^\dagger] = 1$ , and  $S(\theta)$  is the thermal operator.  $S(\theta) \equiv \exp[\theta(a^\dagger \tilde{a}^\dagger - a\tilde{a})]$  and has a similar form to that of a two-mode squeezing operator except for the tilde mode, and  $\theta$  is a parameter related to the temperature by  $\tanh\theta = \exp(-\frac{\hbar\omega}{2kT})$ .

$|0(\beta)\rangle$  is named as the thermal vacuum state.

Let  $\text{Tr} = \text{tr}\tilde{\text{r}}$ . Then

$$\begin{aligned}\text{tr}(A\rho_{\text{th}}) &= \text{Tr}[A|0(\beta)\rangle\langle 0(\beta)|] \\ &= \text{tr}[A\tilde{\text{r}}|0(\beta)\rangle\langle 0(\beta)|],\end{aligned}\quad (5)$$

and the average photon number of the thermal state  $\rho_{\text{th}}$  is

$$n_c = \text{Tr}[a^\dagger a |0(\beta)\rangle\langle 0(\beta)|] = \sinh^2 \theta. \quad (6)$$

Here we should emphasize that  $\tilde{\text{r}}|0(\beta)\rangle\langle 0(\beta)| \neq \langle 0(\beta)|0(\beta)\rangle$ ,  $|0(\beta)\rangle$  involves both real and fictitious modes  $a$  and  $\tilde{a}$ . Equations (3) and (4) show that the worthwhile convenience in Eq. (4) is at the expense of introducing a field (or called a tilde-conjugate) in the extended Hilbert space, i.e., the original optical state  $|n\rangle$  in the Hilbert space  $\mathcal{H}$  is accompanied by a tilde state  $|\tilde{n}\rangle$  in  $\tilde{\mathcal{H}}$ . A similar rule holds for the operators: Bose annihilation operator  $a$  acting on  $\mathcal{H}$  has an image  $\tilde{a}$  acting on  $\tilde{\mathcal{H}}$ . These operators in  $\mathcal{H}$  are commutative with those in  $\tilde{\mathcal{H}}$ .

To realize our purpose, we introduce the squeezed thermal vacuum state, as  $S_1(r)|0(\beta)\rangle$ , where  $S_1(r) = \exp[r(a^2 - a^{\dagger 2})/2]$  is the single-mode squeezing operator for the real mode with  $r$  being the squeezing parameter. Considering Eq. (4) and the Baker-Hausdorff lemma

$$S_1(r)a^\dagger S_1^\dagger(r) = a^\dagger \cosh r + a \sinh r, \quad (7)$$

we then obtain

$$\begin{aligned}S_1(r)|0(\beta)\rangle &= \text{sech } \theta \text{ sech}^{1/2} r \exp[(a^\dagger \cosh r + a \sinh r) \\ &\quad \tilde{a}^\dagger \tanh \theta] \times \exp\left(-\frac{a^{\dagger 2}}{2} \tanh r\right) |0\tilde{0}\rangle, \quad (8)\end{aligned}$$

where we have used  $S_1(\lambda)|0\rangle = \text{sech}^{1/2} \lambda \exp(-a^{\dagger 2}/2 \tanh \lambda)|0\rangle$ . Furthermore, note that  $e^{\tau a \tilde{a}^\dagger} a^\dagger e^{-\tau a \tilde{a}^\dagger} = a^\dagger + \tau \tilde{a}^\dagger$  and operators  $A$  and  $B$  satisfy the conditions  $[A, [A, B]] = [B, [A, B]] = 0$ , we have  $e^{A+B} = e^A e^B e^{-[A, B]/2}$ . Thus, Eq. (8) can be expressed as

$$\begin{aligned}S_1(r)|0(\beta)\rangle &= \text{sech } \theta \text{ sech}^{1/2} r \exp\left[\frac{\tanh \theta}{\cosh r} a^\dagger \tilde{a}^\dagger \right. \\ &\quad \left. + \frac{\tanh r}{2} (\tilde{a}^{\dagger 2} \tanh^2 \theta - a^{\dagger 2})\right] |0\tilde{0}\rangle. \quad (9)\end{aligned}$$

Next, we shall use Eq. (9) to derive the average of operator  $e^{fa^\dagger a}$  under the squeezed thermal vacuum state  $S_1(r)|0(\beta)\rangle$ , which is a bridge for our calculations. Notice that  $e^{f/2a^\dagger a} a^\dagger e^{-f/2a^\dagger a} = a^\dagger e^{f/2}$  and  $e^{-f/2a^\dagger a} a e^{f/2a^\dagger a} = a e^{f/2}$ . Hence we have

$$\begin{aligned}e^{f/2a^\dagger a} S_1(r)|0(\beta)\rangle &= \text{sech } \theta \text{ sech}^{1/2} r \exp\left[\frac{\tanh \theta}{\cosh r} a^\dagger \tilde{a}^\dagger e^{f/2} \right. \\ &\quad \left. + \frac{\tanh r}{2} (\tilde{a}^{\dagger 2} \tanh^2 \theta - a^{\dagger 2} e^f)\right] |0\tilde{0}\rangle, \quad (10)\end{aligned}$$

which leads to

$$\begin{aligned}\langle e^{fa^\dagger a} \rangle &\equiv \langle 0(\beta)| S_1^\dagger(r) e^{fa^\dagger a} S_1(r) |0(\beta)\rangle \\ &= (C e^{2f} - 2B e^f + A)^{-1/2}, \quad (11)\end{aligned}$$

where we have set  $A = n_c^2 + (2n_c + 1) \cosh^2 r$ ,  $B = n_c(n_c + 1)$ , and  $C = n_c^2 - (2n_c + 1) \sinh^2 r$ . Moreover we have used the completeness relation of coherent state  $\int d^2 z d^2 \tilde{z} |z\tilde{z}\rangle\langle z\tilde{z}|/\pi^2 = 1$ , where  $|z\rangle$  and  $|\tilde{z}\rangle$  are the coherent states in real and fictitious modes, respectively, and the following formula<sup>[27]</sup>

$$\begin{aligned}\int \frac{d^2 z}{\pi} \exp(\zeta |z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) \\ = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta \xi \eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right], \quad (12)\end{aligned}$$

whose convergent condition is  $\text{Re}(\zeta \pm f \pm g) < 0$  and  $\text{Re}(\zeta^2 - 4fg)/(\zeta \pm f \pm g) < 0$ . Equation (11) is very important for the calculation of PND and normalization of PASTs and PSSTs.

Interestingly the standard generating function of Legendre polynomials<sup>[28]</sup> is given by

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{m=0}^{\infty} P_m(x) t^m. \quad (13)$$

Thus comparing Eq. (11) with Eq. (13), we obtain

$$\langle e^{fa^\dagger a} \rangle = A^{-1/2} \sum_{m=0}^{\infty} P_m(B/\sqrt{AC}) (\sqrt{C/A} e^f)^m, \quad (14)$$

which indicates that the average value of  $e^{fa^\dagger a}$  under STS is just the generating function of Legendre polynomials. Next, we shall examine the normalizations and PNDs of PASTs and PSSTs using Eqs. (11) and (14).

The  $m$ -photon-added scheme, denoted by the mapping  $\rho \rightarrow a^{\dagger m} \rho a^m$ , was first proposed by Agarwal *et al.*<sup>[4]</sup>. Here, we introduce the PASTs. Theoretically, the PASTs can be obtained by repeatedly operating the photon creation operator  $a^\dagger$  on a STS so that its density operator is given by

$$\rho_{\text{ad}} = C_{a,m}^{-1} a^{\dagger m} S_1 \rho_{\text{th}} S_1^\dagger a^m, \quad (15)$$

where  $m$  is the number of added photons (a non-negative integer) and  $C_{a,m}^{-1}$  is the normalization constant to be determined.

A quantum state should be normalization for us to describe it fully. Next, we shall employ Eqs. (5), (11), and (14) to realize our aim. According to the normalization condition  $\text{tr}\rho_{\text{ad}} = 1$  and the TFD, we have

$$C_{a,m} = \langle 0(\beta)| S_1^\dagger a^m a^{\dagger m} S_1 |0(\beta)\rangle, \quad (16)$$

which implies that the calculation of normalization factor  $C_{a,m}$  is converted to a matrix element after introducing the thermal vacuum state  $|0(\beta)\rangle$ .

Considering the operator identity<sup>[29]</sup>  $e^{\tau a^\dagger a} = e^{-\tau} \cdot \exp[(1 - e^{-\tau}) a^\dagger a]$ , we obtain

$$\sum_{m=0}^{\infty} \frac{\tau^m}{m!} a^m a^{\dagger m} = :e^{\tau a^\dagger a}: = \left(\frac{1}{1 - \tau}\right)^{a^\dagger a + 1}, \quad (17)$$

where the symbol  $\dot{\cdot}$  denotes antinormal ordering. Thus, using Eqs. (11), (16), and (17), ( $e^f \rightarrow \frac{1}{1-\tau}$ ), we have

$$\sum_{m=0}^{\infty} \frac{\tau^m}{m!} C_{a,m} = [A\tau^2 - 2D\tau + 1]^{-1/2}, \quad (18)$$

where  $D = n_c \cosh 2r + \cosh^2 r$ .

Comparing Eqs. (18) with (13), and taking  $\tau' \rightarrow \sqrt{A}\tau$ , we obtain

$$\sum_{m=0}^{\infty} \tau'^m \frac{C_{a,m}}{m! A^{m/2}} = \sum_{m=0}^{\infty} P_m(D/\sqrt{A}) \tau'^m, \quad (19)$$

thus the normalization constant of PASTSs is given by

$$C_{a,m} = m! A^{m/2} P_m(D/\sqrt{A}), \quad (20)$$

which is identical to the result in Ref. [30]. Note that for the case of no-photon-addition with  $m = 0$ ,  $C_{a,0} = 1$ , as expected. In the case of an  $m$ -photon-added thermal state (no squeezing) with  $D = n_c + 1$ ,  $A = (n_c + 1)^2$ , and  $P_m(1) = 1$ ,  $C_{a,m} = m! (n_c + 1)^m$ . This result is consistent with that of Eq. (32) in Ref. [31]. In addition, when  $r = 0$ , corresponding to a photon-added thermal state, Eq. (20) only reduces to  $C_{a,m} = m! \cosh^{2m} \theta$ [31].

The PND is a key characteristic of every optical field. The PND, i.e., the probability of finding  $n$  photons in a quantum state described by the density operator  $\rho$ , is  $\mathcal{P}(n) = \text{tr}[|n\rangle\langle n| \rho]$ . Similar to Eq. (20), considering that  $a^m |n\rangle = \sqrt{n!/(n-m)!} |n-m\rangle$  and  $|n\rangle = a^{\dagger n} / \sqrt{n!} |0\rangle$ , the PND of the PASTSs can be calculated as

$$\begin{aligned} \mathcal{P}_a(n) &= C_{a,m}^{-1} \text{tr} \left[ |n\rangle\langle n| a^{\dagger m} S_1 \rho_{\text{th}} S_1^{\dagger} a^m \right] \\ &= \frac{n! C_{a,m}^{-1}}{(1!)^2} \langle 0(\beta) | S_1^{\dagger} a^{\dagger l} | 0\rangle \langle 0 | a^l S_1 | 0(\beta) \rangle, \end{aligned} \quad (21)$$

which leads to

$$\sum_{l=0}^{\infty} \tau^l \frac{l!}{n!} C_{a,m} \mathcal{P}_a(n) = \langle 0(\beta) | S_1^{\dagger} e^{a^{\dagger} a \ln \tau} S_1 | 0(\beta) \rangle, \quad (22)$$

where  $l = n - m$  and vacuum projector operator  $|0\rangle\langle 0| =: e^{-a^{\dagger} a}:$  and operator identity  $e^{\lambda a^{\dagger} a} =: \exp[(e^{\lambda} - 1)a^{\dagger} a]:$  are used.

Using Eq. (11) again ( $e^f \rightarrow \tau$ ) and comparing Eq. (22) with Eq. (13) we obtain

$$\sum_{l=0}^{\infty} \tau^l \frac{l!}{n!} C_{a,m} \mathcal{P}_a(n) = A^{-1/2} \sum_{l=0}^{\infty} P_l(B/\sqrt{AC}) (\sqrt{C/A}\tau)^l, \quad (23)$$

which leads to the PND of PASTSs

$$\mathcal{P}_a(n) = \frac{n! C_{a,m}^{-1} (C/A)^{(n-m)/2}}{(n-m)! \sqrt{A}} P_{n-m}(B/\sqrt{AC}), \quad (24)$$

which is a Legendre polynomial with a condition  $n \geq m$ , this result implies that the photon-number ( $n$ ) involved in PASTSs is always not less than the photon-number

( $m$ ) operated on the STSs, and that no photon distribution exists when  $n < m$ . When  $m=0$  corresponding to the STS, the PND of STS is also a Legendre distribution[32].

Next, we discuss the PSSTSs, defined as

$$\rho_{\text{sb}} = C_{s,m}^{-1} a^m S_1 \rho_{\text{th}} S_1^{\dagger} a^{\dagger m}, \quad (25)$$

where  $m$  is the subtracted photon number (a non-negative integer) and  $C_{s,m}$  is a normalized constant.

Similarly to deriving Eq. (20), we have

$$C_{s,m} = \langle 0(\beta) | S_1^{\dagger} a^{\dagger m} a^m S_1 | 0(\beta) \rangle. \quad (26)$$

Hence employing  $e^{\lambda a^{\dagger} a} =: \exp[(e^{\lambda} - 1)a^{\dagger} a]:$  and Eq. (11) ( $e^f \rightarrow 1 + \tau$ ), we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{\tau^m}{m!} C_{s,m} &= \langle 0(\beta) | S_1^{\dagger} e^{a^{\dagger} a \ln(1+\tau)} S_1 | 0(\beta) \rangle \\ &= [C\tau^2 - 2E\tau + 1]^{-1/2}, \end{aligned} \quad (27)$$

where  $E = \frac{1}{2} [(2n_c + 1) \cosh 2r - 1]$ . By comparing Eq. (27) with Eq. (13), we yield

$$C_{s,m} = m! C^{m/2} P_m(E/\sqrt{C}), \quad (28)$$

which is the normalization factor of PSSTSs. When  $r = 0$ , corresponding to photon-subtracted thermal state, Eq. (28) only reduces to  $C_{s,m} = m! \sinh^{2m} \theta$ [31].

Similarly to deriving Eq. (24), the PND of PSSTSs is given by

$$\begin{aligned} \mathcal{P}_s(n) &= C_{s,m}^{-1} \langle 0(\beta) | S_1^{\dagger} a^{\dagger m} | n\rangle \langle n | a^m S_1 | 0(\beta) \rangle \\ &= \frac{1}{n!} C_{s,m}^{-1} \langle 0(\beta) | S_1^{\dagger} : a^{\dagger m+n} a^{m+n} e^{-a^{\dagger} a} : S_1 | 0(\beta) \rangle, \end{aligned} \quad (29)$$

so ( $k = m + n$ )

$$\sum_{k=0}^{\infty} \frac{\tau^k}{k!} n! C_{s,m} \mathcal{P}_s(n) = \langle 0(\beta) | S_1^{\dagger} e^{a^{\dagger} a \ln \tau} S_1 | 0(\beta) \rangle, \quad (30)$$

which leads to the PND of PSSTSs

$$\mathcal{P}_s(n) = \frac{(m+n)!}{n! C_{s,m} \sqrt{A}} (C/A)^{m+n/2} P_{m+n}(B/\sqrt{AC}), \quad (31)$$

which a Legendre polynomial. The result is similar to that of Ref. [32].

In conclusion, we present a new concise approach for normalizing  $m$ -photon-added (-subtracted) STSs and deriving the PNDs, which improve the methods used in Refs. [30, 32]. We convert the thermal state to a pure state in doubled Fock space in which the calculations of ensemble averages under a mixed state  $\rho$ , i.e.  $\langle A \rangle = \text{tr}(A\rho)/\text{tr}(\rho)$ , is replaced by an equivalent expectation value with a pure state  $|0(\beta)\rangle$ , i.e.,  $\langle A \rangle = \langle 0(\beta) | A | 0(\beta) \rangle$ . The average value of  $e^{f a^{\dagger} a}$  under STS is just the generating function of Legendre polynomials. Based on this remarkable result, the normalization and PNDs of  $m$ -photon-added (or subtracted) STSs are easily obtained as the Legendre polynomials. The generating function of the Legendre polynomials and the average value of  $e^{f a^{\dagger} a}$  under STS are used throughout the

calculations.

This work was supported by the National Natural Science Foundation of China (No. 60978009), the Major Research Plan of the National Natural Science Foundation of China (No. 91121023), the National “973” Project of China (No. 2011CBA00200), the Natural Science Foundation of Jiangxi Province of China (No. 2010GQW0027), and the Sponsored Program for Cultivating Youths of Outstanding Ability in Jiangxi Normal University.

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