Ray transfer matrix perturbation for an optical component with aberration

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The perturbation theory of matrices is applied to ray transfer matrices (RTMs) to describe an optical component with aberration. A quantitative description of the perturbation extent corresponding to aberration strength is provided using condition numbers and absolute errors for the perturbed RTM. An application to a single small aberration is presented, and the results are compared with those of the diffraction theory of aberrations.

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In geometrical imaging theory, a perfect optical system provides a one-to-one correspondence between object and image points and object and image planes. An optical component essentially maps a point in its object space to its conjugate point in the image space, with rays depicting the action of the component. Theory and practical applications are expected to differ because geometric optics is an approximation. These differences are in the form of image imperfections or aberrations that have been studied extensively[1−3]. The primary aberrations of an optical element[1] and the stability of resonators and periodic focusing systems[2,3] and misaligned cascaded optical elements[4] have been described using ray transfer matrices (RTMs). These matrices are used to describe and analyze optical elements and systems[4].

We propose an approach to quantifying imaging imperfection using perturbed RTMs. The approach allows the decoupling of the optical component and any associated imperfection, which is particularly important in tolerance specifications and component performance. In this letter, we present a matrix perturbation approach to describe an optical component with known aberration. We begin with the following standard form of a perturbed matrix $\mathbf{M}_{\mathrm{A}}$[5]:

$$\mathbf{M}_{\mathrm{A}} = \mathbf{M} + \mathbf{E}, \tag{1}$$

where $\mathbf{M}$ is the invertible RTM of an optical component with no aberration and $\mathbf{E}$ is the perturbation to $\mathbf{M}$. This expression describes the same optical component but with an aberration represented by the small changes in $\mathbf{M}$ due to $\mathbf{E}$. We will take $||\mathbf{E}||_2 << ||\mathbf{M}||_2$ as the condition for the smallness of the perturbation, where $||\cdot||_2$ is the matrix 2-norm[5]. To assess the effect of the aberration, we quantify the perturbation in Eq. (1).

One way of quantifying the perturbation is to use eigenvalues, a popular means of analyzing systems described by matrices. In optics, eigenvalue analysis has been used in the stability of periodic focusing elements and alignment of optical resonators[4,6,7]. However, the main drawback of using eigenvalues is that they are defined using a mathematical singularity[8]. Hence, their appropriateness in describing something physical is scrutinized. For this reason, we use other means of assessment and turn to scalar metrics used predominantly in matrix perturbation theory[5]. Some examples of these metrics include the relative error, perturbed eigenvalue condition number, absolute error $ae(\mathbf{M}_{\mathrm{A}},\mathbf{M})$ defined as[5]

$$ae(\mathbf{M}_{\mathrm{A}}, \mathbf{M}) = ||\mathbf{M}_{\mathrm{A}} - \mathbf{M}||_2, \tag{2}$$

and the matrix condition number $\kappa(\mathbf{M})$ defined as[5]

$$\kappa(\mathbf{M}) = ||\mathbf{M}||||\mathbf{M}^{-1}||_2. \tag{3}$$

In this letter, the latter two metrics shall be used to quantify the perturbation extent. The imaging is said to be perfect when no perturbation to $\mathbf{M}$ exists, that is, $\mathbf{E}$=0. A large $ae(\mathbf{M}_{\mathrm{A}}, \mathbf{M})$ therefore amounts to greater perturbation, which, in turn, is associated with greater imaging imperfection.

To illustrate the use of the metrics $ae(\mathbf{M}_{\mathrm{A}},\mathbf{M})$ and $\kappa(\mathbf{M})$ in assessing imaging imperfections, we consider a Gaussian beam passing through a lens of fixed focal length. The input beam waist is incident at the center of the lens, and the output beam waist and width are observed on a viewing screen whose distance from the lens can be varied. This experiment is equivalent to a fixed focal plane and a varying focal length and can be used to model defocus. We will be determining the beam width and waist position.

In Fig. 1(a), we plot the full-width at half-maximum (FWHM) of the beam in various defocus and focus states. The inset of Fig. 1(a) shows that the beam width changes as the focal plane is moved away from the focal length of the lens. The gray values indicate intensity, and the beam is filtered to avoid saturation. The surface profiles in Fig. 1(b) show a beam in the unfocused and focused states, indicating that beam narrowing does not occur in

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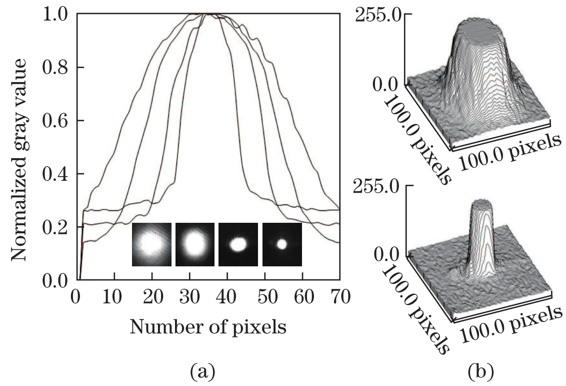

Fig. 1. Focusing of a Gaussian beam. In the sequence of captured beam profiles, the focal plane is moved toward the lens with the last image of the beam profile corresponding to a focused beam. Consistently, the first two or three pixels are not illuminated by the beam as evidenced by the zero gray values. (a) Beam width expands when beam is viewed away from focus. The beam waist is the narrowest part of the beam corresponding to focus. (b) Surface profiles of beam in unfocused (top) and focused (bottom) states.

a particular transverse direction.

Pixel counting is a convenient, straightforward procedure for describing beam expansion using, for example, the focused beam as reference. Such analysis measures how large the beam is compared with the reference. One limitation of pixel counting is the capturing element resolution, which is important when studying infinitesimal beam expansion. Another way of measuring the spot size evolution is through FWHM[4,6]. Unlike purely measuring the beam width through pixels, this quantity makes use of an intrinsic property of the beam and, in this regard, is qualitatively similar to $\kappa(\mathbf{M}_\mathrm{A})$, which contains information regarding the optical component and perturbation.

Assuming that the lens images imperfectly by shifting its focus along the optic axis, we can express the RTM for small changes in f in standard form

$$\mathbf{M}_\mathrm{A}=\begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f+\Delta f} & 1 \end{pmatrix}\approx\begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ \Delta f/f^2 & 0 \end{pmatrix}$$

$$=\mathbf{M}+\mathbf{E}, \tag{4}$$

where $\mathbf{M}$ is the RTM of a thin lens[4,6,7] and Δf is the small change in f contained in the perturbation matrix $\mathbf{E}$. This lens may represent a tunable lens whose focal length can be varied continuously by essentially changing the amount of defocus[9]. From Eqs. (2) and (4), we can obtain $ae(\mathbf{M}_\mathrm{A},\mathbf{M}) = ||\mathbf{E}||_2 = [(\Delta f)\ /\ f^2]$. This result indicates, for this example, the linearity of $||\mathbf{E}||_2$ with Δf. Note that $\Delta f = 0$ when no defocus exists. Moreover, we arrive at the consistent result $||\mathbf{E}||_2 = 0$, suggesting that $||\mathbf{E}||_2$ can be used to describe the shift in beam waist. As shown in Fig. 2(a), the change in the focus location is a monotonically increasing function of $||\mathbf{E}||_2$. Thus, the focus does not change when $\mathbf{E} = 0$ and $||\mathbf{E}||_2$ grows with defocus Δf. The red line in Fig. 2(a) is a linear fit indicating that within reasonable measurable tolerances, $||\mathbf{E}||_2$, as a first approximation, is linear in defocus.

Using Eqs. (3) and (4), we have $\kappa(\mathbf{M}_\mathrm{A}) = 2 + [(f + \Delta f)^{-2}]$, which illustrates an important characteristic of these metrics. The physically greater magnitude of Δf corresponds to moving away from the focal point. One disadvantage of quantitative metrics is that position information is removed. This information is important, for example, when one needs to know if the image is formed in front or behind an optical component. In our example, one can move toward the lens or away from it. Unlike the absolute error, the condition number $\kappa(\mathbf{M}_\mathrm{A})$ takes into account both the change in the RTM and the RTM itself. The same goes for FWHM, which does not merely measure the beam width in terms of illuminated pixels. The change in beam width is a monotonically decreasing function of $\kappa(\mathbf{M}_\mathrm{A})$, as shown in Fig. 2(b). For the focused beam, $\Delta f = 0$ and $\kappa(\mathbf{M}_\mathrm{A}) = \kappa(\mathbf{M})$ and $\kappa(\mathbf{M}) \to 2$ as $\Delta f \to \infty$. In this limit, $\mathbf{M}_\mathrm{A}$ tends to the 2×2 identity matrix representing ray refraction between two media of the same refractive indices[4,6,7]. The red line in Fig. 2(b) is a linear fit, indicating that, as a first approximation, beam width changes can be described by $\kappa(\mathbf{M}_\mathrm{A})$.

To describe the sensitivity of $\mathbf{M}$ to perturbations, we consider the condition number $\kappa(\mathbf{M}_\mathrm{A})$. If $||\mathbf{M}^{-1}||_2||\mathbf{E}||_2 < 1$, then one has $||\mathbf{M}_\mathrm{A}^{-1}||_2(1 - ||\mathbf{M}^{-1}||_2||\mathbf{E}||_2) \leqslant ||\mathbf{M}^{-1}||_2$[10]. This result implies that $\kappa(\mathbf{M}_\mathrm{A})(1 - ||\mathbf{E}||_2||\mathbf{M}^{-1}||_2) \leqslant \kappa(\mathbf{M})\ [1 + (ae(\mathbf{M}_\mathrm{A}, \mathbf{M})/||\mathbf{M}||_2)]$, with $\kappa(\mathbf{M}_\mathrm{A}) = \kappa(\mathbf{M})$, if $\mathbf{E} = 0$. Assuming $||\mathbf{E}||_2||\mathbf{M}^{-1}||_2$, $||\mathbf{E}||_2/||\mathbf{M}||_2 < \varepsilon \in (0, 1)$, we can obtain

$$1-2\varepsilon<\frac{\kappa\left(\mathbf{M}_\mathrm{A}\right)}{\kappa\left(\mathbf{M}\right)}\leqslant\frac{1+\varepsilon}{1-\varepsilon}. \tag{5}$$

Equation (5) provides an estimate for $\kappa(\mathbf{M}_\mathrm{A})$ using bounds that are functions of $\mathbf{E}$. In Fig. 3, we plot the upper and lower bounds of $(\kappa(\mathbf{M}_\mathrm{A})/\kappa(\mathbf{M}))$ as a function of $\varepsilon \in (0, 1)$. It illustrates how smaller aberration strength yields a correspondingly small deviation of $\kappa(\mathbf{M}_\mathrm{A})$ from $\kappa(\mathbf{M})$. The refined restrictions on $||\mathbf{E}||_2$ are interpreted as a measure of how small the aberration is. However, Eq. (5) only provides upper and lower bounds for $\kappa(\mathbf{M}_\mathrm{A})$ and is inadequate for exactly determining the trend of $\kappa(\mathbf{M}_\mathrm{A})$, which is a function of $\mathbf{E}$. In Fig. 2(b), the narrow beams indicated by low normalized values are associated with low $\kappa(\mathbf{M}_\mathrm{A})$. Thus, as the beam expands, the quantity $(\kappa(\mathbf{M}_\mathrm{A})/\kappa(\mathbf{M}))$ increases, making this treatment qualitatively dissimilar to the Strehl ratio in the diffraction theory of aberrations[1,2].

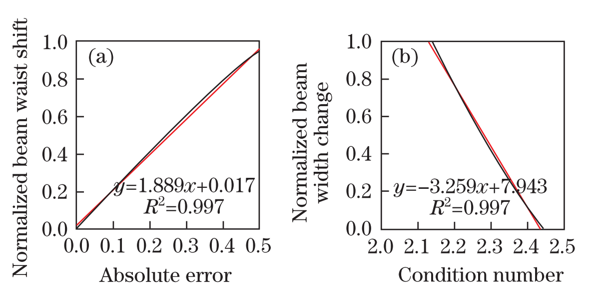

Fig. 2. (Color online) Quantitative measures of image imperfection using $f = 1$ unit with defocus increments of $\Delta f = 0.10$ units for six increments. (a) Beam waist shift representing a variation of focal length and (b) beam width described by condition number.

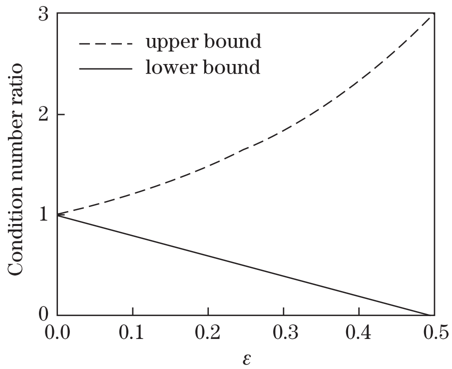

Fig. 3. Upper and lower bounds for the ratio $((\kappa(\mathbf{M}_\mathrm{A})/\kappa(\mathbf{M})))$ for $\varepsilon \in (0, 0.5)$. The effect of ε on the bounds of $\kappa(\mathbf{M}_\mathrm{A})$ is more evident with larger ε values.

In conclusion, using $\mathbf{M}_\mathrm{A}$ has more physical significance because it contains information on $\mathbf{M}$ and the perturbation. One finds $\mathbf{M}_\mathrm{A}$ operationally, whereas $\mathbf{M}$ is known *a priori* and $\mathbf{E}$ is calculated from Eq. (2). The imperfection in the perturbation is different from the optical component. Thus, scalar metrics describe the extent of image imperfection relative to an optical component with no imperfection. Furthermore, in systems with two or more components, the proposed perturbation approach can be used to determine how much imperfection is associated with each system component.

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