Scattering of an unpolarized Bessel beam by spheres

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The scattering process of an unpolarized Bessel beam through spherical scatterers is investigated. We derive the analytical solutions of scattered fields of x- and y-polarized Bessel beams using a sphere, after which the dimensionless scattering function for an unpolarized Bessel beam is obtained. The dimensionless scattering function is applicable to spherical scatterers of any size on the beam axis or near it. Through numerical simulations, we demonstrate that extreme points exist in the direction or neighboring direction of the conical angle for spherical scatterers on the beam axis, whereas the existence of extreme points depends on the ratio between the spherical scatterers size and central spot size of the Bessel beam.

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Bessel beams were first introduced by Durnin[1,2]. These non-diffracting beams can maintain the same intensity profile at any plane orthogonal to their propagation directions, and their initial intensity profiles can be reconstructed even when they have been disturbed by an obstacle under free propagation[3]. The special properties of Bessel beams have generated much interest from various domains, including optical acceleration[4,5], optical manipulation[6,7], nonlinear optics[8,9], and optical interconnection and alignment[10,11]. Light scattering is an important issue in various scientific and engineering applications, and extensive studies on scattering of light beams by different shaped particles, particularly spherical particles, have been carried out. For example, light scattering by a sphere located arbitrarily in a Gaussian beam has been described by Gouesbet *et al.*[12]. Barton *et al.*[13] have reported on theoretical expressions and numerical calculations for internal and near surface electromagnetic fields for a spherical particle irradiated by a focused Gaussian beam. However, only Marston has investigated the scattering by spheres centered on the beam axis[14] of Bessel beams. Nevertheless, research concerning the scattering of Bessel beams is very limited in current literature.

In this letter, we first derive the analytical solutions of scattered fields of x- and y-polarized Bessel beams by a sphere. Next, dimensionless scattering function for an unpolarized Bessel beam is obtained using analytical solutions. The dimensionless scattering function is applicable to spherical scatterers on the beam axis or near it. Through numerical simulation, we demonstrate that extreme points exist in the direction or neighboring direction of the conical angle. For spherical scatterers on the beam axis, the existence of the extreme points depends on the ratio between the size of the spherical scatterers and the size of the central spot of the Bessel beam. This phenomenon can be explained better through quantum theory.

In processing scattering questions[15−17], multiple expansion of radiation field in terms of spherical vector wave functions (SVWFs) are often required. Every solution to the vector Helmholtz equation can be expressed as a linear combination of SVWFs[18]. The SVWFs have the following form:

$$\mathbf{M}_{mn}^{\kappa} = z_n^{(\kappa)}(kr)\exp(\mathrm{i}m\phi)\,[\mathrm{i}\pi_{mn}(\cos\ \theta)\hat{\mathbf{e}}_{\theta} - \tau_{mn}(\cos\ \theta)\hat{\mathbf{e}}_{\phi}], \quad (1)$$

and

$$\begin{aligned}\mathbf{N}_{mn}^{\kappa} = {} & \frac{z_n^{(\kappa)}(kr)}{kr}\exp(\mathrm{i}m\phi)n(n+1)P_n^m(\cos\theta)\hat{\mathbf{e}}_r \\ & + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}(kr)}[krz_n^{(\kappa)}(kr)]\ \exp(\mathrm{i}m\phi) \\ & \times[\tau_{mn}(\cos\theta)\hat{\mathbf{e}}_{\theta} + \mathrm{i}\pi_{mn}(\cos\theta)\hat{\mathbf{e}}_{\phi}], \end{aligned} \quad (2)$$

where $z_n^{(\kappa)}(kr)$ represents all kinds of the spherical Bessel functions ($\kappa = 1, 2, 3, 4$); $P_n^m(\cos\theta)$ represents the associated Legendre polynomials of degree n and order m; $\pi_{mn}(\cos\theta) = mP_n^m(\cos\theta)/\sin\theta$, $\tau_{mn}(\cos\theta) = \mathrm{d}P_n^m(\cos\theta)/\mathrm{d}\theta$; $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_\phi$ are the unit vectors of the spherical coordinate system; k is the wave number of medium, and time dependence $\exp(-\mathrm{i}\omega t)$ is omitted.

We considered a sphere at position (x_0, y_0, z_0) in a rectangular coordinate system exposed to a polarized Bessel beam propagating along the z-axis. The polarized Bessel beam can be expressed in the following integral form[19]:

$$\mathbf{E}_i(\mathbf{r}) = E_0\int_0^{2\pi}\mathbf{e}_v(\alpha,\beta)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\mathrm{d}\beta, \quad (3)$$

where E_0 is the peak amplitude, $\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$ denotes a plane wave with wave vector $\mathbf{k} = (k, \alpha, \beta)$, $\mathbf{e}_v(\alpha, \beta)$ represents the unit polarization vectors, v represents polarization directions x and y, and parameter α is being called conical angle of the Bessle beam. For the x- and y-polarized Bessel beams, $\mathbf{e}_v(\alpha,\ \beta)$ has the following form, respectively:

$$\begin{aligned}\mathbf{e}_x(\alpha,\beta) &= \sin\alpha\cos\beta\hat{\mathbf{e}}_r(\alpha,\beta) + \cos\alpha\cos\beta\hat{\mathbf{e}}_\theta(\alpha,\beta) \\ &\quad - \sin\beta\hat{\mathbf{e}}_\phi(\alpha,\beta), \\ \mathbf{e}_y(\alpha,\beta) &= \sin\alpha\sin\beta\hat{\mathbf{e}}_r(\alpha,\beta) + \cos\alpha\sin\beta\hat{\mathbf{e}}_\theta(\alpha,\beta) \\ &\quad + \cos\beta\hat{\mathbf{e}}_\phi(\alpha,\beta).\end{aligned}$$

In a spherical system with its origin at the center of the sphere, a polarized plane wave can be represented in terms of SVWFs as[15]

$$\mathbf{e}_\nu(\alpha,\beta)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} D_{mn}\Big[p'_{mn}\mathbf{M}^1_{mn}(k\mathbf{r}) + q'_{mn}\mathbf{N}^1_{mn}(k\mathbf{r})\Big], \tag{4}$$

where

$$D_{mn} = \frac{(2n+1)(n-m)!}{4n(n+1)(n+m)!},$$

$$p'_{mn} = -4\mathrm{i}^{n+1}\exp(-\mathrm{i}m\beta)\mathbf{e}_\nu(\alpha,\beta)[\pi_{mn}(\cos\alpha)\hat{\mathbf{e}}_\theta(\alpha,\beta) - \mathrm{i}\tau_{mn}(\cos\alpha)\hat{\mathbf{e}}_\phi(\alpha,\beta)],$$

$$q'_{mn} = -4\mathrm{i}^{n+1}\exp(-im\beta)\mathbf{e}_\nu(\alpha,\beta)[\tau_{mn}(\cos\alpha)\hat{\mathbf{e}}_\theta(\alpha,\beta) - \mathrm{i}\pi_{mn}(\cos\alpha)\hat{\mathbf{e}}_\phi(\alpha,\beta)].$$

By substituting Eq. (4) into Eq. (3) and performing integration over β, the expansion of the polarized Bessel beam in terms of SVWFs is obtained as[18]

$$\mathbf{E}_i(\mathbf{r}) = E_0\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\Big[p^\nu_{mn}\mathbf{M}^1_{mn}(k\mathbf{r}) + q^\nu_{mn}\mathbf{N}^1_{mn}(k\mathbf{r})\Big], \tag{5}$$

where

$$\left\{\begin{array}{c} p^\nu_{mn} \\ q^\nu_{mn}\end{array}\right\} = -4D_{mn}\mathrm{i}^{n+1}\exp(\mathrm{i}kz_0\cos\alpha)\left[\cos\alpha\left\{\begin{array}{c}\pi_{mn}\\ \tau_{mn}\end{array}\right\}I^\nu_+ + \left\{\begin{array}{c}\tau_{mn}\\ \pi_{mn}\end{array}\right\}I^\nu_-\right],$$

$$I^x_\pm = \pi\exp[\mathrm{i}(1-m)\phi_0]J_{1-m}(\rho_0) \pm \pi\exp[-\mathrm{i}(m+1)\phi_0]J_{-1-m}(\rho_0),$$

$$I^y_\pm = -\pi\mathrm{i}\exp[\mathrm{i}(1-m)\phi_0]J_{1-m}(\rho_0) \pm \pi\mathrm{i}\exp[-\mathrm{i}(m+1)\phi_0]J_{-1-m}(\rho_0),$$

$$\rho_0 = k\sqrt{x_0^2+y_0^2}\sin\alpha,$$

and $\phi_0 = \arctan(y_0/x_0) + \pi/2$.

The corresponding magnetic field of the polarized Bessel beam is expressed by

$$\mathbf{H}_i(\mathbf{r}) = \frac{\mathbf{k}}{\mathrm{i}\omega\mu}E_0\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\Big[q^\nu_{mn}\mathbf{M}^1_{mn}(k\mathbf{r}) + p^\nu_{mn}\mathbf{N}^1_{mn}(k\mathbf{r})\Big]. \tag{6}$$

We can also expand the scattered field $(\mathbf{E}_\mathrm{s},\mathbf{H}_\mathrm{s})$ in the ambient medium and the field $(\mathbf{E}_1,\mathbf{H}_1)$ inside the sphere in terms of SVWFs as

$$\mathbf{E}_\mathrm{s}(\mathbf{r}) = E_0\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[a^\nu_{mn}\mathbf{M}^3_{mn}(k\mathbf{r}) + b^\nu_{mn}\mathbf{N}^3_{mn}(k\mathbf{r})\right], \tag{7}$$

$$\mathbf{H}_\mathrm{s}(\mathbf{r}) = \frac{kE_0}{\mathrm{i}\omega\mu}{}_0\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[b^\nu_{mn}\mathbf{M}^3_{mn}(k\mathbf{r}) + a^\nu_{mn}\mathbf{N}^3_{mn}(k\mathbf{r})\right], \tag{8}$$

$$\mathbf{E}_1(\mathbf{r}) = \boldsymbol{E}_0\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[c^\nu_{mn}\mathbf{M}^1_{mn}(k_1\mathbf{r}) + d^\nu_{mn}\mathbf{N}^1_{mn}(k_1\mathbf{r})\right], \tag{9}$$

$$\mathbf{H}_1(\mathbf{r}) = \frac{k_1E_0}{\mathrm{i}\omega\mu_1}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[d^\nu_{mn}\mathbf{M}^1_{mn}(k_1\mathbf{r}) + c^\nu_{mn}\mathbf{N}^1_{mn}(k_1\mathbf{r})\right], \tag{10}$$

where k_1 and k are the wave numbers in the sphere and ambient medium, respectively, and μ_1 and μ are the corresponding magnetic permeabilities. Using the boundary conditions for electromagnetic field at the spherical scatterer surface, i.e., $r = a$, $E_{i\theta} + E_{s\theta} = E_{1\theta}$ and $H_{i\theta} + H_{s\theta} = H_{1\theta}$, the coefficients, a^ν_{mn} and b^ν_{mn} can be solved as

$$a^\nu_{mn} = \frac{\mu j_n(x)[\eta x j_n(\eta x)]' - \mu_1 j_n(\eta x)[xj_n(x)]'}{\mu_1 j_n(\eta x)[xh^1_n(x)]' - \mu h^1_n(x)[\eta xj_n(\eta x)]'}p^\nu_{mn},$$

$$b^\nu_{mn} = \frac{\mu\eta^2 j_n(\eta x)[xj_n(x)]' - \mu_1 j_n(x)[\eta xj_n(\eta x)]'}{\mu_1 h^1_n(x)[\eta xj_n(\eta x)]' - \mu\eta^2 j_n(\eta x)[xh^1_n(x)]'}q^\nu_{mn},$$

where the prime indicates differentiation with respect to the argument in parentheses, the size parameter $x = k\alpha$, and η is the refractive index of the sphere relative to the ambient medium.

In the free space far from the spherical scatterer, $kr \gg n^2$, $h^{(1)}_n \approx (-\mathrm{i})^{n+1}\exp(\mathrm{i}kr)/(kr)$, accordingly,

$$\mathbf{M}^3_{mn} \approx \frac{1}{kr}(-\mathrm{i})^n\exp(\mathrm{i}kr)\exp(\mathrm{i}m\phi)[\pi_{mn}(\cos\theta)\hat{\mathbf{e}}_\theta + \mathrm{i}\tau_{mn}(\cos\theta)\hat{\mathbf{e}}_\phi], \tag{11}$$

$$\mathbf{N}^3_{mn} \approx \frac{1}{kr}(-\mathrm{i})^n\exp(\mathrm{i}kr)\exp(\mathrm{i}m\phi)[\tau_{mn}(\cos\theta)\hat{\mathbf{e}}_\theta + \mathrm{i}\pi_{mn}(\cos\theta)\hat{\mathbf{e}}_\phi]. \tag{12}$$

By substituting Eqs. (11) and (12) into the scattered field Eq. (7) and by writing the scattered field intensity as

$$I(r,\theta,\phi) = \frac{E_0^2}{k^2r^2}F_\nu(\theta,\phi), \tag{13}$$

then $F_\nu(\theta,\phi)$, which is the dimensionless scattering function, can be obtained from Eq. (13) as

$$F_\nu(\theta,\phi) = |S^\nu_1(\theta,\phi)|^2 + |S^\nu_2(\theta,\phi)|^2, \tag{14}$$

where $S_1^\nu(\theta,\phi)$ and $S_2^\nu(\theta,\phi)$ are the scattering amplitude functions given by

$$S_1^\nu(\theta,\phi) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}(-\mathrm{i})^n \exp(\mathrm{i}m\phi)$$
$$[a_{mn}^\nu \pi_{mn}(\cos\theta) + b_{mn}^\nu \tau_{mn}(\cos\theta)],$$
$$S_2^\nu(\theta,\phi) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}(-\mathrm{i})^n \exp(\mathrm{i}m\phi)$$
$$[a_{mn}^\nu \tau_{mn}(\cos\theta) + b_{mn}^\nu \pi_{mn}(\cos\theta)].$$

For a completely unpolarized Bessel beam that can be viewed as the incoherent superposition of two orthogonally polarized Bessel beams with equal amplitudes, its corresponding dimensionless scattering function is

$$F(\theta,\phi) = F_x(\theta,\phi) + F_y(\theta,\phi). \tag{15}$$

It is obvious that the dimensionless scattering function describing the intensity of scattered radiation in any given direction is only relevant to the scattering angle θ and the azimuth angle ϕ.

In numerical calculations, we used the following parameters: $\lambda = 1.55$ μm and $\alpha = \pi/6$ for the unpolarized Bessel beam; $x_0 = y_0 = z_0 = 0$ for the spherical scatterer position in the Bessel beam; $\mu_1 = \mu = 1$ for the magnetic permeability of the spherical scatterer and its ambient medium. When the spherical scatterer is on the axis of the unpolarized Bessel beam, the azimuth angle value can be selected arbitrarily because of the axial symmetry of the dimensionless scattering function. The dimensionless scattering function is an infinite series, and in order to keep enough terms to yield a sufficiently accurate approximation, the summation upper limit must be set as $N = x + 4x^{1/3} + 2$, where x refers to the size parameter[20].

Figure 1 shows the dimensionless scattering function for spherical scatterers with a radius $a = 2\lambda$ and different refractive indices. Figure 2 shows the dimensionless scattering function for spherical scatterers with a refractive index $\eta = 1.33$ and different radii. The two figures indicate that for larger spherical scatterers, there exist extreme points in the direction or neighboring direction of the conical angle that are not affected by different refractive indices and radii of the spherical scatterers. This phenomenon is also found and explained geometrically by Marston[14]. The wave-particle duality of light means that this phenomenon can be explained better through quantum theory.

Obviously, the refractive indices of spherical scatterers do not affect the existence of the extreme points in the direction or neighboring direction of the conical angle. The spherical scatterers are on the axis of the Bessel beam; hence, this phenomenon must be associated with the size of both the scatterers and the central spot of the Bessel beam. Let σ denote the ratio between

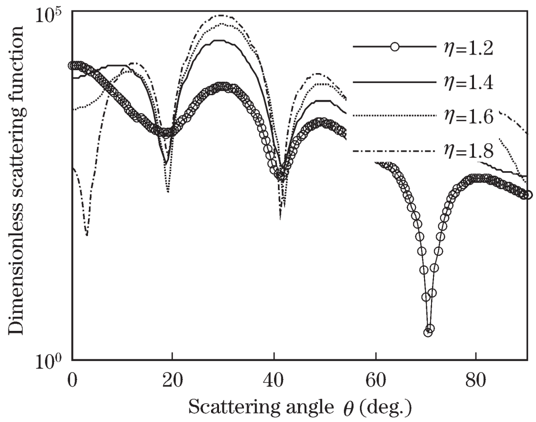

Fig. 1. Dimensionless scattering function for spherical scatterers with a radius $a = 2\lambda$ and different refractive indices η on the axis of a Bessel beam.

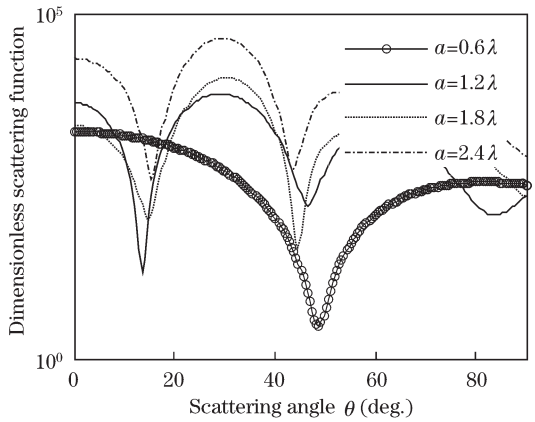

Fig. 2. Dimensionless scattering function for spherical scatterers with a refractive index $\eta = 1.33$ and different radii a on the axis of a Bessel beam.

a spherical scatterer radius and the central spot radius of the Bessel beam, then $\sigma = a/\rho = ak\sin\alpha/2.405$, where $\rho = 2.405/(k\sin\alpha)$ is the central spot radius of the Bessel beam[1].

Figure 3 shows that an extreme point emerges in the direction or neighboring direction of the conical angle when $\sigma > 5/4$. Figure 4 shows that the extreme point vanishes when $\sigma < 5/4$, indicating that $\sigma \approx 5/4$ can be thought of approximately as the critical condition for the existence of the extreme point in the direction or neighboring direction of the conical angle. Further numerical calculations verify that the above conclusion is always efficacious for any different Bessel beams, with different wavelengths and conical angles. In addition, a larger conical angle requires a larger σ for its extreme point to be seen.

Writing the critical condition $\sigma \approx 5/4$ in another form, $ak\sin\alpha/2.405 \approx 5/4$, and multiplying the Planck constant $\hbar = h/(2\pi)$ on both sides, a new version of the critical condition is obtained after some simple algebra calculations:

$$2a\Delta p_\perp \approx h, \tag{16}$$

where $\Delta p_\perp = \hbar k\sin\alpha$ and $2a$ refers to the diameter of

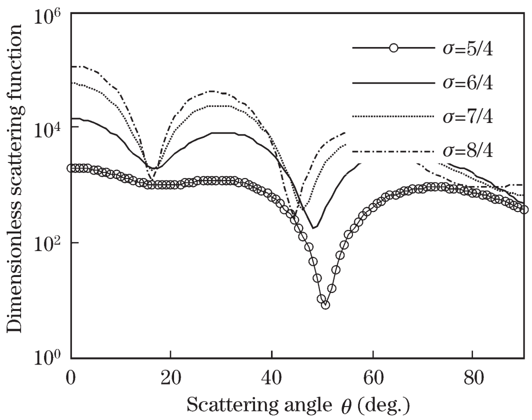

Fig. 3. Dimensionless scattering function at different σ ($\sigma \geq 5/4$), σ denotes the ratio between a spherical scatterer radius and the central spot radius of a Bessel beam.

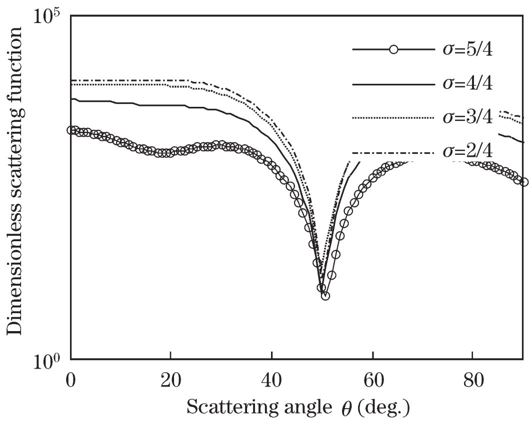

Fig. 4. Dimensionless scattering function at different σ ($\sigma \leq 5/4$).

the spherical scatterer. From Eq. (16), it can be seen that the critical condition for the existence of an extreme point in the direction or neighboring direction of the conical angle is a representation of the well-known Heisenberg uncertainty principle. According to the quantum theory, in the case of $\sigma < 5/4$, $2a\Delta p_{\perp} < h$, which means the photons interacting with the sphere are in the same state, moving forward along the beam axis; hence, the forward scattering along the beam axis is peaked and the extreme point disappears in the direction or neighboring direction of the conical angle. In the case of $\sigma > 5/4$, $2a\Delta p_{\perp} > h$, the photons interacting with the sphere are not only in the forward direction but also in the direction of the conical angle of the Bessel beam. Therefore, it is possible that the forward scattering along the beam axis is peaked. At the same time, the extreme point exists in the direction or neighboring direction of the conical angle of the Bessel beam.

In this letter, we investigate the scattering process of an unpolarized Bessel beam by spheres. Analytical solutions of scattered fields of x- and y-polarized Bessel beams by a sphere have been derived. Utilizing the analytical solutions, dimensionless scattering function for an unpolarized Bessel beam has been obtained. The dimensionless scattering function is applicable to spherical scatterers on the axis of an unpolarized Bessel beam or near it. Through numerical simulation, we have demonstrated that there are extreme points in the direction or neighboring direction of the conical angle for spherical scatterers on the beam axis. In addition, the existence of the extreme points has been found to depend on the ratio between size of the spherical scatterers and the size of the central spot of a Bessel beam.

The appearance and disappearance of an extreme point in the direction or neighboring direction of the conical angle can be explained through the application of the quantum theory in the scattering process of a Bessel beam by spheres according to the wave-particle duality of light. The propagation of Bessel beam is not in conflict with the Heisenberg uncertainty principle[2]. In fact, this principle also plays an important role in the light scattering process of unpolarized Bessel beams by spheres.

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