Correlation function of an optical bistable system with cross-correlated additive white noise and multiplicative colored noise

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Considering an optical bistable system with cross-correlated additive white noise and multiplicative colored noise, we study the effects of correlation between the noises on the correlation function $C(s)$ using the unified colored noise approximation and the Stratonovich decoupling ansatz formalism. The effects of the self-correlation time τ of the multiplicative colored noise and the correlation intensity λ between the two noises are studied with numerical calculation. It is found that $C(s)$ increases with the increase of the self-correlation time τ, but decreases with the increase of the correlation intensity λ. At large value of τ, there is almost no change for $C(s)$ when τ changes.

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It has been found that a nonlinear stochastic system can have far-reaching consequence and cause interesting phenomena. Though various noises are presented simultaneously in some stochastic processes, noises are assumed to have different origins and are treated as independent random variables in most of the previous investigations[1−5]. However, Fulinski *et al.* pointed out that noises in some stochastic processes might have a common origin and thus would not be independent[6]. Since then, many researchers considered the cross-correlation between the two noises when studying the statistical fluctuation properties, and the obtained results were in better agreement with the experimental results[7−13]. In 1994, Bartussek *et al.* studied the property of an optical bistable system driven only by multiplicative noise[14]. In 2003, Mei *et al.* studied the effects of correlations between additive and multiplicative noise on the relaxation time in a bistable system driven by cross-correlated noises[15]. Luo *et al.* investigated stochastic resonance in a bistable nonlinear system when both the multiplicative noise and the coupling between the additive and multiplicative noise are colored with different values of noise correlation time[16]. In 2004, Cheng *et al.* studied the stationary intensity distribution of a single-mode laser cubic model driven by colored pump noise with cross-correlation between the real and imaginary parts of the quantum noise[17]. In 2006, Ning *et al.* investigated the transient properties of an optical bistable system driven by multiplicative white noise and additive white noise[18]. In this letter, the correlation function of an optical bistable system with cross-correlated additive white noise and multiplicative colored noise is studied.

A model for purely absorptive optical bistability in an optical cavity was introduced by Bonifacio and Lugiato[19]. For the input light amplitude y and the transmitted amplitude x, they derived the equation of motion for the dimensionless variables as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - x - \frac{2cx}{1+x^2} = -\frac{\mathrm{d}U(x)}{\mathrm{d}x}, \tag{1}$$

where the potential $U(x) = -\int (y - x - \frac{2cx}{1+x^2})dx$. The parameter c is proportional to the inversion of the population of the atomic levels. For large c, the input-output characteristics show the bistability. The potential $U(x)$ has two minima when the system exhibits optical bistability[20].

For large value of c, choose input amplitude y to be y_0 within the regime of bistability, and take into account fluctuations of the input amplitude y and the inversion c, we assume these fluctuations are fast and can be modeled by Gaussian noise, which are expressed as

$$y \to y_0 + \eta(t), \tag{2}$$
$$c \to c + \xi(t). \tag{3}$$

The transmitted light amplitude is thus described by

$$\frac{dx}{dt} = y_0 - x - \frac{2cx}{1+x^2} + \frac{2x}{1+x^2}\xi(t) + \eta(t), \tag{4}$$

where $\xi(t)$ and $\eta(t)$ are correlated in the forms as

$$\langle\eta(t)\rangle = \langle\xi(t)\rangle = 0, \tag{5}$$
$$\langle\xi(t)\xi(t')\rangle = \frac{Q}{2\tau}\exp(-\frac{|t-t'|}{\tau}), \tag{6}$$
$$\langle\eta(t)\eta(t')\rangle = D\delta(t-t'), \tag{7}$$
$$\langle\eta(t')\xi(t)\rangle = \langle\xi(t')\eta(t)\rangle = \lambda\sqrt{QD}\delta(t-t'), \tag{8}$$

where Q and D are the intensities of noises of $\xi(t)$ and $\eta(t)$, $\langle\rangle$ denotes the ensemble average, τ is the self-correlation time of the multiplicative colored noise, and λ is the correlation intensity between the two noises with $|\lambda| \le 1$.

According to Eq. (4), the corresponding Fokker-Planck equation is

$$\frac{\partial P(x,t)}{\partial t} = L_{\mathrm{FP}}P(x,t), \tag{9}$$

$$L_{\mathrm{FP}} = -\frac{\partial}{\partial x}F(x,\tau) + \frac{\partial^2}{\partial x^2}G(x,\tau). \tag{10}$$

The drift coefficient $F(x,\tau)$ and the diffusion coefficient $G(x,\tau)$ are given by

$$F(x,\tau)=\frac{f(x)}{R(x,\tau)}+\frac{K'(x)}{R^2(x,\tau)}-\frac{R(x,\tau)}{R^3(x,\tau)},\quad(11)$$

$$G(x,\tau)=\frac{K(x)}{R^2(x,\tau)},\quad(12)$$

where

$$f(x)=y_0-x-\frac{2cx}{1+x^2},\quad(13)$$

$$K(x)=Q\frac{4x^2}{(1+x^2)^2}+2\lambda\sqrt{QD}\frac{2x}{1+x^2}+D,\quad(14)$$

$$R(x,\tau)=1-\tau\left[f'(x)-\frac{1-x^2}{(1+x^2)x}f(x)\right].\quad(15)$$

only considering the stationary state, the steady-state probability density of Eq. (9) can be obtained as

$$P_{\rm st}(x)=N\frac{R(x,\tau)}{\sqrt{K(x)}}\exp\left[\int\frac{f(x)R(x,\tau)}{K(x)}{\rm d}x\right],\quad(16)$$

where N is the normalization constant.

According to the steady-state probability density expression [Eq. (16)] of the optical bitable system, the effects of τ and λ on $P_{\rm st}(x)$ can be studied with the numerical computation. $P_{\rm st}(x)$ with different values of τ and λ are plotted in Figs. 1 and 2.

Figure 1 shows the steady-state probability density $P_{\rm st}(x)$ as a function of x with different values of τ. It is seen that there are two peaks in $P_{\rm st}(x)$, and the left peak is narrower than the right one. From Fig. 1, we can also find that the larger τ is, the higher the left peak is, and the lower the right one is. Figure 2 shows the steady-state probability density $P_{\rm st}(x)$ as a function of x with different values of λ. It is clear that the larger λ is, the lower the left peak is, and the higher the right one is.

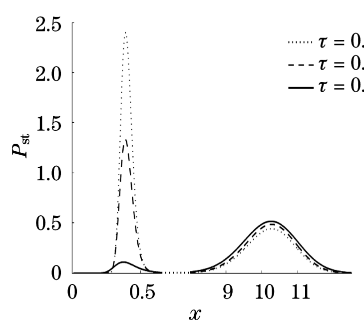

Fig. 1. Steady-state probability density $P_{\rm st}(x)$ as a function of x with different values of τ. ($y_0=14.2$, $c=20.02$, $D=0.32$, $Q=0.35$, and $\lambda=0.36$).

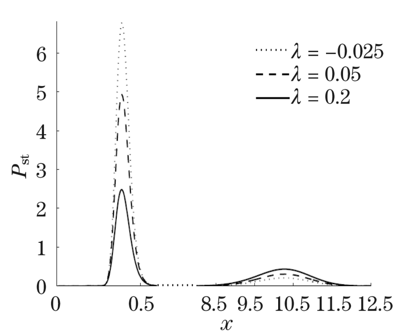

Fig. 2. Steady-state probability density $P_{\rm st}(x)$ as a function of x with different values of λ. ($y_0=14.2$, $c=20.02$, $D=0.32$, $Q=0.35$, and $\tau=0.4$.).

For a nonlinear stochastic system, the correlation function is

$$C(s)=\frac{K(s)}{\langle(x(t)-\langle x(t)\rangle)^2\rangle_{\rm st}},\quad(17)$$

where $K(s)$ is the two-time correlation function and can be expressed as[17]

$$K(s)=\langle x(t+s)x(t)\rangle_{\rm st}-\langle x(t)\rangle_{\rm st}^2.\quad(18)$$

The expectation value of the nth power of x is defined as

$$\langle x^n\rangle_{\rm st}=\int_0^{+\infty}x^nP_{\rm st}(x){\rm d}x.\quad(19)$$

Using the two-time correlation probability density $\omega(x,t+s;x',t)$, we get

$$\begin{aligned}\langle x(t+s)x(t)\rangle_{\rm st}&=\iint x\omega(x,t+s;x',t)x'{\rm d}x{\rm d}x'\\&=\iint xP_{\rm tran}(x,t+s;x',t)\\&\quad P_{\rm st}(x)x'{\rm d}x{\rm d}x'.\end{aligned}\quad(20)$$

Because the transition probability density $P_{\rm tran}(x,t+s;x',t)=\exp(L_{\rm FP}s)\delta(x-x')$, we can obtain

$$\langle x(t+s)x(t)\rangle_{\rm st}=\int x\exp(L_{\rm FP}s)xP_{\rm st}(x){\rm d}x.\quad(21)$$

Using Eq. (21), the derivative of the stationary correlation function $K(s)$ is

$$\begin{aligned}\frac{{\rm d}K(s)}{{\rm d}s}&=\int xL_{\rm FP}\exp(L_{\rm FP}s)xP_{\rm st}(x){\rm d}x\\&=\int(L_{\rm FP}^{+}x)\exp(L_{\rm FP}s)xP_{\rm st}(x){\rm d}x.\end{aligned}\quad(22)$$

In terms of the adjoint operator $L_{\rm FP}^{+}$ of Eq. (10), using the Stratonovich approximate[21], we get

$$\frac{1}{K(s)}\frac{{\rm d}K(s)}{{\rm d}s}=\frac{\langle x(t)L_{\rm FP}^{+}x(t)\rangle_{\rm st}}{K(0)},\quad(23)$$

and

$$K(0)=\langle(\delta x)^2\rangle_{\rm st}=\langle(x(t)-\langle x(t)\rangle)^2\rangle_{\rm st}.\quad(24)$$

The solution of Eq. (22) is $K(s)=K(0)\exp(-\mu s)$. The correlation function is

$$C(s)=\frac{K(s)}{K(0)}=\exp(-\mu s),\quad(25)$$

where

$$\mu=\frac{\langle\delta x(L_{FP}^{+})^{-1}\delta x\rangle_{\rm st}}{\langle(\delta x)^2\rangle_{\rm st}}.\quad(26)$$

Making use of the correlation function of Eq. (25), the effects of both τ and λ on the system can be analyzed by

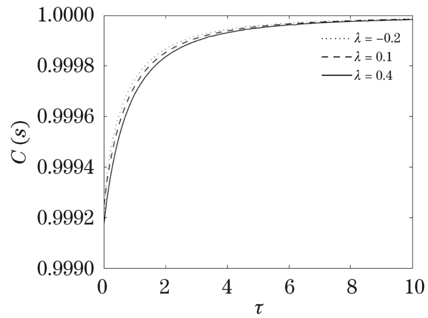

Fig. 3. Correlation function $C(s)$ as a function of τ with different values of λ. $y_0 = 14.2$, $c = 20.02$, $D = 0.32$, and $Q = 0.35$.

the numerical calculation.

Figure 3 shows the curves of $C(s)$ as a function of the self-correlation time τ of the multiplicative noise for different values of λ. It is known that $C(s)$ is a measure of correlation between x fluctuations at time t and $t+s$. As shown, we can find that $C(s)$ increases with the increase of the self-correlation time τ. In other words, the decay rate of the x fluctuation becomes slower and slower with the increase of the self-correlation time τ. At large value of τ, there is almost no change for $C(s)$ when τ changes. We can also find that the smaller the λ is, the larger the $C(s)$ becomes.

In conclusion, we investigate the effects of the self-correlation time τ and the correlation intensity λ on the statistical properties of optical bistable system. We find that the correlation function $C(s)$ increases with the increase of the self-correlation time and decreases with the increase of the correlation intensity λ.

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