A new method of studying the statistical properties of speckle phase

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Received January 26, 2008

A new theoretical method with generality is proposed to study the statistical properties of the speckle phase. The general expression of the standard deviation of the speckle phase about the first-order statistics is derived according to the relation between the phase and the complex speckle amplitude. The statistical properties of the speckle phase have been studied in the diffraction fields with this new theoretical method.

OCIS codes: 030.6140, 000.5490.

doi: 10.3788/COL20090701.0005.

Traditional theoretical method of evaluating the extent of the statistical distributions of speckle phase is to calculate the standard deviation of phase which is based on the probability density function of speckle phase. However, it is very difficult and even impossible to derive the strict analytical expression. Thus approximate analytical methods were developed to evaluate the extent of the speckle phase distributions. With regard to the first-order statistics, Takai *et al.* used the phase angle defined from the equiprobability density ellipse to evaluate approximately the extent of phase distributions of the Gaussian speckle[1]. But this method is only valid in the cases that the coordinate origin in the complex plane of speckle amplitude must be located at or outside the equiprobability density ellipses. Wang used the phase angle defined from the two equiprobability density ellipses to evaluate approximately the extent of the free statistical distributions of phase difference of the Gaussian speckle[2]. But this method seems to be only valid in some relatively small regions at which the equiprobability density ellipses are located in the complex plane of the speckle amplitude with the variation of some parameters of the speckle fields. For relatively large regions, this method seems to be wrong to explain the free statistical properties of the phase difference. So, we cannot use these analytical methods to study widely or correctly the statistical properties of the speckle phase including the fully and partially developed speckle fields. In this letter, a new method of studying the speckle phase is proposed. The general expression of the standard deviation of the speckle phase about the first-order statistics is derived according to the relation between the phase and the complex speckle amplitude. The first-order statistical properties of speckle phase have been investigated in the diffraction fields.

The complex speckle amplitude at a point in the speckle fields is expressed as

$$A = A_\mathrm{r} + \mathrm{i}A_\mathrm{i} = |A|\exp(\mathrm{i}\theta), \tag{1}$$

where A_r and A_i are the real and imaginary parts of the speckle fields and

$$|A| = \sqrt{{A_\mathrm{r}}^2 + {A_\mathrm{i}}^2}, \quad \theta = \tan^{-1}\left(\frac{A_\mathrm{i}}{A_\mathrm{r}}\right) \tag{2}$$

are the amplitude of A and the speckle phase, respectively. $\langle A\rangle$ is the ensemble mean value of the complex speckle amplitude, i.e., the specular component, in which $\langle\cdots\rangle$ denotes the ensemble mean. $\langle A\rangle$ is expressed as

$$\langle A\rangle = \langle A_\mathrm{r}\rangle + \mathrm{i}\langle A_\mathrm{i}\rangle = |\langle A\rangle|\exp\left(\mathrm{i}\bar{\theta}\right), \tag{3}$$

where $\langle A_\mathrm{r}\rangle$ and $\langle A_\mathrm{i}\rangle$ are the real and imaginary parts of $\langle A\rangle$ and

$$|\langle A\rangle| = \sqrt{\langle A_\mathrm{r}\rangle^2 + \langle A_\mathrm{i}\rangle^2}, \quad \bar{\theta} = \tan^{-1}\left(\frac{\langle A_\mathrm{i}\rangle}{\langle A_\mathrm{r}\rangle}\right) \tag{4}$$

are the average amplitude and the mean value of speckle phase of $\langle A\rangle$, respectively. Define $\langle A\rangle^*$ as the complex conjugate value of $\langle A\rangle$. The product of A and $\langle A\rangle^*$, i.e.,

$$A^p = A\langle A\rangle^*, \tag{5}$$

can also be expressed as

$$A^p = {A_\mathrm{r}}^p + \mathrm{i}{A_\mathrm{i}}^p = |A^p|\exp\left(\mathrm{i}\theta_\mathrm{d}\right), \tag{6}$$

where ${A_\mathrm{r}}^p$ and ${A_\mathrm{i}}^p$ are the real and imaginary parts of A^p, respectively. According to Eq. (5), we can get

$${A_\mathrm{r}}^p = A_\mathrm{r}\langle A_\mathrm{r}\rangle + A_\mathrm{i}\langle A_\mathrm{i}\rangle, \tag{7}$$

$${A_\mathrm{i}}^p = A_\mathrm{i}\langle A_\mathrm{r}\rangle - A_\mathrm{r}\langle A_\mathrm{i}\rangle. \tag{8}$$

$|A^p|$ is the module of A^p and

$$\theta_\mathrm{d} = \theta - \bar{\theta} = \tan^{-1}\left(\frac{{A_\mathrm{i}}^p}{{A_\mathrm{r}}^p}\right) \tag{9}$$

is the deviation of speckle phase. The ensemble mean values of the real and imaginary parts of A^p are calculated from Eqs. (7) and (8), respectively, as

$$\langle {A_\mathrm{r}}^p\rangle = \langle A_\mathrm{r}\rangle^2 + \langle A_\mathrm{i}\rangle^2 = |\langle A\rangle|^2, \tag{10}$$

$$\langle {A_\mathrm{i}}^p\rangle = 0. \tag{11}$$

According to Eqs. (10) and (11), the mean value of the phase deviation θ_d equals zero, and is expressed as

$$\bar{\theta}_\mathrm{d} = \overline{\theta - \bar{\theta}} = \tan^{-1}\left(\frac{\langle {A_\mathrm{i}}^p\rangle}{\langle {A_\mathrm{r}}^p\rangle}\right) = 0. \tag{12}$$

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We use the root-mean-square (RMS) value of the imaginary part of the complex of A^p of Eq. (8), i.e., $\left\langle (A_\mathrm{i}{}^p)^2 \right\rangle^{1/2}$, to calculate the approximate value of the standard deviation of the speckle phase, which is expressed as

$$\Delta\theta_\mathrm{a} = \tan^{-1}\left(\frac{\left\langle (A_\mathrm{i}{}^p)^2 \right\rangle^{1/2}}{\langle A_\mathrm{r}{}^p \rangle}\right), \tag{13}$$

in which $\left\langle (A_\mathrm{i}{}^p)^2 \right\rangle^{1/2}$ is calculated, from Eq. (8), as

$$\left\langle (A_\mathrm{i}{}^p)^2 \right\rangle^{1/2} = \Big(\left\langle \Delta A_\mathrm{r}{}^2 \right\rangle \langle A_\mathrm{i} \rangle^2 + \left\langle \Delta A_\mathrm{i}{}^2 \right\rangle \langle A_\mathrm{r} \rangle^2 - 2 \langle \Delta A_\mathrm{r} \Delta A_\mathrm{i} \rangle \langle A_\mathrm{r} \rangle \langle A_\mathrm{i} \rangle\Big)^{1/2}. \tag{14}$$

When the surface roughness increases to an enough large value, the speckle fields will become fully developed, the ensemble mean value of the real and imaginary parts of the complex speckle amplitude A and the value of $\langle \Delta A_\mathrm{r} \Delta A_\mathrm{i} \rangle$ will approach zero[3,4]. From Eqs. (10) and (14), we may prove that the value of $\left\langle (A_\mathrm{i}{}^p)^2 \right\rangle^{1/2} / \langle A_\mathrm{r}{}^p \rangle$ will approach infinity, $\Delta\theta_\mathrm{a}$ will approach $\pi/2$. In the case of fully developed speckle field, the speckle phase obeys the homogeneous statistical distributions, the probability density function of speckle phase is $P(\theta) = 1/(2\pi)$[3,4], the standard deviation of the speckle phase is $\pi/\sqrt{3}$. Therefore, through modifying Eq. (13), the standard deviation of speckle phase is given in the relatively exact form as

$$\Delta\theta = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\left\langle (A_\mathrm{i}{}^p)^2 \right\rangle^{1/2}}{\langle A_\mathrm{r}{}^p \rangle}\right). \tag{15}$$

Now we calculate the statistical parameters in Eqs. (10), (14) and (15). The optical configuration for producing speckle fields in the diffraction region is shown in Fig. 1. The transparent diffuser placed in the object plane with the ξ-η coordinate system is illuminated by a Gaussian laser beam with the beam waist position in the object plane. The speckle fields are produced in the diffraction region. The observation plane denoted by the x-y coordinate system is set in the diffraction region with a distance z away from the object plane. The complex amplitude at a point in the observation plane is given by

$$A(z, \mathbf{x}) = \iint_{-\infty}^{+\infty} E(\boldsymbol{\xi}) T(\boldsymbol{\xi}) K(z, \boldsymbol{\xi}, \mathbf{x}) \mathrm{d}^2\boldsymbol{\xi}, \tag{16}$$

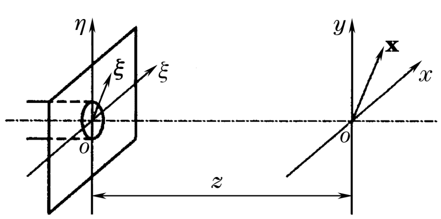

Fig. 1. Optical configuration for producing speckle fields in the diffraction region.

where

$$E(\boldsymbol{\xi}) = \exp\left(-\frac{|\boldsymbol{\xi}|^2}{\omega_0^2}\right) \tag{17}$$

is the Gaussian amplitude of the illuminating laser beam and ω_0 is the beam width of the laser light in the beam waist position.

$$T(\boldsymbol{\xi}) = \exp[\mathrm{i}\phi(\boldsymbol{\xi})] \tag{18}$$

is the transmittance function of the diffuser, $\phi(\boldsymbol{\xi})$ represents the microscopic random phase variation for the laser beam transmitting through the object. $K(z, \boldsymbol{\xi}, \mathbf{x})$ is the propagation function from the object plane to the observation plane and is given by

$$K(z, \boldsymbol{\xi}, \mathbf{x}) = \frac{1}{\lambda z} \exp\left(\frac{\mathrm{i}\pi}{\lambda z} |\boldsymbol{\xi} - \mathbf{x}|^2\right). \tag{19}$$

In Eq. (19), λ is the wavelength of the incident laser beam. We assume that the random phase $\phi(\boldsymbol{\xi})$ is a stationary Gaussian random variable with zero mean. Substituting Eqs. (17)—(19) into Eq. (16), the ensemble mean values of the real and imaginary parts of the complex speckle amplitude are obtained, respectively, as

$$\langle A_\mathrm{r}(z, \mathbf{x}) \rangle = \frac{1}{\sqrt{1+\hat{z}^2}} \exp\left(-\frac{1}{2}\sigma_\phi^2\right) \times \exp\left(-|\hat{\mathbf{x}}|^2\right) \cos\left[\tan^{-1}\left(\frac{1}{\hat{z}}\right) + \hat{z}|\hat{\mathbf{x}}|^2\right], \tag{20}$$

$$\langle A_\mathrm{i}(z, \mathbf{x}) \rangle = \frac{1}{\sqrt{1+\hat{z}^2}} \exp\left(-\frac{1}{2}\sigma_\phi^2\right) \times \exp\left(-|\hat{\mathbf{x}}|^2\right) \sin\left[\tan^{-1}\left(\frac{1}{\hat{z}}\right) + \hat{z}|\hat{\mathbf{x}}|^2\right], \tag{21}$$

where

$$\hat{z} = z/z_0, \tag{22}$$

$$z_0 = \pi\omega_0^2/\lambda, \tag{23}$$

$$\hat{\mathbf{x}} = \mathbf{x}/\omega(z), \tag{24}$$

$$\omega(z) = \omega_0\sqrt{1+(z/z_0)^2}. \tag{25}$$

In Eqs. (20) and (21), σ_ϕ denotes the standard deviation of the random phase $\phi(\boldsymbol{\xi})$ and is called as the optical roughness of the diffuse object. The circle function is assumed for the spatial correlation function of the random phase variation and is given by

$$\rho_\phi(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) = \mathrm{circ}\left(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|/\alpha\right), \tag{26}$$

where α is the correlation length. Under these circumstances, the variance of the real and imaginary parts, and the covariance between the real and imaginary parts of the complex speckle amplitude A are calculated and expressed, respectively, as

$$\left\langle \Delta A_{\mathrm{r}}{}^2 \right\rangle = \frac{1}{4N\hat{z}^2}\left[1-\exp\left(-\sigma_\phi^2\right)\right]\left\{1-\frac{\hat{z}}{\sqrt{1+\hat{z}^2}}\exp\left(-\sigma_\phi^2\right)\exp\left(-2\left|\hat{\mathbf{x}}\right|^2\right)\cos\left[\tan^{-1}\left(\frac{1}{\hat{z}}\right)+2\hat{z}\left|\hat{\mathbf{x}}\right|^2\right]\right\}, \tag{27}$$

$$\left\langle \Delta A_{\mathrm{i}}{}^2 \right\rangle = \frac{1}{4N\hat{z}^2}\left[1-\exp\left(-\sigma_\phi^2\right)\right]\left\{1+\frac{\hat{z}}{\sqrt{1+\hat{z}^2}}\exp\left(-\sigma_\phi^2\right)\exp\left(-2\left|\hat{\mathbf{x}}\right|^2\right)\cos\left[\tan^{-1}\left(\frac{1}{\hat{z}}\right)+2\hat{z}\left|\hat{\mathbf{x}}\right|^2\right]\right\}, \tag{28}$$

$$\left\langle \Delta A_{\mathrm{r}}\Delta A_{\mathrm{i}} \right\rangle = -\frac{1}{4N\hat{z}\sqrt{1+\hat{z}^2}}\left[1-\exp\left(-\sigma_\phi^2\right)\right]\exp\left(-\sigma_\phi^2\right)\exp\left(-2\left|\hat{\mathbf{x}}\right|^2\right)\sin\left[\tan^{-1}\left(\frac{1}{\hat{z}}\right)+2\hat{z}\left|\hat{\mathbf{x}}\right|^2\right]. \tag{29}$$

In Eqs. (27)—(29), $N = \pi\omega_0^2/(\pi\alpha^2)$ is the number of scatters involved within the illumination region over the diffuser. Substituting Eqs. (20), (21), and (27)—(29) into Eqs. (10), (14), and (15), the standard deviation of the speckle phase in the diffraction fields is obtained as

$$\Delta\theta = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{\sqrt{1+\hat{z}^2}}{2\hat{z}\sqrt{N}}\left[\left(1-\exp\left(-\sigma_\phi^2\right)\right)\right.\right.$$
$$\left.\left.\times\left(\exp\left(\sigma_\phi^2\right)+\frac{\hat{z}}{\sqrt{1+\hat{z}^2}}\exp\left(-2\left|\hat{\mathbf{x}}\right|^2\right)\cos\left(\tan^{-1}\left(\frac{1}{\hat{z}}\right)\right)\right)\right]^{1/2}\exp\left(\left|\hat{\mathbf{x}}\right|^2\right)\right\}. \tag{30}$$

When the normalized distance of $\hat{z}$ increases and reaches to an enough large value, the observation point will be located in the far-fields diffraction region. In this case, Eq. (30) is simplified as

$$\Delta\theta = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{1}{2\sqrt{N}}\left[\left(1-\exp\left(-\sigma_\phi^2\right)\right)\left(\exp\left(\sigma_\phi^2\right)+\exp\left(-2\left|\hat{\mathbf{x}}\right|^2\right)\right)\right]^{1/2}\exp\left(\left|\hat{\mathbf{x}}\right|^2\right)\right\}. \tag{31}$$

Now we investigate the statistical properties of the speckle phase in the diffraction region. For convenience, we investigate the one-dimensional coordinate case. As seen from Eq. (30), the standard deviation of speckle phase is related to the four parameters respectively: the optical roughness σ_ϕ of the diffuser, the normalized distance $\hat{z}$ from the object plane to the observation plane, the normalized position $\hat{x}$ of the observation point in the observation plane, the number of the scatters N contributing to the speckle fields.

Figure 2 shows $\Delta\theta$ as a function of the optical roughness σ_ϕ for several values of $\hat{x}$. As shown in Fig. 2, when $\sigma_\phi = 0$, there is no random phase turbulence caused by the object, therefore, $\Delta\theta = 0$, which means that there is no random phase fluctuation in the optical fields. The value of $\Delta\theta$ increases with the increase of σ_ϕ. When σ_ϕ reaches to an enough large value, the speckle fields become fully developed, $\Delta\theta$ approaches the constant value of $\pi/\sqrt{3}$. Figure 3 shows $\Delta\theta$ as a function of $\hat{z}$. As seen from Fig. 3, when the observation point approaches to the diffuser, the value of $\Delta\theta$ increases. That is because the fluctuation of the speckle fields increase when $\hat{z}$ decreases. When $\hat{z}$ increases and reaches to an enough large value, the observation point will be located in the far-field diffraction region, $\Delta\theta$ will approach the value of the far-field case which is expressed by Eq. (31). It should be pointed here that Eq. (15) is not valid when $\hat{z}$ approaches zero. This is due to the limitation of the diffraction theory of the optical fields. Figure 4 shows $\Delta\theta$ as a function of $\hat{x}$ for several values of N. As seen from Fig. 4, when $\hat{x} = 0$, $\Delta\theta$ reaches to the relatively smallest value, $\Delta\theta$ increases with the increase of $\hat{x}$, when $\hat{x}$ increases and reaches to an enough large value, the speckle fields will become fully developed. In this case, $\Delta\theta$ reaches to the

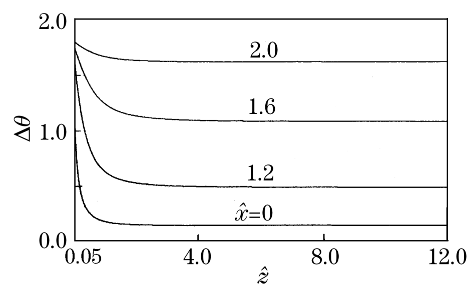

Fig. 3. Standard deviation $\Delta\theta$ as a function of the normalized distance $\hat{z}$ with $\sigma_\phi = 1.0$, $N = 40$ for several values of $\hat{x} = 0$, 1.2, 1.6, and 2.0.

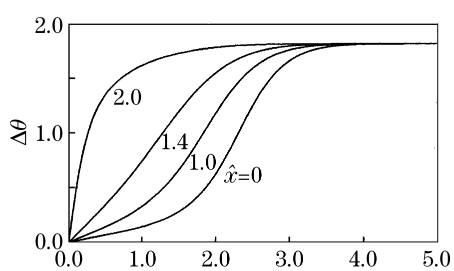

Fig. 2. Standard deviation $\Delta\theta$ as a function of the optical roughness σ_ϕ with $\hat{z} = 8$, $N = 40$ for several values of $\hat{x} = 0$, 1.0, 1.4, and 2.0.

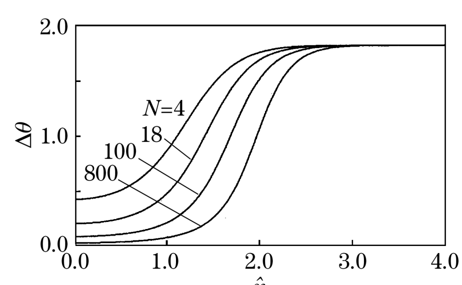

Fig. 4. Standard deviation $\Delta\theta$ as a function of the normalized position $\hat{x}$ with $\sigma_\phi = 1.0$, $\hat{z} = 8$ for several values of $N = 4$, 18, 100, and 800.

constant value of $\pi/\sqrt{3}$. As seen also, the value of $\Delta\theta$ decreases with the increase of the scatter number N, because the increase of the number of the scatters within the illumination region over the object causes the decrease of the fluctuation of speckle fields.

In summary, a new theoretical method is proposed to study the statistical properties of speckle phase. The general expression of the standard deviation of speckle phase is obtained according to the relations between the phase and the complex speckle amplitude. The statistical properties of speckle phase have been studied in the diffraction region by using this new method. It is shown that the standard deviation of speckle phase is related to the four parameters of the speckle fields. The statistical properties of the speckle fields in the diffraction region are well shown and thoroughly understood through this new method. It should be pointed here that this method may also be used to study the statistical properties of phase difference of the speckle fields. The theoretical results will be reported in another article.

References

1. N. Takai, H. Kadono, and T. Asakura, Opt. Eng. **25,** 627 (1986).
2. Q. Wang, Opt. Commun. **120,** 1 (1995).
3. A. Ishimaru, *Wave Propagation and Scattering in Random Media, Vol.1, Single Scattering and Transport Theory* (Academic Press, New York, 1978) Part 1, Chap.4.
4. J. W. Goodman, in *Laser Speckle and Related Phenomena,* J. C. Dainty (ed.) (Springer-Verlag, Berlin, 1975) p.9.