

Composition of Airy disc

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The description of a plane wave diffracted by a circular aperture is directly started from the Maxwell's equations. Based on the vector angular spectrum representation of Maxwell's equations, the diffracted plane wave is decomposed into the TE and TM terms. The analytical TE and TM terms in the far field are presented by stationary phase. As the TE and TM terms are orthogonal to each other in the far field, their sum constitutes the so-called Airy disc pattern. Therefore, this research reveals the composition of Airy disc, which is beneficial to deepen and enhance the recognition of the classical diffraction problem.

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A plane wave diffracted by a circular aperture is a typical diffraction problem which has been treated by many different methods^[1-8]. Only when the radius of aperture is of the order of the light wavelength, vector diffraction theory can be used to describe the optical propagation through the aperture. Otherwise, scalar diffraction theory is usually adopted. The scalar description of the diffracted plane wave is simple, but it maybe misses some physical information. Although the radius of aperture is much larger than the light wavelength of the incident plane wave, the description of a plane wave diffracted by a circular aperture is directly started from the Maxwell's equations in this letter. Moreover, the vector angular spectrum method is used to resolve the Maxwell's equations. As the Maxwell's equations can be separated into transverse and longitudinal field equations in homogeneous media, the diffracted wave is decomposed into the TE and TM terms. The TE term denotes the electric field transverse to the propagation axis, and the TM term denotes the associated magnetic field transverse to the propagation axis^[9,10]. As the divergence condition of the electric field should be satisfied and the polarized direction of every plane wave component must be perpendicular to its own wave vector, the TE and TM terms of the diffracted plane wave are unique. By using the method of stationary phase^[11,12], the analytical TE and TM terms in the far field are presented. Furthermore, the TE and TM terms are orthogonal to each other in the far field. Consequently, the sum of the light intensities of the TE and TM constitutes the Airy disc pattern.

A linearly polarized plane wave described by

$$\begin{pmatrix} E_x(\mathbf{r}) \\ E_y(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \exp(ikz) \quad (1)$$

passes through a co-axial circular aperture with radius R , where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = (x^2 + y^2 + z^2)^{1/2}$, and

$$A_x(p, q) = \frac{1}{\lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_x(x_0, y_0, 0) \exp[-ik(px_0 + qy_0)] dx_0 dy_0 = \frac{R \cos \alpha J_1(kRb)}{\lambda b}, \quad (6)$$

$$A_y(p, q) = \frac{1}{\lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_y(x_0, y_0, 0) \exp[-ik(px_0 + qy_0)] dx_0 dy_0 = \frac{R \sin \alpha J_1(kRb)}{\lambda b}, \quad (7)$$

where $b = (p^2 + q^2)^{1/2}$. J_1 is the first order Bessel function of the first kind. The longitudinal component stems from

$k = 2\pi/\lambda$, λ is the wavelength of the incident plane wave.

Jones vector $\begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$ describes the linearly polarized state. α is the linearly polarized angle and ranges from 0° to 180° . The time-dependent factor $\exp(-i\omega t)$ is omitted in Eq. (1), and ω is the circular frequency. A suitable Cartesian coordinate system is constructed as follows. The circular aperture plane is selected as the x - y plane, and the center of the circular aperture is the origin. The z -axis is taken to be the propagation axis. Assuming that the radius R is much larger than the wavelength λ , the optical field just behind the aperture reads as

$$\begin{pmatrix} E_x(x_0, y_0, 0) \\ E_y(x_0, y_0, 0) \end{pmatrix} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \text{circ}(\zeta), \quad (2)$$

where $\zeta = \rho_0/R$, $\rho_0 = (x_0^2 + y_0^2)^{1/2}$, and the aperture function is given by

$$\text{circ}(\zeta) = \begin{cases} 1 & 0 \leq \zeta < 1 \\ 0 & \zeta > 1 \end{cases}. \quad (3)$$

According to the vector angular spectrum representation of Maxwell's equations^[13,14], the diffracted plane wave propagating toward the half free space $z \geq 0$ is obtained by

$$\mathbf{E}(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{A}(p, q) \exp[ik(px + qy + mz)] dp dq, \quad (4)$$

where $m = (1 - p^2 - q^2)^{1/2}$, and the vector angular spectrum is expressed as

$$\mathbf{A}(p, q) = A_x(p, q) \left(\mathbf{i} - \frac{p}{m} \mathbf{k} \right) + A_y(p, q) \left(\mathbf{j} - \frac{q}{m} \mathbf{k} \right), \quad (5)$$

where $A_x(p, q)$ and $A_y(p, q)$ are the transverse components of the vector angular spectrum and found to be

the transversality condition of the optical field $\nabla \cdot \mathbf{E}(\mathbf{r}) = 0$, where the dot denotes scalar product. According to the vectorial structure of electromagnetic wave^[6–8], three unit vectors \mathbf{s} , \mathbf{e}_1 , and \mathbf{e}_2 can be defined in the frequency domain as

$$\mathbf{s} = p\mathbf{i} + q\mathbf{j} + m\mathbf{k}, \quad \mathbf{e}_1 = (q\mathbf{i} - p\mathbf{j})/b, \quad \mathbf{e}_2 = m(p\mathbf{i} + q\mathbf{j})/b - b\mathbf{k}. \quad (8)$$

Moreover, the three unit vectors form a mutually perpendicular right-handed system

$$\mathbf{s} \times \mathbf{e}_1 = \mathbf{e}_2, \quad \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{s}, \quad \mathbf{e}_2 \times \mathbf{s} = \mathbf{e}_1. \quad (9)$$

In this system, the vector angular spectrum $\mathbf{A}(p, q)$ can be decomposed into two terms:

$$\mathbf{A}(p, q) = [\mathbf{A}(p, q) \cdot \mathbf{e}_1]\mathbf{e}_1 + [\mathbf{A}(p, q) \cdot \mathbf{e}_2]\mathbf{e}_2. \quad (10)$$

Therefore, the diffracted plane wave can be decomposed into the TE and TM terms:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\text{TE}}(\mathbf{r}) + \mathbf{E}_{\text{TM}}(\mathbf{r}), \quad (11)$$

with

$$\mathbf{E}_{\text{TE}}(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{R(q \cos \alpha - p \sin \alpha) J_1(kRb)}{\lambda b^3} (q\mathbf{i} - p\mathbf{j}) \exp[ik(px + qy + mz)] dp dq, \quad (12)$$

$$\mathbf{E}_{\text{TM}}(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{R(p \cos \alpha + q \sin \alpha) J_1(kRb)}{\lambda b^3 m} (pm\mathbf{i} + qm\mathbf{j} - b^2\mathbf{k}) \exp[ik(px + qy + mz)] dp dq. \quad (13)$$

By taking the curl of Eq. (13), the corresponding magnetic field of the TM term turns out to be

$$\mathbf{H}_{\text{TM}}(\mathbf{r}) = -\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{R(p \cos \alpha + q \sin \alpha) J_1(kRb)}{\lambda b^3 m} (q\mathbf{i} - p\mathbf{j}) \exp[ik(px + qy + mz)] dp dq, \quad (14)$$

where ε_0 and μ_0 are the electric permittivity and magnetic permeability, respectively. Here the TE term denotes that the longitudinal electric field is equal to zero, and the TM term means the longitudinal magnetic field being zero. The electromagnetic field of the diffracted plane wave can be decomposed into the transverse and longitudinal components in the spatial domain, while the decomposition of the electromagnetic field into the TE and TM terms is carried out in terms of the frequency domain.

In the far field, the evanescent plane waves have completely disappeared. Thus, the integrals in Eqs. (12) and (13) should be restrained within the range of $0 \leq b \leq 1$. As the condition $kr \rightarrow \infty$ is satisfied in the far field regime, the analytical TE and TM terms in the far field can be presented. According to the method of stationary phase^[11,12], the surface integral of Eq. (12) is shown to have the asymptotic value^[15]:

$$\mathbf{E}_{\text{TE}}(\mathbf{r}) = \frac{i2\pi}{kr} \sum_j \frac{\varepsilon_j}{(|\tau_j \beta_j - \gamma_j^2|)^{1/2}} \mathbf{M}(p_j, q_j) \exp[ikr f(p_j, q_j, x, y)] \quad \text{as } kr \rightarrow \infty, \quad (15)$$

where

$$\mathbf{M}(p, q) = \frac{R(q \cos \alpha - p \sin \alpha) J_1(kRb)}{\lambda b^3} (q\mathbf{i} - p\mathbf{j}), \quad f(p, q, x, y) = (px + qy + mz)/r. \quad (16)$$

The stationary points (p_j, q_j) are solutions of the following simultaneous equations:

$$\left. \frac{\partial f(p, q, x, y)}{\partial p} \right|_{\substack{p=p_j \\ q=q_j}} = 0, \quad \left. \frac{\partial f(p, q, x, y)}{\partial q} \right|_{\substack{p=p_j \\ q=q_j}} = 0. \quad (17)$$

Thus, we can obtain

$$p_1 = x/r, \quad q_1 = y/r. \quad (18)$$

There is only one stationary point. As a result, the parameters τ_1 , β_1 , γ_1 , and ε_1 turn out to be

$$\begin{aligned} \tau_1 &= \left. \frac{\partial f^2(p, q, x, y)}{\partial p^2} \right|_{\substack{p=p_1 \\ q=q_1}} = -(1 + x^2/z^2), & \beta_1 &= \left. \frac{\partial f^2(p, q, x, y)}{\partial q^2} \right|_{\substack{p=p_1 \\ q=q_1}} = -(1 + y^2/z^2), \\ \gamma_1 &= \left. \frac{\partial f^2(p, q, x, y)}{\partial p \partial q} \right|_{\substack{p=p_1 \\ q=q_1}} = -xy/z^2, & \varepsilon_1 &= -1. \end{aligned} \quad (19)$$

Therefore, the analytical TE term in the far field reads

$$\mathbf{E}_{\text{TE}}(\mathbf{r}) = -\frac{iRz \sin(\varphi - \alpha) J_1(kR\rho/r)}{\rho^2 r} (y\mathbf{i} - x\mathbf{j}) \exp(ikr), \quad (20)$$

where $\rho = (x^2 + y^2)^{1/2}$, and $\varphi = \arctan(y/x)$. Similarly, the analytical TM term in the far field can be obtained as

$$\mathbf{E}_{\text{TM}}(\mathbf{r}) = -\frac{iR \cos(\varphi - \alpha) J_1(kR\rho/r)}{\rho^2 r} (xz\mathbf{i} + yz\mathbf{j} - \rho^2\mathbf{k}) \exp(ikr). \quad (21)$$

Apparently, the TE and TM terms are orthogonal to each other in the far field. The light intensity of the TE and TM terms are found to be

$$I_{\text{TE}}(\mathbf{r}) = \frac{R^2 z^2 \sin^2(\varphi - \alpha) J_1^2(kR\rho/r)}{\rho^2 r^2}, \quad (22)$$

$$I_{\text{TM}}(\mathbf{r}) = \frac{R^2 \cos^2(\varphi - \alpha) J_1^2(kR\rho/r)}{\rho^2}. \quad (23)$$

The beam spot of TE term is located at the orientation perpendicular to the direction of linearly polarized angle. While, the beam spot of TM term is located at the direction of linearly polarized angle. The light intensity of the diffracted plane wave is given by

$$\begin{aligned} I(\mathbf{r}) &= I_{\text{TE}}(\mathbf{r}) + I_{\text{TM}}(\mathbf{r}) \\ &= \frac{R^2}{\rho^2 r^2} J_1^2(kR\rho/r) [z^2 + \rho^2 \cos^2(\varphi - \alpha)] \\ &= \frac{R^2 z^2}{\rho^2 r^2} J_1^2(kR\rho/r). \end{aligned} \quad (24)$$

Equation (24) is just the expression of Airy disc. As the radius is much larger than the wavelength, the optical field is confined to the paraxial region, which results in the last step in the above equation. Accordingly, the Airy disc is composed of two orthogonal terms and independent of the linearly polarized angle.

For the sake of intuition, the light intensity distributions of the diffracted plane wave and its TE and TM terms are depicted in Figs. 1 and 2. The radius of the

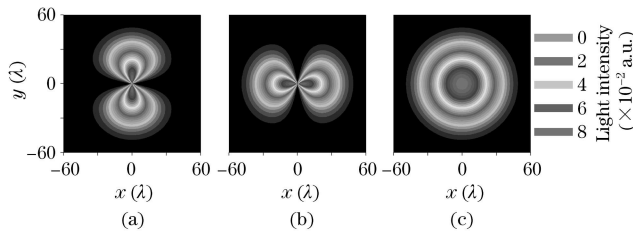


Fig. 1. Light intensity distributions of the diffracted plane wave and its TE and TM terms at the reference plane $z = 1000\lambda$. $R = 10\lambda$ and $\alpha = 0^\circ$. (a) The TE term; (b) the TM term; (c) the diffracted plane wave.

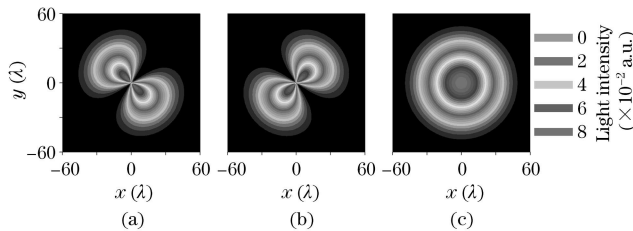


Fig. 2. Light intensity distributions of the diffracted plane wave and its TE and TM terms at the reference plane $z = 1000\lambda$. $R = 10\lambda$ and $\alpha = 45^\circ$. (a) The TE term; (b) the TM term; (c) the diffracted plane wave.

aperture is $R = 10\lambda$. The reference plane is $z = 1000\lambda$. α is set to 0° in Fig. 1 and 45° in Fig. 2. The intensity distribution of TE term is relatively centralized in the direction perpendicular to the linearly polarized angle. While that of TM term is relatively centralized in the direction of the linearly polarized angle. The beam spots of the TE and TM terms are similar to figure of eight. In the figures, the sum of (a) and (b) is just the Airy disc pattern.

In conclusion, the description of a linearly polarized plane wave diffracted by a circular aperture is directly started from the Maxwell's equations. In terms of frequency domain, the diffracted plane wave is decomposed into the TE and TM terms. In the far field, analytical TE and TM terms are presented by stationary phase. The beam spot of TE term is located at the orientation perpendicular to the direction of linearly polarized angle, and that of TM term is located at the direction of linearly polarized angle. The TE and TM terms are orthogonal in the far field. The sum of the TE and TM terms is just the Airy disc. This research reveals the composition of the Airy disc pattern and maybe promotes the recognition of the light propagation through a circular aperture.

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References

1. J. A. Stratton and L. J. Chu, Phys. Rev. **56**, 99 (1939).
2. O. Mitrofanov, M. Lee, J. W. P. Hsu, L. N. Pfeiffer, K. W. West, J. D. Wynn, and J. F. Federici, Appl. Phys. Lett. **79**, 907 (2001).
3. X. Deng, B. Wu, and S. Xiao, Acta Opt. Sin. (in Chinese) **21**, 1432 (2001).
4. G. Zhou, D. Zhao, and S. Wang, Acta Photon. Sin. (in Chinese) **31**, 874 (2002).
5. G. Zhou, D. Zhao, and S. Wang, Acta Opt. Sin. (in Chinese) **23**, 22 (2003).
6. G. D. Gillen and S. Guha, Am. J. Phys. **72**, 1195 (2004).
7. S. Guha and G. D. Gillen, Opt. Express **13**, 1424 (2005).
8. K. Duan and B. Lü, Opt. Laser Technol. **37**, 193 (2005).
9. R. Martínez-Herrero, P. M. Mejías, S. Bosch, and A. Carnicer, J. Opt. Soc. Am. A **18**, 1678 (2001).
10. H. Guo, J. Chen, and S. Zhuang, Opt. Express **14**, 2095 (2006).
11. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).
12. W. H. Carter, J. Opt. Soc. Am. **62**, 1195 (1972).
13. C. G. Chen, P. T. Konkola, J. Ferrera, R. K. Heilmann, and M. L. Schattenburg, J. Opt. Soc. Am. A **19**, 404 (2002).
14. G. Zhou and S. Wang, Chinese J. Lasers (in Chinese) **32**, 67 (2005).
15. M. Born and E. Wolf, *Principles of Optics* (4th edn.) (Pergamon, New York, 1970).