Propagation of Helmholtz-Gauss beams in weak turbulent atmosphere

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Based on the Rytov approximation of light propagation in weak turbulent atmosphere, the closed-form expressions of field and average irradiance of each one of the four fundamental families of Helmholtz-Gauss (HzG) beams: cosine-Gauss beams, stationary Mathieu-Gauss beams, stationary parabolic-Gauss beams, and Bessel-Gauss beams, which are propagating in weak turbulent atmosphere, are obtained. The results show that the field and average irradiance can be written as the product of four factors: complex amplitude depending on the z-coordinate only, a Gaussian beam, a factor of complex phase perturbation induced by atmospheric turbulence, and a complex scaled version of the transverse shape of the non-diffracting beam. The effect of weak atmospheric turbulence on irradiance distribution of the HzG beam can be ignored.

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Propagation of light through atmospheric turbulence has recently attracted renewed attention due to the emergence of high-capacity free-space optical communication systems. Propagation in the atmosphere is significantly influenced by turbulence. Therefore, it is usually desired to find ways to reduce the turbulent effects on the propagating optical beam[1−5]. A great number of studies about using partially coherent light sources as a method for reducing the turbulent effects and improving the system performance have been reported. Non-diffracting beams as new light sources have attracted attention since Durnin first reported the generation of Bessel beams in 1987[6]. Thereafter, several exact non-diffracting solutions of the wave equation have also been reported, for instance Mathieu beams in elliptic coordinates[7,8] and parabolic beams in parabolic coordinates[9]. It was observed that the disturbance of turbulent atmosphere on non-diffracting light beam is less than that of conventional light beams[10]. In this paper, we study the propagation of a non-diffracting Helmholtz-Gauss (HzG) beam in weak turbulent atmosphere.

Let us suppose that a monochromatic wave $E(\vec{r}_\rho, z)$ with time dependence $\exp(-j\omega t)$ propagates in z direction which has a disturbance across the plane $z = 0$ given by

$$E_0(\vec{r}_\rho) = \exp\left(-r_\rho^2/w_0^2\right) W(\vec{r}_\rho, k_\rho), \quad (1)$$

where $\vec{r}_\rho = (x, y) = (\rho, \varphi)$ denotes the transverse coordinates, $W(\vec{r}_\rho, k_\rho)$ is the transverse pattern of an ideal non-diffracting beam $W(\vec{r}_\rho, k_\rho)\exp(jk_z z)$, and w_0 is the waist size of a Gaussian envelope. The transverse (k_ρ) and longitudinal (k_z) components of the wave vector $\vec{k}$ satisfy the relation $\vec{k}^2 = \vec{k}_\rho^2 + \vec{k}_z^2$.

The transverse distribution $W(\vec{r}_\rho, k_\rho)$ of the ideal non-diffracting beam fulfills the two-dimensional (2D) Helmholtz equation and can be expressed as a superposition of plane waves whose transverse wave numbers k_ρ are restricted to a single value, that is[11]

$$W(\vec{r}_\rho, k_\rho) = \int_{-\pi}^{\pi} \tilde{E}(\varphi) \exp\left[jk_\rho (x\cos\varphi + y\sin\varphi)\right] \mathrm{d}\varphi, \quad (2)$$

where $\tilde{E}(\varphi)$ is the angular spectrum of the ideal non-diffracting beam. Because non-diffracting beams can be expanded in terms of plane waves, $E(\vec{r})$ is given by

$$E(\vec{r}) = \frac{w_0}{w(z)} \exp\left[-\frac{r_\rho^2}{w^2(z)} + j\left(kz + \frac{kr_\rho^2}{2R(z)} - \Theta(z)\right)\right] \times \exp\left[\frac{r_0^2}{w^2(z)} - \frac{r_0^2}{w_0^2}\right] \exp\left[-j\frac{kr_0^2}{2R(z)}\right] W\left(\frac{\vec{r}_\rho}{\mu}, k_\rho\right), \quad (3)$$

with $r^2 = r_\rho^2 + z^2$, $w(z) = w_0\left(1 + z^2/z_\mathrm{R}^2\right)^{1/2}$, $\mu^{-1} = [w_0/w(z)]\exp[-j\Theta(z)]$, $R(z) = z + z_\mathrm{R}^2/z$, $\Theta(z) = \arctan(z/z_\mathrm{R})$, $r_0 = k_\rho w_0^2/2$. Here $z_\mathrm{R} = kw_0^2/2$ is the usual Rayleigh range of a Gaussian beam[11]. Equation (**??**) is a solution of the homogeneous Helmholtz equation under the paraxial regime throughout the whole space.

Based on the Rytov approximation[2,3,12], we give the expression for the field $E(\vec{r})$ at any point in the atmospheric turbulence half-space $z > 0$ as

$$E(\vec{r}) = \frac{w_0}{w(z)} \exp\left[-\frac{r_\rho^2}{w^2(z)} + j\left(kz + \frac{kr_\rho^2}{2R(z)} - \Theta(z)\right)\right] \times \exp\left[\frac{r_0^2}{w^2(z)} - \frac{r_0^2}{w_0^2}\right] \exp\left[-j\frac{kr_0^2}{2R(z)}\right] \times W\left(\frac{\vec{r}_\rho}{\mu}, k_\rho\right) \exp\left[\psi_1(\vec{r})\right], \quad (4)$$

where ψ_1 is the complex phase perturbation due to weak turbulence and can be expressed as

$$\psi_1 = \chi + jS, \quad (5)$$

in which χ and S are the log amplitude and the phase, respectively, at a point in the output plane.

The average irradiance can be written as

$$I(\vec{r}) = \left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2r_\rho^2}{w^2(z)}\right] \exp\left[\frac{2r_0^2}{w^2(z)} - \frac{2r_0^2}{w_0^2}\right] \times W^2\left(\frac{\vec{r}_\rho}{\mu}, k_\rho\right) \exp\left[-\frac{1}{2}D_\psi(\vec{r})\right], \quad (6)$$

where $D_\psi(\vec{r}) = D_\psi(\vec{r}_\rho, z) = \langle \psi(\vec{r}) + \psi^*(\vec{r}) \rangle$ is the spherical-wave structural function[13].

For Tatarskii spectrum of refractive-index fluctuations of atmospheric turbulence, i.e. $\phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3}\exp\left(-\kappa^2/\kappa_m^2\right)$, $\kappa_m = 5.92/l_0$, here l_0 is the inner scale and C_n^2 is refractive-index structure constant, the spherical-wave structural function is given by $D_\psi(\vec{r}_\rho, z) = \left[0.545C_n^2k^2z\right]^{5/6} r_\rho^2$.

By Eqs.(4) and (6), we investigate the propagation characteristics of the HzG beams in weak turbulent atmosphere.

One of the simplest non-diffracting beams in Cartesian coordinates is the ideal cosine field

$$W(\vec{r}_\rho, k_\rho) = \cos(k_\rho y) \quad (7)$$

resulting from the superposition of two ideal plane waves $\exp(jk_\rho y)/2$ and $\exp(-jk_\rho y)/2$. The expression for a cosine-Gauss (CG) beam is given by

$$E(\vec{r}) = \frac{w_0}{w(z)} \exp\left[j\left(kz + \frac{k\left(r_\rho^2 - r_0^2\right)}{2R(z)} - \Theta(z)\right)\right] \times \exp\left[\frac{r_0^2 - r_\rho^2}{w^2(z)} - \frac{r_0^2}{w_0^2}\right] \cos\left(\frac{y}{\mu}k_\rho\right) \exp\left[\psi_1(\vec{r})\right]. \quad (8)$$

The average irradiance is represented as

$$I(\vec{r}) = \left[\frac{w_0}{w(z)}\right]^2 \times \exp\left[\frac{2\left(r_0^2 - 0.25\left[0.545C_n^2k^2z\right]^{5/6} r_\rho^2 - r_\rho^2\right)}{w^2(z)} - \frac{2r_0^2}{w_0^2}\right] \times \cos^2\left(\frac{y}{\mu}k_\rho\right). \quad (9)$$

Bessel beams are exact non-diffracting solutions of the scalar wave equation in circular cylindrical coordinates[6]. The transverse field of the mth-order Bessel beam reads as

$$W(\vec{r}_\rho, k_\rho) = J_m(k_\rho y) \exp(jm\varphi), \quad (10)$$

where $J_m(\cdot)$ is the mth-order Bessel function. Applying Eq. (4), we find the expression for the Bessel-Gauss beams to be

$$E(\vec{r}) = \frac{w_0}{w(z)} \exp\left[j\left(kz + \frac{k\left(r_\rho^2 - r_0^2\right)}{2R(z)} - \Theta(z)\right)\right] \times \exp\left[\frac{r_0^2 - r_\rho^2}{w^2(z)} - \frac{r_0^2}{w_0^2}\right] J_m\left(\frac{k_\rho r_\rho}{\mu}\right) \times \exp(jm\varphi) \exp\left[\psi_1(\vec{r})\right], \quad (11)$$

and the average irradiance is written as

$$I(\vec{r}) = \left[\frac{w_0}{w(z)}\right]^2 \times \exp\left[\frac{2\left(r_0^2 - 0.25\left[0.545C_n^2k^2z\right]^{5/6} r_\rho^2 - r_\rho^2\right)}{w^2(z)} - \frac{2r_0^2}{w_0^2}\right] \times J_m^2\left(\frac{r_\rho}{\mu}k_\rho\right). \quad (12)$$

The third family of non-diffracting beams results from the solution of the wave equation in elliptic cylindrical coordinates[7,8]. Since the transverse pattern of such beams is described by the Mathieu functions, they are called Mathieu beams.

The exact analytical expression for the Mathieu-Gauss (MG) beams of any order has not been reported yet. But, the transverse fields of the mth-order even and odd Mathieu beams can be written as

$$W^{\mathrm{e}}(\vec{r}_\rho, k_\rho) = Je_m(\xi, q)\, ce_m(\eta, q),$$
$$W^{\mathrm{o}}(\vec{r}_\rho, k_\rho) = Jo_m(\xi, q)\, se_m(\eta, q), \quad (13)$$

where $Je_m[\cdot]$ and $Jo_m[\cdot]$ are the mth-order even and odd modified Mathieu functions, respectively, and $ce_m[\cdot]$ and $se_m[\cdot]$ are the mth-order even and odd ordinary Mathieu functions, respectively. The parameter $q = f^2k_\rho^2/4$, $2f$ is the interfocal separation.

From Eqs. (4) and (13), the closed-form expressions for the propagation of the mth-order even and odd MG beams are found to be

$$E(\vec{r})^{\mathrm{e}} = \frac{w_0}{w(z)} \exp\left[j\left(kz + \frac{k\left(r_\rho^2 - r_0^2\right)}{2R(z)} - \Theta(z)\right)\right] \times \exp\left[\frac{r_0^2 - r_\rho^2}{w^2(z)} - \frac{r_0^2}{w_0^2}\right] Je_m\left(\bar{\xi}, q\right) ce_m(\bar{\eta}, q) \times \exp\left[\psi_1(\vec{r})\right], \quad (14a)$$

$$E(\vec{r})^{\mathrm{o}} = \frac{w_0}{w(z)} \exp\left[j\left(kz + \frac{k\left(r_\rho^2 - r_0^2\right)}{2R(z)} - \Theta(z)\right)\right] \times \exp\left[\frac{r_0^2 - r_\rho^2}{w^2(z)} - \frac{r_0^2}{w_0^2}\right] Jo_m\left(\bar{\xi}, q\right) so_m(\bar{\eta}, q) \times \exp\left[\psi_1(\vec{r})\right], \quad (14b)$$

where in a transverse z plane the complex elliptic variables $[\bar{\xi}, \bar{\eta}]$ are determined by

$$x = f_0 (1 + z/z_R) \cosh \bar{\xi} \cos \bar{\eta},$$

$$y = f_0 (1 + z/z_R) \sinh \xi \cosh \bar{\eta}, \qquad (15)$$

where f_0 is the semifocal separation at the waist plane $z = 0$. The average irradiances are given by

$$I(\vec{r})^{\mathrm{e}} = \left[\frac{w_0}{w(z)}\right]^2 \times \exp\left[\frac{2\left(r_0^2 - 0.25\left[0.545 C_n^2 k^2 z\right]^{5/6} r_\rho^2 - r_\rho^2\right)}{w^2(z)} - \frac{2r_0^2}{w_0^2}\right] \times \mathrm{Re}\left[Je_m\left(\bar{\xi}, q\right) ce_m\left(\bar{\eta}, q\right)\right]^2, \qquad (16a)$$

$$I(\vec{r})^{\mathrm{o}} = \left[\frac{w_0}{w(z)}\right]^2 \times \exp\left[\frac{2\left(r_0^2 - 0.25\left[0.545 C_n^2 k^2 z\right]^{5/6} r_\rho^2 - r_\rho^2\right)}{w^2(z)} - \frac{2r_0^2}{w_0^2}\right] \times \mathrm{Re}\left[Jo_m\left(\bar{\xi}, q\right) se_m\left(\bar{\eta}, q\right)\right]^2. \qquad (16b)$$

The parabolic beams are the fourth family of fundamental non-diffracting beams[9]. It was found that the transverse structure of the parabolic beams is described by the parabolic functions and, contrary to Bessel or Mathieu beams, their eigenvalue is continuous instead of discrete.

The parabolic cylindrical coordinates (ξ, η) are defined by[9] $x = \left(\eta^2 - \xi^2\right)/2$ and $y = \xi\eta$, in which the variables range in $\xi \in [0, \infty)$ and $\eta \in (-\infty, \infty)$. The transverse fields of the even and odd parabolic beams are written as[9]

$$W^{\mathrm{e}}(\xi, \eta, k_\rho) = \frac{|\Gamma_1|^2}{\pi\sqrt{2}} P_{\mathrm{e}}\left(\sqrt{2k_\rho}\xi; a\right) P_{\mathrm{e}}\left(\sqrt{2k_\rho}\eta; -a\right), \qquad (17a)$$

$$W^{\mathrm{o}}(\xi, \eta, k_\rho) = \frac{|\Gamma_3|^2}{\pi\sqrt{2}} P_{\mathrm{o}}\left(\sqrt{2k_\rho}\xi; a\right) P_{\mathrm{o}}\left(\sqrt{2k_\rho}\eta; -a\right), \qquad (17b)$$

where $\Gamma_1 = \Gamma\left(\frac{1}{4} + \frac{1}{2}ja\right)$, $\Gamma_3 = \Gamma\left(\frac{3}{4} + \frac{1}{2}ja\right)$ and the parameter a represents the order of the beam and can be assumed any real value in the range $(-\infty, \infty)$. The functions $P_{\mathrm{e}}(\cdot)$ and $P_{\mathrm{o}}(\cdot)$ are the even and odd solutions to the parabolic cylindrical differential equation $\left[\mathrm{d}^2/\mathrm{d}x^2 + \left(x^2/4 - a\right)\right] P(x, y) = 0$.

Applying Eq. (4) and also noting that $(x/\mu, y/\eta) \to \left(\xi/\sqrt{\mu}, \eta/\sqrt{\mu}\right)$ for parabolic coordinates, we find the expression for the even and odd parabolic-Gauss (PG) beams to be

$$E(\vec{r}, a)^{\mathrm{e}} = \frac{w_0}{w(z)} \exp\left[j\left(kz + \frac{k\left(r_\rho^2 - r_0^2\right)}{2R(z)} - \Theta(z)\right)\right] \times \exp\left[\frac{r_0^2 - r_\rho^2}{w^2(z)} - \frac{r_0^2}{w_0^2}\right] \frac{|\Gamma_1|^2}{\pi\sqrt{2}} P_{\mathrm{e}}\left(\sqrt{2k_\rho}\xi; a\right) \times P_{\mathrm{e}}\left(\sqrt{2k_\rho}\eta; -a\right) \exp\left[\psi_1(\vec{r})\right],$$

$$E(\vec{r}, a)^{\mathrm{o}} = \frac{w_0}{w(z)} \exp\left[j\left(kz + \frac{k\left(r_\rho^2 - r_0^2\right)}{2R(z)} - \Theta(z)\right)\right] \times \exp\left[\frac{r_0^2 - r_\rho^2}{w^2(z)} - \frac{r_0^2}{w_0^2}\right] \frac{|\Gamma_3|^2}{\pi\sqrt{2}} P_{\mathrm{o}}\left(\sqrt{2k_\rho}\xi; a\right) \times P_{\mathrm{o}}\left(\sqrt{2k_\rho}\eta; -a\right) \exp\left[\psi_1(\vec{r})\right].$$

And we have

$$I(\vec{r})^{\mathrm{e}} = \left[\frac{w_0}{w(z)}\right]^2 \times \exp\left[\frac{2\left(r_0^2 - 0.25\left[0.545 C_n^2 k^2 z\right]^{5/6} r_\rho^2 - r_\rho^2\right)}{w^2(z)} - \frac{2r_0^2}{w_0^2}\right] \times \mathrm{Re}\left[\frac{|\Gamma_1|^2}{\pi\sqrt{2}} P_{\mathrm{e}}\left(\sqrt{2k_\rho}\xi; a\right) P_{\mathrm{e}}\left(\sqrt{2k_\rho}\eta; -a\right)\right]^2, \qquad (18a)$$

$$I(\vec{r})^{\mathrm{o}} = \left[\frac{w_0}{w(z)}\right]^2 \times \exp\left[\frac{2\left(r_0^2 - 0.25\left[0.545 C_n^2 k^2 z\right]^{5/6} r_\rho^2 - r_\rho^2\right)}{w^2(z)} - \frac{2r_0^2}{w_0^2}\right] \times \mathrm{Re}\left[\frac{|\Gamma_3|^2}{\pi\sqrt{2}} P_{\mathrm{o}}\left(\sqrt{2k_\rho}\xi; a\right) P_{\mathrm{o}}\left(\sqrt{2k_\rho}\eta; -a\right)\right]^2. \qquad (18b)$$

We will now discuss the effect of atmospheric turbulence on irradiance distribution of the HzG beam. Because $W(\vec{r}_\rho, k_\rho) \exp(jk_z z)$ is an ideal non-diffracting beam, we can use the factor $|W(\vec{r}_\rho, k_\rho)|^2$ to normalize the irradiance distribution of the HzG beam, namely

$$\tilde{I}\left(\vec{r}, C_n^2\right) = \frac{I\left(\vec{r}, C_n^2\right)}{|W|^2} = \left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2r_\rho^2}{w^2(z)}\right] \exp\left[\frac{2r_0^2}{w^2(z)} - \frac{2r_0^2}{w_0^2}\right] \times \exp\left[-\frac{1}{2}\left[0.545 C_n^2 k^2 z\right]^{5/6} r_\rho^2\right]. \qquad (19)$$

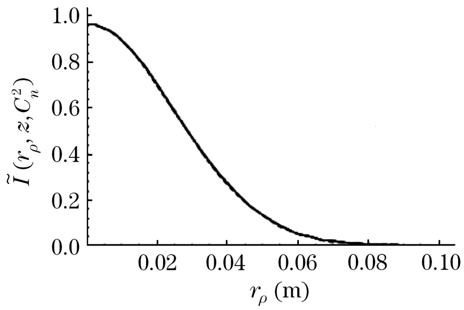

Fig. 1. Irradiance distribution of HzG beams in atmospheric turbulence. The curves for $C_n^2 = 0$, 1×10^{-17}, 5×10^{-17}, 1×10^{-16}, 5×10^{-16} $\text{m}^{-2/3}$ are approximately overlapping.

The normalized irradiance distribution $\tilde{I}_n(\vec{r}) = \tilde{I}_n\left(r_\rho, z, C_n^2\right)$ (here we have set the normalized irradiance distribution $\tilde{I}_n(\vec{r})$ to follow the relationship $\tilde{I}_n(0, z, 0) = 1$) as a function of r_ρ is depicted in Fig. 1 for $C_n^2 = 0$, 1×10^{-17}, 5×10^{-17}, 1×10^{-16}, 5×10^{-16} $\text{m}^{-2/3}$, and $z = 1000$ m, $\lambda = 0.6328$ μm, $w_0 = 0.05$ m, $k_\rho = 10^5$ m^{-1}. Figure 1 shows that the curves of irradiance distribution are approximately overlapping. It is shown that the effect of weak atmospheric turbulence on irradiance distribution of the HzG beam can be ignored.

In conclusion, a detailed analysis of the propagation in weak atmospheric turbulence of an arbitrary non-diffracting beam whose disturbance in the plane $z = 0$ is modulated by a Gaussian envelope has been presented. We have found that the HzG beams which are propagating through the whole space can be described by a simple and elegant closed-form expression composed of an amplitude factor depending on the z-coordinate, a Gaussian beam, a turbulent fluctuation factor and a scaled version of the transverse shape of the ideal non-diffracting beam. Our analysis revealed the conservation of non-diffracting behavior of the HzG beams propagation in the weak atmospheric turbulence region.

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