# Theoretical analysis and simulation of conjugate heights for dual-conjugate AO system in lidar

**Xueke Ding (丁学科), Jian Rong (荣 健), Hong Bai (白 宏), Xiu Wang (王 秀), Jin'e Shen (申金娥), and Fang Li (李 芳)**

*School of Physical Electronic, University of Electronic Science and Technology of China, Chengdu 610054*

Received May 24, 2007

A multi-conjugate adaptive optics (MCAO) can offer a possibility of widening field of view (FOV) characterized by the isoplanatic angle, and the choose of conjugate height becomes a basic problem for MCAO, which influences the size of iosplanatic angle. Considering the application of lidar, the isoplanatic angle's expressions of two deformable mirrors (DMs) MCAO for uplink and downlink are deduced. The effects of conjugate heights for dual-conjugate AO are thoughtfully discussed, and the isoplanatic angles are further analyzed. The results show that the isopanatic angle varies with the conjugate height and reaches the maximum as the conjugate height is at the optimal altitude. Moreover, the optimal conjugate height changes with the propagation distance.

*OCIS codes:* 010.1080, 010.1330, 010.3640.

An adaptive optics (AO) performs a real-time correction of the atmospheric turbulence effects on image formation. Thus, ground-based telescopes equipped with AO offer nowadays high resolution images of fainter and fainter objects. This success has brought, however, new challenges. Mostly because it is difficult to find a suitable bright reference source for wave-front measurements within the isoplanatic patch, which will lead a small field of view (FOV). In 1987, Beckers *et al.* proposed that the atmospheric turbulence was compensated with several deformable mirrors (DMs) conjugated to different heights, called multi-conjugate adaptive optics (MCAO) systems, offering a possibility of widening FOV. It extends application of AO systems in attain, point and track (ATP), satellite communications, and astronomical exploration[1−4]. Despite of the existence of theoretical calculations of MCAO systems performance[5], the issue of the FOV gain is still not clear. Owing to the inherent complexity of the problem, the off-axis Strehl ratio depends on many system parameters. A case study of a particular MCAO performance was found[6]. A more general attempt to calculate the MCAO FOV gain by segmenting the atmospheric refractive-index structure's constant profile $C_n^2$ into several slabs, placing DM at the center of each slab, and summing up the remaining anisoplanatic effects was made by Yan *et al.*[7,8]. However it is sometimes assumed that turbulence is concentrated on a few thin layers that exactly conjugate the DM planes. This assumption leads to the unlimited FOV. Hence calculating the realistic FOV gain still remains unresolved.

The FOV of classical AO systems (single-DM) is usually characterized by the isoplanatic angle $\theta_0$ defined by Fried[9]. So we propose a generalized parameter $\theta_m$ for an $M$-DM MCAO system that shares the desirable properties of $\theta_0$, depending on the $C_n^2$ profile and DM conjugate height. Choosing the conjugate height is a basic problem for MCAO, which influences the size of isoplanatic angle. In view of the application of lidar, the isoplanatic angle expressions of dual-conjugate AO (two-DM MCAO) for uplink and downlink are deduced. Through simulation, the choose of conjugate height is thoughtfully discussed and the isoplanatic angle is further analyzed.

In an AO system, the correct effect is evaluated by the residual phase variance, and some parameters correlated with the influence on wave-front by turbulence are based on this, such as coherence length, isoplanatic angle, and scintillation index.

It is noted that every turbulent layer is corrected by all DMs. And the residual wave-front phase after compensation can be written as[10]

$$\varepsilon\left(r,\theta\right)=\phi\left(r-\theta h\right)-\sum_{m=1}^{M}\varphi_m\left(r-\theta H_m\right), \tag{1}$$

which defines our basic model for a MCAO system. $\phi\left(r-\theta h\right)$ is the perturbation phase caused by turbulence, which is assumed to be known. The problem is thus to find a set of corrections $\varphi_m\left(r-\theta H_m\right)$ which can give the best-compensated image quality. We adopt a minimum-variance approach of the control algorithms, which leads to the following expression for the aperture-averaged residual phase variance produced by a single turbulent layer:

$$\left\langle\varepsilon_p^2\right\rangle=2\pi\int_0^\infty fW_\phi\left(f\right)\left|G\left(f\right)\right|^2p\left(f\right)\mathrm{d}f, \tag{2}$$

where $f$ is the spatial frequency ($f$ = 1/period, not a wave number). And $W_\phi\left(f\right)$ is the power spectrum of phase disturbance, given as

$$W_\phi\left(f\right)=0.38\lambda^{-2}\left(f^2+L_0^{-2}\right)^{-11/6}C_n^2\left(h\right)\mathrm{d}h, \tag{3}$$

where $p\left(f\right)$ is the high-pass piston filter for a circular aperture of radius $R$, expressed as

$$p\left(f\right)=1-\left[\frac{J_1\left(2\pi Rf\right)}{\pi Rf}\right]^2, \tag{4}$$

where $\left|G\left(f\right)\right|^2$ is the spatial filter, depending on $\theta$, conjugate height of DM ($H_m$), layer altitude $h$, DM response $r\left(f\right)$, and the spatial filters $g_m\left(f\right)$, defined as

$$\left|G\left(f\right)\right|^2=1-2\mathbf{b}^{\mathrm{T}}g+g^{\mathrm{T}}\mathbf{A}g, \tag{5}$$

 

where the vector **b** and the matrix **A** are

$$\mathbf{b}_m = r(f)\,J_0(2\pi f x_m),$$

$$\mathbf{A}_{mm'} = r^2(f)\,J_0[2\pi f(x_m - x_{m'})],$$

$$x_m = \theta(H_m - h).$$

The combined effect of all atmospheric layers is finally found as an integral over altitude:

$$\langle\varepsilon_{\mathrm{p}}^2\rangle = \int_0^{h_{\max}} C_n^2(h)F'(h)\,\mathrm{d}h, \quad (6)$$

where

$$F'(h) = 2.40\lambda^{-2}\int_0^{\infty} f\left(f^2 + L_0^{-2}\right)^{-11/6} p(f)\,|G(f)|^2\,\mathrm{d}f. \quad (7)$$

If we use a control algorithm without spatial filtering and assume in addition that DMs have infinite resolution [$r(f) = 1$], that the telescope has an infinite aperture diameter, and that the turbulence outer scale is infinite, then the piston removal becomes irrelevant [$p(f) = 1$], and the residual phase variance can be written as

$$\langle\varepsilon_{\mathrm{p}}^2\rangle = 2.905\times 4\pi^2\lambda^2\,|\theta|^{5/3}\,C_n^2(h)\,F_m(h)\,\mathrm{d}h, \quad (8)$$

where $F_m(h)$ is the weighting function depending on conjugate height ($H_m$) for each DM, given by

$$F_m(h) = \hat{\mathbf{b}}^{\mathrm{T}}\mathbf{c} - 0.5\mathbf{c}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{c}, \quad (9)$$

where the vector **c** with elements $c_m$ is the control algorithm. The vector $\hat{\mathbf{b}}$ and the matrix $\hat{\mathbf{A}}$ are

$$\hat{\mathbf{b}}_m = |h - H_m|^{5/3}, \quad \hat{\mathbf{A}}_{mm'} = |H_m - H_{m'}|^{5/3}.$$

For one DM, **c** is 1. And for tow DMs, **c** is given by

$$\mathbf{c} = 0.5\,|H_2 - H_1|^{-5/3} \times \left(|h - H_2|^{5/3} - |h - H_1|^{5/3} + |H_2 - H_1|^{5/3}\right). \quad (10)$$

Now the residual phase variance summed over the whole atmosphere takes the form

$$\langle\varepsilon^2\rangle = (|\theta|/\theta_m)^{5/3}, \quad (11)$$

which is similar to Fried's expression[9] for a classical AO. The generalized isoplanatic angle $\theta_m$ is an angular radius of a field where the residual phase variance reaches 1 rad, calculated as

$$\theta_m^{-5/3} = 2.905k^2(\sec\beta)^{8/3}\times\int_0^Z C_n^2(h)\,F_m(h)\,\mathrm{d}h, \quad (12)$$

where $\beta$ is the source zenith angle, $Z$ is the maximum altitude for propagation.

For a ground lidar system which is a downlink path, the downlink laser beam propagation can be accurately modeled by a plane wave, because the atmospheric turbulence is near the observing system and far from the observed object[11]. Then the isoplanatic angle $\theta_m$ is given by

$$\theta_m = \left[2.905k^2(\sec\beta)^{8/3}\times\int_0^Z C_n^2(h)\,F_m(h)\,\mathrm{d}h\right]^{-3/5}. \quad (13)$$

For an air-borne lidar system which is an uplink path, contrarily, the uplink beam wave model can be modeled by a spherical wave[12]. Thus the isoplanatic angle $\theta_m$ can be represented as

$$\theta_m = \left[2.905k^2(\sec\beta)^{8/3} \times\int_0^Z C_n^2(h)\,(h/Z)^{5/3}F_m(h)\,\mathrm{d}h\right]^{-3/5}. \quad (14)$$

The conjugate height function $F_2(h)$ for a tow-DM MCAO can be computed from Eqs. (9) and (10) as

$$F_2(h) = 0.5\,|h - H_1|^{5/3} + 0.5\,|h - H_2|^{5/3} - 0.25\,|H_2 - H_1|^{5/3} - 0.25\,|H_2 - H_1|^{-5/3}\left(|h - H_1|^{5/3} - |h - H_2|^{5/3}\right)^2. \quad (15)$$

The choose of conjugate height ($H_m$), namely the set of turbulent layer to be corrected for each DM, is very important for the MCAO system, which influences the gain of isoplanatic angle that characterizes the isoplanatic FOV size. So it is necessary to discuss the conjugate height. We refer to 10.6-$\mu$m infrared as incident wave, and cite the well-known Hufnagel turbulence profile used by Valley and Wandzura (HV profile)[13]

$$C_n^2(h) = 5.94\times 10^{-53}(v/27)^2\,h^{10}e^{-h/1000} + 2.7\times 10^{-16}e^{-h/1500} + 1.7\times 10^{-14}e^{-h/100}, \quad (16)$$

where $h$ is in meters, and $v$ is room mean square (rms) wind speed in meters per second varying with time, altitude, and location ($v$ = 21 m/s). Usually an air-borne system flies within the stratosphere, which would not be beyond an altitude of 20 km. Though a space-borne system, such as the satellite remote sensing, operates over 400 km above sea level, the atmospheric turbulence is very weak beyond an altitude of 20 km, which hardly effects on laser propagation. It is assumed that the total height of turbulent layer is 20 km.

A dual-conjugate AO is a simple MCAO system with two DMs, which corresponds to two-layer conjugate heights that must be considered respectively. When one of the two-layer conjugate heights is fixed, what will the relationship between the isoplanatic angles and the other layer be? As the first conjugate height is at an altitude of 1 km near the ground ($H_1$ = 1 km), the isoplanatic angle versus the second conjugate height ($H_2$) is shown in Fig. 1. And when the second layer is fixed at an altitude

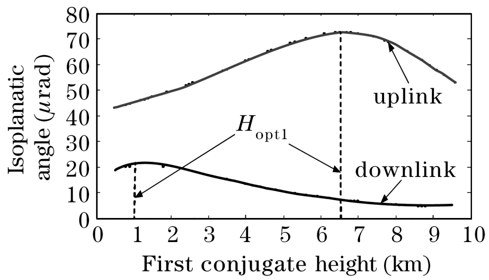


Fig. 1. Isoplanatic angle versus conjugate height for the first layer when $H_2 = 19$ km.

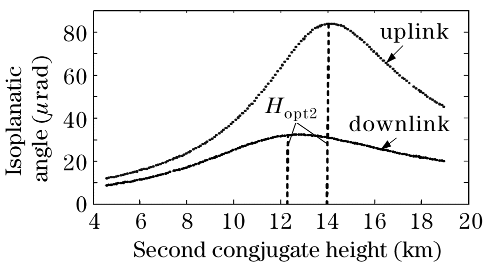


Fig. 2. Isoplanatic angle versus conjugate height for the second layer when $H_1 = 1$ km.

of 19 km ($H_2 = 19$ km) near the end of turbulent layer, the isoplanatic angle versus the first conjugate height is shown in Fig. 2. These figuses show the isoplanatic angle, which has a maximum value as the conjugate height is placed at the optimal position called optimal conjugate height ($H_{\rm opt}$), varies with the conjugate height. The first optimal conjugate altitudes ($H_{\rm opt1}$) for uplink and downlink are 6.5 and 1 km, and the second optimal conjugate heights ($H_{\rm opt2}$) are 14 and 12.5 km, respectively. As a consequence, when the laser propagation distance is 20 km, for an air-borne lidar, the two-layer optimal altitudes (the first and second layers) are 6.5 and 14 km, and for a ground lidar, the two-layer optimal altitudes are 1 and 12.5 km, respectively.

Then, the most important thing is to seek the positions of optimal conjugate height. The further study of the optimal conjugate height are given in Figs. 3 and 4, which show the optimal conjugate height (for the first and second layers) increases with the propagation distance from 10 to 20 km. The first optimal altitude in downlink changes slightly toward the propagation distance, whose scale is between 0.78 and 1.35 km. However, the first

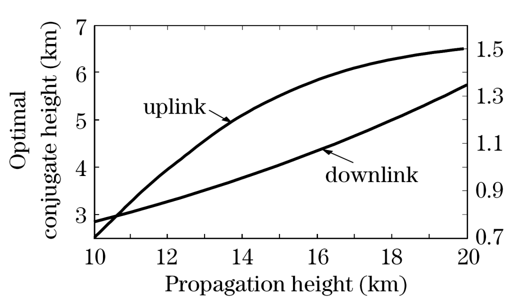


Fig. 3. First optimal conjugate height as propagation distance when the second conjugate height is 1 km from the end of propagation height ($H_2 = x - 1$ km). Coordinates: left $y$-axis is for the uplink system, and right $y$-axis is for the downlink system.

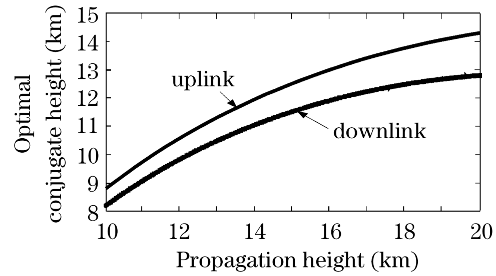


Fig. 4. Contrary to Fig. 3, the second optimal conjugate as propagation distance when the first conjugate height is 1 km above the ground ($H_1 = 1$ km).

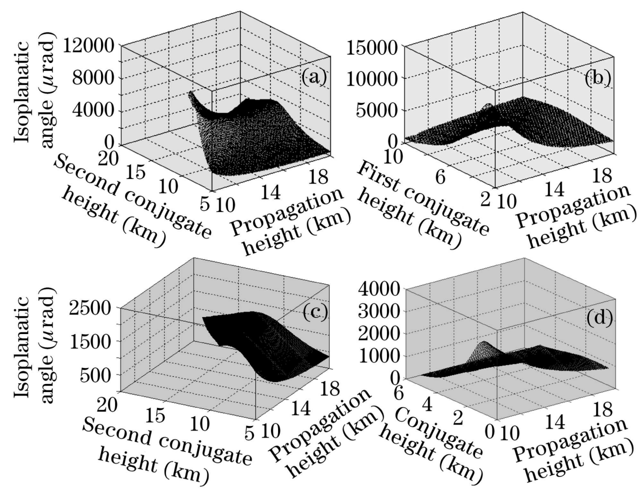


Fig. 5. Isoplanatic angle versus conjugate height (**x**-axis) and propagation distance from 10 to 20 km. The difference of the four figures ((a)—(d)) is what $x$-axis represents: (a) $H_2$ for uplink when $H_1 = 1$ km; (b) $H_1$ for uplink when $H_2 = y - 1$ km; (c) $H_2$ for downlink when $H_1 = 1$ km; (d) $H_1$ for downlink when $H_2 = y - 1$ km.

optimal altitude in uplink changes greatly with increasing distance changing from 2.5 to 6.5 km. That is because the strong turbulent layer near ground is near the pupil plane of the downlink system and far from the pupil plane of the uplink system. For the second optimal conjugate heights, there are minor differences between the downlink and the uplink, and the scale changes with the propagation distance from 8 to 14 km. That seems probably due to the fact that the conjugate height for the second DM, far from the ground, is within weak turbulent region.

To further study, how the three parameters (isoplanatic angle, conjugate height, and propagation distance) are related to each other is analyzed by three-dimensional (3D) simulation, shown in Fig. 5. From Figs. 5(a)—(d), the conclusions are drawn as follows.

- Under equal conditions, the isoplanatic angle for the uplink system is larger than that for the downlink system, because the atmospheric turbulence is far from the uplink observing system;
- When the two-layer conjugate height is fixed, the isoplanatic angle, on the whole, decreases with propagation distance, with a little fluctuation, which implies that a dual-conjugate AO system should have an optimal detecting height;
- When the propagation distance is changeless, the isoplanatic angle varies with the conjugate height. And the

value reaches maximum as the conjugate height is at the optimal altitude. Moreover, the optimal conjugate height changes with the propagation distance, which is in accordance with the above-mentioned results;

• The isoplanatic angle changes with the first conjugate height more greatly than the second one, which indicates that the first conjugate height can play a great role on widening the isoplanatic angle. That is because the first conjugate height is within the strong turbulent region, which plays the main role in correction.

This work was supported by the National Natural Science Foundation of China under Grant No. 60572079. X. Ding's e-mail address is jx_dxk@126.com.

## References


1. J. Farinato, R. Ragazzoni, and E. Diolaiti, Proc. SPIE **5490,** 1229 (2004).
2. J. Kolb, Proc. SPIE **6272,** 62723J1 (2006).
3. E. Meyer, W. Gaessler, S. A. Kellner, E. Diolaiti, S. Egner, R. Ragazzoni, and J. Farinato, Proc. SPIE **6272,** 62723Q1 (2006).
4. N. Hubin, M. L. Louarn, M. Sarazin, A. Tokovinin, and E. Viard, Proc. SPIE **4007,** 510 (2006).
5. D. C. Johnston and B. M. Welsh, Opt. Soc. Am. A **11,** 394 (1994).
6. T. Fusco, J.-M. Conan, V. Michau, L. M. Mugnier, and G. Rousset, Proc. SPIE **3763,** 125 (1999).
7. J. Yan, R. Zhou, and X. Yu, Opt. Eng. **32,** 2161 (1993).
8. J. Yan, R. Zhou, and X. Yu, Opt Eng. **33,** 2942 (1994).
9. D. L. Fried, Opt. Soc. Am. **72,** 52 (1982).
10. J. Stone, P. H. Hu, S. P. Mils, and S. Ma, Opt. Soc. Am. A **11,** 347 (1994).
11. V. P. Lukin and B. V. Fortes, Proc. SPIE **3494,** 191 (1998).
12. D. L Fried, Opt. Soc. Am. **56,** 1382 (1996).
13. G. C. Valley and S. M. Wandzura, Opt. Soc. Am. **69,** 712 (1979).