Aharonov-Bohm effect in spherical billiard

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Using Gutzwiller's periodic orbit theory, we study the quantum level density of a spherical billiard in the presence of a magnetic flux line added at its center, especially discuss the influence of the magnetic flux strength on the quantum level density. The Fourier transformed quantum level density of this system has allowed direct comparison between peaks in the level density and the length of the periodic orbits. For particular magnetic flux strength, the amplitude of the peaks in the level density decreased and some of the peaks disappeared. This result suggests that Aharonov-Bohm effect manifests itself through the cancellation of periodic orbits. This phenomenon will provide a new experimental testing ground for exploring Aharonov-Bohm effect.

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The connection between the quantum effect of a particle and its classical motion in a given potential has attracted much interest. Advances in lithographic techniques and crystal growth have made it possible to produce very small and clean devices, such as nanodevices[1]. The electron in such devices is confined to two or more spatial dimensions through gate voltage, which can be considered as a quantum billiard. Quantum billiard have attracted many interest in many fields[2−5]. Many theoretical methods have been developed to study the dynamical behaviour of a quantum billiard. Among them, periodic orbit theory (POT)[6] has rapidly become one of the most useful and intuitive semiclassical methods as it can be employed to make very direct connections between the energy spectrum of a quantum system and the periodic orbit of the corresponding classical system. In the previous studies, many researchers have focused their attention on the two-dimensional (2D) billiard system, such as the square billiard, 2D triangular billiard, circular or annular billiard etc.[7−12]. As for the three-dimensional (3D) billiard, the research is very little. In Ref. [13], we studied the correspondence between the quantum spectra of the cubic billiard and the length of the classical orbits of this system. But as to the influence of the magnetic flux on the quantum spectra of the 3D system, none has given the discussion.

In this paper, we investigate a 3D spherical billiard with a singular magnetic flux line added at its center. In this system, the Aharonov-Bohm effect should be taken into account[14]. As we all know, the Aharonov-Bohm flux line has had a rather spectacular effect on physics. It provides a compelling example of quantum nonlocality. Largely because of this example, it is now clearly understood that charged quantum particles have phase interference effects originating from a magnetic field that vanishes in all regions accessible to the particle. Although this purely quantum effect has no directly corresponding classical physics, it is related to the scattering of a classical charge neutral wave from a vortex. The results show that the effect of the magnetic flux can be described by the addition of the Aharonov-Bohm phase to the classical action, which leads to a drastic modifications to the quantum level density.

First, we consider a billiard moving in a 3D infinite well potential with the radius R (without the magnetic flux). The potential is described by

$$V(r,\theta,\phi)=\begin{cases}0, & 0<r\le R, 0\le\theta\le\pi, \text{ and } 0\le\phi\le\pi\\ \infty, & \text{otherwise}\end{cases}. \tag{1}$$

The solution of the corresponding 3D Schrödinger equation is given by[15]

$$\psi(r,\theta,\varphi)=C_{kl}j_l(kr)Y_{lm}(\theta,\varphi), \tag{2}$$

where C_{kl} is the normalized constant, $j_l(kr)$ are the spherical Bessel functions of order l. They are related to the cylindrical Bessel functions of half-integral order by $j_l(kr)=\sqrt{\pi/2kr}J_{l+1/2}(kr)$. The wave number k is related to the energy by $k=\sqrt{2\mu E/\eta^2}$ and the energy eigenvalues are quantized under the boundary conditions at the infinite wall at $r=R$, namely $j_l(kR)=0$. The quantized energies are given by

$$E_{(n,l)}=\frac{\eta^2k^2_{(n,l)}}{2\mu}=\frac{\eta^2a^2_{(n,l)}}{2\mu R^2}, \tag{3}$$

where $a_{(n,l)}$ denotes the zeros of the Bessel function of order l and n counts the number of the radial nodes. Because of the spherical symmetry in this billiard system, each such state has a degeneracy given by $d_{(n,l)}=2l+1$.

In the periodic orbit theory, one of the most important physical quantities is the density of energy states. As in Ref. [6], the energy state density can be split into a smoothly, slowly varying part ($\rho_0(E)$) and some oscillatory terms, which are dominated by the classical periodic orbits whose actions, $S_\gamma(E)$, correspond to periodic orbit or closed paths

$$\sum_{n=1}^{\infty}\delta(E-E_n)\equiv\rho(E)$$

$$=\rho_0(E)+\sum_{p=1}^{\infty}\sum_{\gamma}\rho_{\gamma,p}\cos\left[p\left(\frac{S_\gamma(E)}{\eta}-\phi_\gamma\right)\right], \tag{4}$$

where each periodic orbit is characterized by a label $\gamma = 1, \cdots, \infty$ and p denotes all possible repetitions of such trajectories ($p = 1, \cdots, \infty$), ϕ_γ is the phase modification in the path integral.

The classical periodic orbits in the spherical well have been discussed at length[16]. They consist of planar trajectories characterized by two integers, p and q ($p \geq 2q$), where p describes the number of hits on the walls during one period, and q is the number of revolutions, the orbit encircles the center during the fundamental period. This condition can be written as $p\Phi = q \cdot 2\pi$, where Φ is the angle subtended by the chord length between successive bounces with the spherical containing walls. The length of a primitive orbit is given by

$$L(p,q) = p[2R\sin(\frac{\Phi}{2})] = p[2R\sin(\frac{\pi q}{p})]. \tag{5}$$

If p and q have common factors, such a (p, q) pair can be thought of as representing multiple repetitions of the primitive trajectories. Because of the momentum conservation, the motion of the particle is determined by its direction and position. All the periodic orbits for the spherical billiard are regular polygons except the diameter orbit (see Fig. 1). Some of the path lengths and their repetitions with $L(p, q) \leq 14.0R$ are given in Table 1.

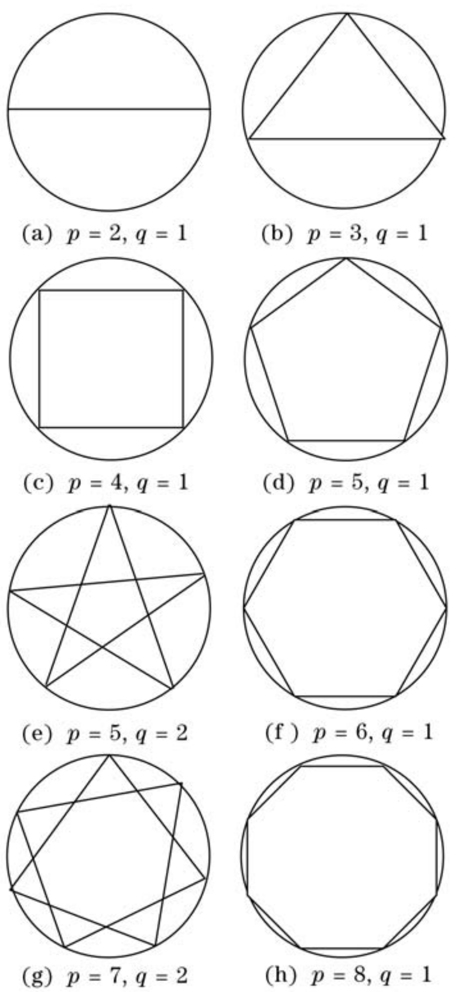

Fig. 1. Some periodic orbits in the spherical billiard. (a) Diameter orbit; (b)—(h) regular polygans orbits.

Table 1. Values of the Path Length $L/R \leq 14.0$ for Periodic Orbits in the Spherical Billiard

(p, q)	$L(p, q)$ and Recurrence Below $L/R \leq 14.0$	(p, q)	$L(p, q)$ and Recurrence Below $L/R \leq 14.0$
$(2, 1)$	$4.00, 8.00, 12.00$	$(5, 2)$	9.510
$(3, 1)$	$5.196, 10.392$	$(7, 2)$	10.945
$(4, 1)$	$5.657, 11.314$	$(9, 2)$	11.570
$(5, 1)$	$5.878, 11.756$	$(11, 2)$	11.894
$(6, 1)$	$6.00, 12.00$	$(13, 2)$	12.082
$(7, 1)$	$6.074, 12.148$	$(15, 2)$	12.202
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\infty, 1)$	$6.283, 12.566$	$(\infty, 2)$	12.566
		$(7, 3)$	13.649

What interested us most is the oscillatory term of Eq. (4). In k-space, we write the equivalent density of states as

$$\sum_{n=1}^{\infty} \delta(k - k_n) \equiv \rho(k)$$

$$= \rho_0(k) + \sum_{p=1}^{\infty}\sum_{\gamma} \rho_{\gamma,p} \cos[p(kL_\gamma - \phi_\gamma)]. \tag{6}$$

By taking the Fourier transform of the above equation, we get the Fourier transformed quantum energy level density:

$$\rho(L) \equiv \sum_{nl} \frac{d_{(n,l)}}{k_{(n,l)}^{3/2}} \mathrm{e}^{ik_{(n,l)}L}. \tag{7}$$

The weighting factor in the denominator is added to increase the sensitivity to lengthen classical paths; the degeneracy factor $d_{(n,l)} = 2l + 1$ simply counts the number of levels for each value of $k_{(n,l)}$. If we evaluate the Fourier transform $\rho(L)$ of the wave number spectrum by Eq. (7), we should find a series of sharp peaks corresponding to the lengths of classical periodic orbits.

Next, we consider the spherical Aharonov-Bohm billiard. This system consists of a charged point particle moving in a spherical enclosure in three dimensions, with a magnetic flux line added at its center and perpendicular to the x-y plane. This is equivalent to a charged particle enclosed by a spherical boundary interacting with magnetic field of an infinitely thin and long solenoid enclosing a finite flux ϕ. For the 3D billiard, the Lagrangian is given by

$$L = \frac{1}{2}m\vec{v}^{\,2} + \frac{e}{c}\vec{v} \cdot \vec{A}, \tag{8}$$

where $\vec{A}$ is the vector potential. In cylindrical coordinates, assuming the magnetic flux line lies along the z-axis, it can be written as $\vec{A} = \frac{\phi}{2\pi\rho}\vec{e}_\theta$. Here ϕ is the magnetic flux through the solenoid. This corresponds to a magnetic field B which has a δ function singularity at the origin and is zero everywhere else. Substituting

$\vec{A}$ into Eq. (8), the Lagrangian in spherical coordinates (r, θ, φ) is simplified to

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2) + \frac{e\phi}{2\pi c}\dot{\theta}. \qquad (9)$$

As a consequence of the well-known Aharonov-Bohm effect[14], the vector potential A adds to the quantum phase along a path from r_1 to r_2 by an amount $\Delta\alpha = \frac{e}{\eta}\int_{r_1}^{r_2}\vec{A}(r)\cdot \mathrm{d}\vec{r}$ and the wave function of the particle acquires a flux-dependent phase change upon rotation around the solenoid. For a nonzero flux ϕ, the particle is classically not allowed to penetrate the flux line, because its energy would become infinite. For $r \neq 0$, the Lorentz force on the particle is always zero and the classical equations of motion remain unchanged. The geometries of the classical orbit with $p > 2q$ therefore do not change, they are the same as ones given in Fig. 1 except the diameter orbit. However, the quantum spectrum does depend on the flux ϕ.

For integer values of the canonical angular momentum Λ, the energy eigenvalues and wave functions are determined by solving the corresponding radial part of the Schrödinger equation, with the dimensionless flux strength $\alpha = e\phi/(2\pi\eta c)$,

$$\frac{\eta^2}{2m}\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}(\Lambda-\alpha)^2\right)\psi_i = \varepsilon_i\psi_i. \qquad (10)$$

The boundary condition is $\psi_i(R) = 0$. The quantity $(\Lambda - \alpha)^2$ can take fractional values, if we assume that α can take a continuous range of values between 0 and 1. The boundary condition together with the condition of normalizability of the quantal wave functions finally yields the energy eigenvalue as $\varepsilon_i = \varepsilon_{|\Lambda-\alpha|,n} = E_0 X^2_{|\Lambda-\alpha|,n}$, in which $E_0 = \eta^2/(2mR^2)$, $j_{|\Lambda-\alpha|}(X_{|\Lambda-\alpha|,n}) = 0$.

From the above formula, we find that the presence of the flux line in the spherical billiard simply changes the order of the Bessel functions from integer to fraction, the symmetry $\alpha \leftrightarrow 1-\alpha$ in the quantum spectrum allows the restriction to $0 \leq \alpha \leq 0.5$. For integer flux $\alpha = 0, 1, 2, \cdots$, the quantum spectrum is unaltered by the flux line. Therefore, the formula for the Fourier transform of the quantum energy level density (Eq. (7)) is still valid except that the energy E is replaced by ε.

Using Eq. (7), we calculate the Fourier transformed quantum level density $|\rho(L)|^2$ (using the lowest eigenvalue $N = 1000$). Figure 2 plots the results of $|\rho(L)|^2$ versus $L(p,q) \leq 14.0R$ with different magnetic flux strengths α. Figure 2(a) is the quantum energy level density without the magnetic flux line, $\alpha = 0$. This case is equivalent to the simple spherical billiard[16]. There are many peaks in the plot and each peak corresponds to the length of one periodic orbit given in Table 1. With the increase of the flux strength α, the heights of these peaks are reduced. At $\alpha = 0.25$, all the periodic orbits with $L > 4R$ are disappeared. For $\alpha > 0.25$, they revive again. Their disappearance at $\alpha = 0.25$ can be seen as a consequence of the Aharonov-Bohm effect. We also find two prominent characteristics. First, the peak at $L = 4R$ exists for all $\alpha > 0$, except it decreases at $\alpha = 0.25$. Second, there appears a new signal at $L = 2R$,

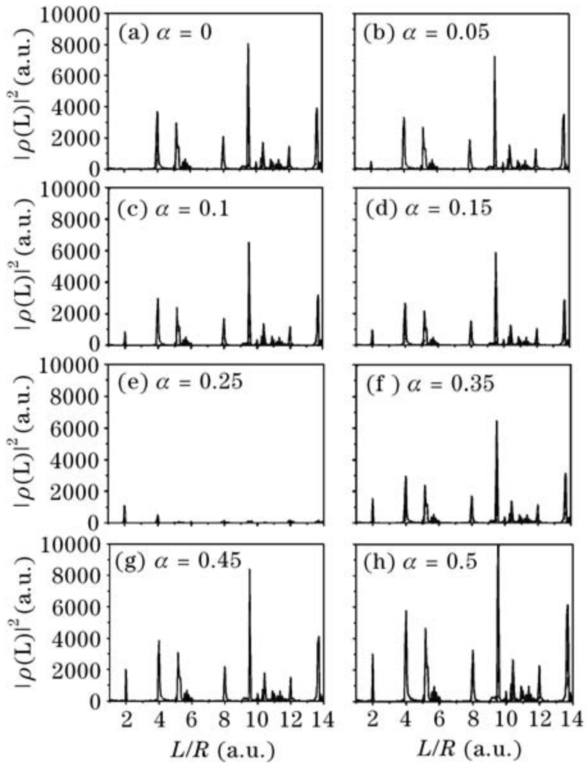

Fig. 2. Fourier transformed quantum level density $|\rho(L)|^2$ versus $L(p,q) \leq 14.0R$ with different magnetic flux strengths α (using the lowest eigenvalue $N = 1000$).

corresponding to a reflected half –diameter orbit, with an amplitude increasing till up to $\alpha = 0.5$. The reasons of these phenomena can be interpreted as follows: when a wave hits the flux line, it is diffracted; part of it is reflected and part is transmitted. Therefore, the two signals at $L = 2R$ and $L = 4R$ correspond to the reflected and transmitted waves, respectively. The ratio of the reflected to transmitted waves depends on the flux strength, it is approximately 1:2. The fact that the $L = 4R$ peak is suppressed at $\alpha = 0.25$ is obviously due to the possibility of the wave to bypass the flux line on either side: the two events have opposite phases, so that their contributions cancel exactly at $\alpha = 0.25$ as in the classical Aharonov-Bohm experiment. For $\alpha = 0.25$, the tiny signals at $L = 4R$ and hardly visible ones at $L = 6R$ and $8R$ are actually the higher harmonics with the repetitions of the reflected half-diameter orbit.

In summary, we have calculated the quantum level density of a spherical billiard with a magnetic flux line added at its center. The results show that the magnetic flux line has significant influence on the quantum level density. The Fourier transformed quantum level density of this system has allowed direct comparison between peaks in the level density and the length of the periodic orbits. From which we can testify the importance of the correspondence between the quantum effect and the classical motion. From Table 1 we can see, when the magnetic flux added to the billiard, its classical motion does not change. But because of the Aharonov-Bohm effect, the wave function of a particle acquires a phase $2\pi\alpha$ upon each rotation around the flux line. Thus its quantum level density is modified drastically. For particular magnetic flux strength, the amplitude of the peaks in the level density decreased and some of the peaks disappeared. This can be seen as a result of the Aharonov-Bohm effect, which manifests itself through the cancellation of periodic orbits. We hope that our result will provide a new experimental testing ground for exploring Aharonov-Bohm effect and quantum chaos.

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