Fast regularized image interpolation method

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The regularized image interpolation method is widely used based on the vector interpolation model in which down-sampling matrix has very large dimension and needs large storage consumption and higher computation complexity. In this paper, a fast algorithm for image interpolation based on the tensor product of matrices is presented, which transforms the vector interpolation model to matrix form. The proposed algorithm can extremely reduce the storage requirement and time consumption. The simulation results verify their validity.

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Image interpolation can be used in image enlargement and local image zooming. Several common interpolation algorithms have been suggested, such as zero-order interpolation, bi-linear interpolation[1], and cubic convolution interpolation[2]. However, image artifacts like blurring or zigzag on edge may occur when these interpolation schemes are used. In order to reduce the effect of image artifacts, other new methods have been proposed, including directional image interpolation[3], convolution-based interpolation[4], and edge-directed interpolation[5]. These methods take into account the edge information of image, and the vision effect is better than the conventional image interpolation methods.

Yoon *et al.*[6] presented regularized image sequence interpolation by fusing low-resolution (LR) frames. The regularized iterative image interpolation performs good subjective quality, nevertheless, requires lots of running time. In order to reduce the time-consuming, we present a fast regularization image interpolation method of single image based on matrix tensor product.

Let $x_{\mathrm{c}}(p,q)$ represent a two-dimensional (2D) spatially continuous image, and $x(m,n)$ is the corresponding digital image obtained by sampling $x_{\mathrm{c}}(p,q)$, with size $M\times N$, such as

$$x(m,n)=x_{\mathrm{c}}(mT_{\mathrm{v}},nT_{\mathrm{h}}),$$

$$m=0,1,\cdots,M-1;n=0,1,\cdots,N-1, \quad (1)$$

where T_{v} and T_{h} represent the vertical and horizontal sampling intervals respectively. In a similar way, the image with four times LR in both horizontal and vertical directions can be represented as

$$y(m,n)=\frac{1}{4}\sum_{i=0}^{1}\sum_{j=0}^{1}x(2m+i,2n+j),$$

$$m=0,1,\cdots,M/2-1;n=0,1,\cdots,N/2-1. \quad (2)$$

A discrete linear space-invariant degradation model for an $M/2\times N/2$ LR frame obtained by sub-sampling the original $M\times N$ high resolution image frame, can be given as[7−11]

$$\mathbf{y}=\mathbf{H}\mathbf{x}+\mathbf{n}, \quad (3)$$

where the $MN\times 1$ vector $\mathbf{x}$ represents the lexicographically ordered high resolution image frame, and the $MN/4\times 1$ vectors $\mathbf{y}$ and $\mathbf{n}$ represent observed LR and noise image frames, respectively. $\mathbf{H}$ is an $MN/4\times MN$ uniform down-sampling matrix.

The interpolation problem, therefore, can be formulated as solving the least squares problem for $\mathbf{x}$, given the observation $\mathbf{y}$. That is, we find the estimation, $\tilde{\mathbf{x}}$, which satisfies the following optimization problem[6],

$$\mathbf{x}=\arg\min f(\tilde{\mathbf{x}}), \quad (4)$$

where

$$f(\tilde{\mathbf{x}})=\left\|\mathbf{n}\right\|^2=\left\|\mathbf{y}-\mathbf{H}\tilde{\mathbf{x}}\right\|^2. \quad (5)$$

From the regularized image restoration theory, it is well known that solving Eq. (3) is an ill-posed problem. In order to make the problem better-posed, the following functional is minimized,

$$f(\tilde{\mathbf{x}})=\left\|\mathbf{y}-\mathbf{H}\tilde{\mathbf{x}}\right\|^2+\lambda\left\|\mathbf{L}\tilde{\mathbf{x}}\right\|^2, \quad (6)$$

where $\mathbf{L}$ is the regularization operator, which is preferably the three-dimensional (3D) Laplacian operator process, to capture the between-channel information in the interpolation process. The parameter λ is a global regularization parameter.

In order to solve the above equation given in Eq. (5), the successive approximation equation[8] describing the interpolated image $\tilde{\mathbf{x}}$, at the $k+1$ iteration step, is given by

$$\tilde{\mathbf{x}}^{k+1}=\tilde{\mathbf{x}}^{k}+\beta\mathbf{H}^{\mathrm{T}}(\mathbf{y}-\mathbf{H}\tilde{\mathbf{x}}^{k}), \quad (7)$$

where β means the function which controls the convergence rate, and k represents the iteration number. The successive approximation equation of Eq. (6) by using the same method may be represented as

$$\tilde{\mathbf{x}}^{k+1}=\tilde{\mathbf{x}}^{k}+\beta(\mathbf{H}^{\mathrm{T}}\mathbf{y}-(\mathbf{H}^{\mathrm{T}}\mathbf{H}+\lambda\mathbf{L}^{\mathrm{T}}\mathbf{L})\tilde{\mathbf{x}}^{k}). \quad (8)$$

In image processing field, bit map is the most commonly used image format. Consider the bit map with 256 gray levels, that is each pixel in this image must use 8 bits (1 byte) to represent when storing in computer. If using the observation model in Eq. (8), the degrading matrix $\mathbf{H}$ which is with the size of $MN/4 \times MN$, $M^2N^2/4$ bytes are needed to store matrix $\mathbf{H}$. It is impossible for common computer and lots of computation consumption is required to obtain the solution. In order to reduce the store consumption and computational complexity, a new regularized image interpolation is presented.

Now we discuss the tensor product decomposition of the down-sampling matrix $\mathbf{H}$ and regularization matrix $\mathbf{L}$. For a decimation factor of q, the decimation matrix $\mathbf{H}$ consists of q^2 non-zero elements of value $1/q^2$ along each row at appropriate locations and has the form[6]

$$\mathbf{H} = \frac{1}{q^2}\begin{bmatrix} 1\,1\cdots1 & & & 0 \\ & 1\,1\cdots1 & & \\ & & \ddots & \\ 0 & & & 1\,1\cdots1 \end{bmatrix}. \tag{9}$$

Equation (9) can be written as

$$\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2, \tag{10}$$

where $\otimes$ represents the tensor product, $\mathbf{H}_1$ of size $M/2 \times M$ and $\mathbf{H}_2$ of size $N/2 \times N$ represent the one-dimensional (1D) low-pass filtering and sub-sampling by factor of q respectively. As an example, for a decimation factor of $q = 2$ and $M = N$, $\mathbf{H}_1$ and $\mathbf{H}_2$ have the form as

$$\mathbf{H}_1 = \mathbf{H}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}. \tag{11}$$

The regularization matrix $\mathbf{L}$ is a BTTB (block-Toeplitz-Toeplitz-block) matrix of size $MN \times MN$ for a decimation factor of $q = 2$, and is represented as

$$\mathbf{L} = \begin{bmatrix} \mathbf{P} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B} & \mathbf{P} & \mathbf{B} & \ddots & \vdots \\ \mathbf{0} & \mathbf{B} & \mathbf{P} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \mathbf{B} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B} & \mathbf{P} \end{bmatrix}.$$

The structure of regularization matrix $\mathbf{L}$ can be decomposed to

$$\mathbf{L} = \begin{bmatrix} \mathbf{P} & & & \\ & \mathbf{P} & & \\ & & \ddots & \\ & & & \mathbf{P} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B} & & \\ \mathbf{B} & \mathbf{0} & \ddots & \\ & \ddots & \ddots & \mathbf{B} \\ & & \mathbf{B} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \otimes \mathbf{P} + \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix} \otimes \mathbf{B}$$

$$= \mathbf{I}_M \otimes \mathbf{P} - \mathbf{Q} \otimes \mathbf{I}_N, \tag{12}$$

where $\mathbf{I}_M$ and $\mathbf{I}_N$ are the identity matrices of size M and N respectively, the dimension of $\mathbf{P}$ is $N \times N$, and those of $\mathbf{B}$ and $\mathbf{Q}$ are $M \times M$, the expression is given by

$$\mathbf{P} = \begin{bmatrix} 1 & -\frac{1}{4} & 0 & \cdots & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} & \ddots & \vdots \\ 0 & -\frac{1}{4} & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{1}{4} \\ 0 & \cdots & 0 & -\frac{1}{4} & 1 \end{bmatrix},$$

$$\mathbf{B} = -\frac{1}{4}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 0 & \frac{1}{4} & & \\ \frac{1}{4} & 0 & \ddots & \\ & \ddots & \ddots & \frac{1}{4} \\ & & \frac{1}{4} & 0 \end{bmatrix}.$$

By using Eqs. (10) and (12), Eq. (8) is represented as

$$\tilde{\mathbf{x}}^{k+1} = \tilde{\mathbf{x}}^k + \beta((\mathbf{H}_1 \otimes \mathbf{H}_2)^{\mathrm{T}}\mathbf{y} - ((\mathbf{H}_1 \otimes \mathbf{H}_2)^{\mathrm{T}}(\mathbf{H}_1 \otimes \mathbf{H}_2)$$
$$+\lambda(\mathbf{I} \otimes \mathbf{P} - \mathbf{Q} \otimes \mathbf{I})^{\mathrm{T}}(\mathbf{I} \otimes \mathbf{P} - \mathbf{Q} \otimes \mathbf{I}))\tilde{\mathbf{x}}^k), \tag{13}$$

recalling the following properties of the Kronecker product[9]:

$$(\mathbf{A} \otimes \mathbf{B})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}} \otimes \mathbf{B}^{\mathrm{T}})$$
$$(\mathbf{A}_1 \otimes \mathbf{B}_1)(\mathbf{A}_2 \otimes \mathbf{B}_2) = (\mathbf{A}_1\mathbf{A}_2 \otimes \mathbf{B}_1\mathbf{B}_2)$$
$$(\mathbf{A} \otimes \mathbf{B})\mathrm{reshape}(\mathbf{V}) = \mathrm{reshape}(\mathbf{A}\mathbf{V}\mathbf{B}^{\mathrm{T}}), \tag{14}$$

where reshape$(\cdot)$ reorders the entries of a matrix in row-wise order into vector format, and applying these properties to Eq. (13), we have

$$\tilde{\mathbf{x}}^{k+1} = \tilde{\mathbf{x}}^k + \beta((\mathbf{H}_1^{\mathrm{T}} \otimes \mathbf{H}_2^{\mathrm{T}})\mathbf{y} - (\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1 \otimes \mathbf{H}_2^{\mathrm{T}}\mathbf{H}_2$$
$$+\lambda(\mathbf{I} \otimes \mathbf{P}^{\mathrm{T}}\mathbf{P} - \mathbf{Q} \otimes \mathbf{P}^{\mathrm{T}} - \mathbf{Q}^{\mathrm{T}} \otimes \mathbf{P} + \mathbf{Q}^{\mathrm{T}}\mathbf{Q} \otimes \mathbf{I}))\tilde{\mathbf{x}}^k). \tag{15}$$

By using Eq. (14), Eq. (8) is represented as

$$\tilde{\mathbf{X}}^{k+1} = \tilde{\mathbf{X}}^k + \beta(\mathbf{H}_1^{\mathrm{T}}\mathbf{Y}\mathbf{H}_2 - (\mathbf{H}_1^{\mathrm{T}}\mathbf{H}_1\tilde{\mathbf{X}}^k\mathbf{H}_2\mathbf{H}_2^{\mathrm{T}}$$
$$+\lambda(\tilde{\mathbf{X}}^k\mathbf{P}\mathbf{P}^{\mathrm{T}} - \mathbf{Q}\tilde{\mathbf{X}}^k\mathbf{P} - \mathbf{Q}^{\mathrm{T}}\tilde{\mathbf{X}}^k\mathbf{P}^{\mathrm{T}} + \mathbf{Q}^{\mathrm{T}}\mathbf{Q}\tilde{\mathbf{X}}^k))), \tag{16}$$

where $\tilde{\mathbf{X}}^k$ is the kth estimated image matrix. When $k = 0$, that is original value $\tilde{\mathbf{X}}^0 = \beta\mathbf{H}_1^{\mathrm{T}}\mathbf{Y}\mathbf{H}$; $\mathbf{Y}$ is the image matrix. This iteration continues until the cost function stabilizes or

$$\left\|\tilde{\mathbf{X}}^{n+1} - \tilde{\mathbf{X}}^n\right\|_F \Big/ \left\|\tilde{\mathbf{X}}^n\right\|_F < T \tag{17}$$

is satisfied, where T is the threshold value.

The regularization technique involves a tradeoff between fidelity to the data, as measured by the residual norm, and the fidelity to some prior information, as measured by the side constraint norm, the quality of the solution still depends on the value of regularization parameter λ. We adopt the L-curve approach for choosing the regularization parameter[10], it can be shown that the optimal λ for this criterion must satisfy

$$\lambda = \|\mathbf{y} - \mathbf{H}\tilde{\mathbf{x}}\|_2^2 \Big/ \|\mathbf{L}\tilde{\mathbf{x}}\|_2^2. \tag{18}$$

By using the Eqs. (10), (12) and (14), Eq. (18) is represented as

$$\lambda = \frac{\|\mathbf{y} - (\mathbf{H}_1 \otimes \mathbf{H}_2)\tilde{\mathbf{x}}\|_2^2}{\|(\mathbf{I} \otimes \mathbf{P} - \mathbf{Q} \otimes \mathbf{I})\tilde{\mathbf{x}}\|_2^2} = \frac{\left\|\mathbf{Y} - \mathbf{H}_1\tilde{\mathbf{X}}^k\mathbf{H}_2^{\mathrm{T}}\right\|_2^2}{\left\|\tilde{\mathbf{X}}^k\mathbf{P}^{\mathrm{T}} - \mathbf{Q}\tilde{\mathbf{X}}^k\right\|_2^2}. \tag{19}$$

The proposed regularized image interpolation method of Eq. (16) has some advantages, compared with interpolation method based on vector model, that the system storage consumption is reduced by tensor product decomposition for decimation matrix $\mathbf{H}$ and regularization matrix $\mathbf{L}$ and the proposed algorithm can reduce the computational complexity.

Now let us analyze the computational complexity of Eqs. (8) and (16) for each iteration. For simplicity, set $M = N = D$, the total operating numbers of the above two equations are denoted as OP (operating-count)[11]. For example, each element of calculating $\mathbf{H}^{\mathrm{T}}\mathbf{y}$ needs $MN/4$ times of operation. Since there are MN elements in matrix $\mathbf{H}^{\mathrm{T}}\mathbf{y}$, the total OP for calculating $\mathbf{H}^{\mathrm{T}}\mathbf{y}$ is $(MN)^2/4$. The required OPs for other parts of Eq. (8) have the same calculating method with $\mathbf{H}^{\mathrm{T}}\mathbf{y}$, so we get

$$\mathrm{OP} = MN(1+\frac{5}{4}MN + \frac{5}{4}M^2N^2) \approx \frac{5}{4}M^3N^3 = \frac{5}{4}D^6. \tag{20}$$

In the same way, we get the total OP of Eq. (16) as

$$\mathrm{OP} = \frac{3}{2}(M^3 + N^3)+2MN(\frac{19}{8}M+2N+1) \approx 11.8D^3. \tag{21}$$

By comparing Eq. (20) with Eq. (21), it is obvious that the OP of the proposed algorithms will drop exponentially with increment of image size, compared with the traditional algorithms.

To demonstrate the performance of the proposed interpolation algorithms, we present a number of experimental results. Also these results are compared with the traditional algorithms based on the vector model. These results are calculated on a packet computer.

In order to show the validity of proposed algorithms, we start with an original image of size 256×256 pixels. The image is then down-sampled by a factor of 2 to generate a LR image of size 128×128 pixels. The terminate criteria is $T = 0.5 \times 10^{-3}$. Experimental results are shown in Fig. 1. Figure 1 shows the 256×256 high resolution images reconstructed from the 128×128 LR images.

In Table 1, we compare the storage requirement of down-sampling matrix with the method based on the vector model for the test images of size from 16×16 to 256×256. The running time of test images for $2\times$ enlarged images was shown in Tables 1 and 2. Table 2 shows the time consumptions of traditional algorithm based on vector model. Only 16×16 and 32×32 images are used since the traditional algorithm based on vector model needs a long running time to get a solution. Comparing Table 1 with Table 2, we can get some results: firstly, the proposed algorithm can reduce the processing time extremely and obtain resolution in reasonable time; secondly, the time increment factor of traditional algorithm based on vector model is larger, from Table 2 this is scaled by hundreds but the presented algorithm is only several tens. In Table 3, we compare the peak signal-to-noise ratio (PSNR) of reconstructed image between the proposed algorithm and bi-linear interpolation method. We see that the proposed algorithm can acquire better quality.

Fig. 1. Interpolation images. (a) LR image; (b) bi-linear interpolation; (c) proposed method; (d) original HR image.

Table 1. Storage Consumption of Down-Sampling Matrix

LR Image	Vector Model	Matrix Model	Comparison
32×32	4 MB	4 kB	1 k
64×64	64 MB	16 kB	4 k
128×128	1 GB	64 kB	16 k
256×256	16 GB	256 kB	64 k

Table 2. Running Time (Seconds) for $2\times$ Enlarged Images

LR Image	Running Time of Vector Model	Running Time of Matrix Model
16×16	1.213	0.0310
32×32	274.532	0.1410
64×64	–	0.5630
128×128	–	4.0320

Table 3. PSNR (dB) Comparison of Bi-Linear Interpolation and the Proposed Method

Image	Bi-Linear Interpolation	Proposed Method
Lena	27.8990	29.0848
Couple	30.7515	32.2645
Woman	32.1401	34.5829

In this paper, a fast algorithm for image interpolation based on tensor product of matrix is presented. Our algorithm transforms the cost function based on vector model to the matrix form cost function. The proposed algorithm can reduce storage requirement extremely and obtain the optimal solution in a reasonable time. Experiment results show the proposed algorithm can also break the limits of dimensions that traditional iterative algorithm cannot implement.

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